Smith Normal Forms of Strongly Regular graphs

Peter Sin, U. of Florida

UD Discrete Math. Seminar, May 7th, 2018.

Smith normal forms

Collaborators

The coauthors for various parts of this talk are: Andries Brouwer, David Chandler, Josh Ducey, Venkata Raghu Tej Pantangi and Qing Xiang.

Smith normal forms associated with graphs

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Survey article on SNFs in combinatorics by R. Stanley (JCTA 2016).

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Origins and early work on $K(\Gamma)$ include: Sandpile model (Dhar), Chip-firing game (Biggs), Cycle Matroids (Vince).

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A has eigenvalues k, (mult. 1) r (mult. f) , s (mult. g).

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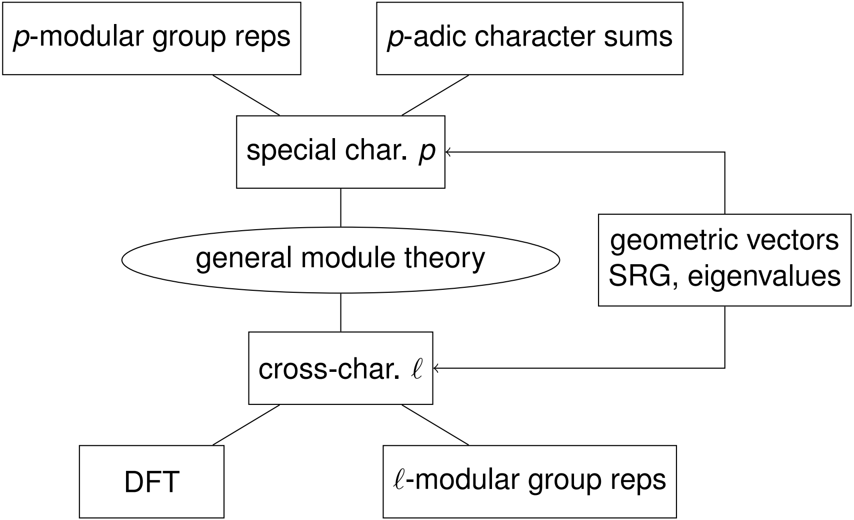
- Van Lint-Schrijver cyclotomic SRGs (Pantangi, 2018)


p-modular group reps
p-adic character sums
special char. p
general module theory
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A or L defines RG-module homomorphism

$$\alpha : R^V \to R^V$$

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$$M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_r = \mathrm{Ker}(\alpha) \supseteq 0.$$

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All quotients $\overline{M}_a/\overline{M}_{a+1}$ are FG-modules, so the number of nonzero e_i is at most the composition length of $\overline{M}$ as a FG-module.

Lemma

Let M, and α be as above. Let d be the ℓ-adic valuation of the product of the nonzero elementary divisors of α, counted with multiplicities. Suppose that we have an increasing sequence of indices $0 < a_1 < a_2 < \cdots < a_h$ and a corresponding sequence of lower bounds $b_1 > b_2 > \cdots > b_h$ satisfying the following conditions.

(a) $\dim_F \overline{M}_{a_j} \geq b_j$ *for* $j = 1, \ldots, h$.

(b) $\sum_{j=1}^{h} (b_j - b_{j+1}) a_j = d$, *where we set* $b_{h+1} = \dim_F \overline{\ker(\phi)}$.

Then the following hold.

(i) $e_{a_j}(\phi) = b_j - b_{j+1}$ *for* $j = 1, \ldots, h$.

(ii) $e_0(\phi) = \dim_F \overline{M} - b_1$.

(iii) $e_i(\phi) = 0$ *for* $i \notin \{0, a_1, \ldots, a_h\}$.

Paley graphs (Chandler-S-Xiang 2015)

Uses: DFT ($\mathbb{F}_q$-action) to get the p'-part, $\mathbb{F}_q^*$-action Jacobi sums and Transfer matrix method for p-part. The following gives the p-part of $K(\Gamma)$.

Theorem

Let $q = p^t$ be a prime power congruent to 1 modulo 4. Then the number of p-adic elementary divisors of $L(\mathrm{Paley}(q))$ which are equal to p^λ, $0 \leq \lambda < t$, is

$$f(t,\lambda) = \sum_{i=0}^{\min\{\lambda, t-\lambda\}} \frac{t}{t-i} \binom{t-i}{i} \binom{t-2i}{\lambda - i} (-p)^i \left(\frac{p+1}{2}\right)^{t-2i}.$$

The number of p-adic elementary divisors of $L(\mathrm{Paley}(q))$ which are equal to p^t is $\left(\frac{p+1}{2}\right)^t - 2$.

Examples

$$K(\mathrm{Paley}(5^3)) \cong (\mathbb{Z}/31\mathbb{Z})^{62} \oplus (\mathbb{Z}/5\mathbb{Z})^{36} \oplus (\mathbb{Z}/25\mathbb{Z})^{36} \oplus (\mathbb{Z}/125\mathbb{Z})^{25}.$$

$$K(\mathrm{Paley}(5^4)) \cong (\mathbb{Z}/156\mathbb{Z})^{312} \oplus (\mathbb{Z}/5\mathbb{Z})^{144} \oplus (\mathbb{Z}/25\mathbb{Z})^{176} \oplus (\mathbb{Z}/125\mathbb{Z})^{144} \oplus (\mathbb{Z}/625\mathbb{Z})^{79}.$$

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For $A(\Gamma)$, we can see that only k is divisible by p, so $S(\Gamma)$ is cyclic of order p^t.

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  Note $A(\Gamma) \equiv -L(\Gamma) \mod (p^{4t})$, so just consider $A(\Gamma)$.

Example: Skew lines in $\mathrm{PG}(3,9)$

Elem. Div.	1	3	3^2	3^4	3^5	3^6	3^8
Multiplicity	361	256	6025	202	256	361	1

Polar graphs

(Pantangi-S 2017) Uses: eigenvalue methods, structure of cross characteristric permutation modules (S-Tiep, 2005). The p-part is cyclic.

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Γ is an $\mathrm{SRG}\left(\begin{bmatrix} 2m \\ 1 \end{bmatrix}_q,\ q\begin{bmatrix} m-1 \\ 1 \end{bmatrix}_q (1+q^{m-1}),\ \begin{bmatrix} 2m-2 \\ 1 \end{bmatrix}_q - 2,\ \begin{bmatrix} 2m-2 \\ 1 \end{bmatrix}_q \right)$.

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$Spec(A) = (k, r, s) = (q\begin{bmatrix} m-1 \\ 1 \end{bmatrix}_q(1+q^{m-1}), q^{m-1}-1, -(1+q^{m-1}))$ with multiplicities
$(1, f, g) = (1,\ \frac{q(q^m-1)(q^{m-1}+1)}{2(q-1)},\ \frac{q(q^m+1)(q^{m-1}-1)}{2(q-1)})$

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- $|S| = |det(A)| = |kr^f s^g|$ and $|K| = t^f u^g / v$(by Kirchhoff's matrix-tree theorem.)

Description of $S(\Gamma)$ for Symplectic polar graph

Theorem

Let $\ell \mid |S|$. Then

(1) *If ℓ is odd prime with $v_\ell(1+q^{m-1}) = a > 0$, then $e_a(\ell) = g+1$, $e_0(\ell) = f$ and $e_i(\ell) = 0$ otherwise.*

(2) *If ℓ is an odd prime with $v_\ell(\left[{m-1 \atop 1}\right]_q) = a$ and $v_\ell(q-1) = b$, we have*

   (i) *If $a > 0$, $b > 0$, $e_{a+b}(\ell) = f$, $e_a(\ell) = 1$, $e_0(\ell) = g$ and $e_i(\ell) = 0$ for $i \neq 0, a+b, a$*

   (ii) *If $b = 0$, $e_a(\ell) = f+1$, $e_0(\ell) = g$ and $e_i(\ell) = 0$ for $i \neq 0, a$*

   (iii) *If $a = 0$, $e_b(\ell) = f$, $e_0(\ell) = g+1$ and $e_i(\ell) = 0$ for $i \neq 0, b$*

(Theorem Cont'd)

(3) If $\ell \mid q$, $e_{v_\ell(q)}(\ell) = 1$, $e_0(q) = f + g$ and $e_i(\ell) = 0$ for $i \neq v_\ell(q)$.

(4) If $\ell = 2$ and q is odd,

(i) If m is even, $e_a(2) = f - g - 1$, $e_{a+b}(2) = g + 1$, $e_0(2) = g + 1$ and $e_i(2) = 0$ for all other i's, where $a = v_2(q-1)$ and $b = v_2(q^{m-1} + 1)$.

(ii) If m is odd, $e_{a+b+1}(2) = g + 1$, $e_{a+b}(2) = f - g - 1$, $e_a(2) = 1$, $e_0(2) = g$ and $e_i(2) = 0$ for all other i's. Here, $v_2(\left[{m-1 \atop 1}\right]_q) = a$, $v_2(q-1) = b$.

Example: $q = 9$, $m = 3$

Γ is an $\mathrm{SRG}(66430, 7380, 818, 820)$. Eigenvalues $(7380, 80, -82)$ with multiplicities $(1, 33579, 32850)$.

$$S = \mathbb{Z}/9\mathbb{Z} \times (\mathbb{Z}/41\mathbb{Z})^{32581} \times (\mathbb{Z}/5\mathbb{Z})^{33580} \times (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/16\mathbb{Z})^{728} \times (\mathbb{Z}/32\mathbb{Z})^{32851}$$

$$K = (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/4\mathbb{Z})^{728} \times (\mathbb{Z}/8\mathbb{Z})^{32851} \times (\mathbb{Z}/41\mathbb{Z}) \times (\mathbb{Z}/91\mathbb{Z})^{32580} \times (\mathbb{Z}/25\mathbb{Z})^{33578} \times (\mathbb{Z}/5\mathbb{Z}) \times (\mathbb{Z}/73\mathbb{Z})^{33579}$$

Thank you for your attention!