Spreads, Ovoids, Opposites and Irreducible Group Representations

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Collaborators

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Ovoids and Spreads

New bounds

Oppositeness and simple modules

Association Schemes

Ovoids and spreads in polar spaces

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- A *partial spread* is a set generators where no two generators have a point in common. It is a *spread* if it covers every point.

Hermitian polar space

- The *Hermitian polar space* $\mathrm{H}(2d-1, q^2)$, for $q = p^t$ a prime power, is given by a non-degenerate Hermitian form f of $\mathbf{F}_{q^2}^{2d}$. The generators of $\mathbf{F}_{q^2}^{2d}$ have dimension d. A simple counting argument shows a partial spread of $\mathrm{H}(2d-1, q^2)$ has size at most $q^{2d-1}+1$. (No spreads exist, as shown bt Segre ($d=2$) and Thas ($d>2$).)

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- When d is odd, Vanhove found a better upper bound of q^d+1 for partial spreads, and both Aguglia and Luyckx showed that partial spreads of that size exist. So we are interested in the case when d is even.

A *generalized n-gon* of order (s, r) is a triple $\Gamma = (\mathcal{P}, \mathcal{L}, \mathrm{I})$, where elements of $\mathcal{P}$ are called *points*, elements of $\mathcal{L}$ are called *lines*, and $\mathrm{I} \subseteq \mathcal{P} \times \mathcal{L}$ is an *incidence relation* between the points and lines, which satisfies the following axioms:

1. Each line is incident with $s + 1$ points.
2. Each point is incident with $r + 1$ lines.
3. The *incidence graph* has diameter n and girth $2n$.

Here the incidence graph is the bipartite graph with $\mathcal{P} \cup \mathcal{L}$ as vertices, $p \in \mathcal{P}$ and $\ell \in \mathcal{L}$ are adjacent if $(p, \ell) \in \mathrm{I}$.

A *partial ovoid* of a generalized n-gon Γ is a set of points pairwise at distance n in the incidence graph. An easy counting argument shows that the size of a partial ovoid of a generalized octagon of order (s, r) is at most $(sr)^2 + 1$. A partial ovoid of a generalized octagon of order (s, r) is called an *ovoid* if it has the maximum possible size $(sr)^2 + 1$. The Ree-Tits octagon $\mathrm{O}(2^t)$ is a generalized octagon of order $(2^t, 4^t)$, so the size of an ovoid is $64^t + 1$. The only known thick finite generalized octagons are the *Ree-Tits octagons* $\mathrm{O}(2^t)$, t odd, and their duals.

Oppositeness

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- An partial ovoid is a clique in the oppositenes graph on points. A *partial spread* in a polar space is a clique in the oppositeness graph on the set of generators.

Lemma

Let $(X, \sim)$ be a graph. Let A be the adjacency matrix of X. Let Y be a clique of X. Then

$$|Y| \leq \begin{cases} \mathit{rank}_p(A) + 1, & \textit{if } p \textit{ divides } |Y| - 1, \\ \mathit{rank}_p(A), & \textit{otherwise.} \end{cases}$$

Proof.

Let J be the all-ones matrix of size $|Y| \times |Y|$. Let I be the identity matrix of size $|Y| \times |Y|$. As Y is a clique, the submatrix A' of A indexed by Y is $J - I$. Hence, the submatrix has p-rank $|Y| - 1$ if p divides $|Y| - 1$, and it has p-rank $|Y|$ if p does not divide $|Y| - 1$. As $\mathrm{rank}_p(A') \leq \mathrm{rank}_p(A)$, the assertion follows. $\square$

A new bound for partial ovoids in the *Ree-Tits octagons*

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Theorem 1

The size of a partial ovoid of the Ree-Tits octagon $\mathrm{O}(2^t)$*, t odd, is at most* $26^t + 1$*. In particular, no ovoids exist.*

Coolsaet and Van Maldeghem (2000) showed that in O(2) a partial ovoid has at most 27 points. They conjectured nonexistence of ovoids.

A new bound for partial spreads in Hermitian spaces

Theorem 2

Let $q = p^t$ with p prime and $t \geq 1$. Let Y be a partial spread of $\mathrm{H}(2d-1, q^2)$, where d is even.

(a) *If $d = 2$, then $|Y| \leq \left(\frac{2p^3+p}{3}\right)^t + 1$.*

(b) *If $d = 2$ and $p = 3$, then $|Y| \leq 19^t$.*

(c) *If $d > 2$, then $|Y| \leq \left(p^{2d-1} - p\frac{p^{2d-2}-1}{p+1}\right)^t + 1$.*

- For $d = 2$ the previous best known bound is $(q^3 + q + 2)/2$ by DeBeule (2008). For fixed p (and let $q = p^t$), the bound in part (a) is $o(q^3)$, which is asymptotically better than the bound of $(q^3 + q + 2)/2$.
- For $d > 2$ the new bound improves all previous bounds if $t > 1$.

Oppositeness modules

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  2. *We have*

$$rank_p(A(q)) = rank_p(A(p))^t.$$

- This result reduces the p-rank problem to the case $q = p$.

- Let $q = 2^t$, t odd. There is a Steinberg endomorphism τ of the algebraic group F_4 (over an algebraic closure of $\mathbf{F}_2$) such that the Ree group $G(q)$ is the subgroup of fixed points of τ, and the subgroup of fixed points of τ^2 is the Chevalley group $\mathrm{F}_4(q)$. The octagon $\mathrm{O}(2^t)$ is the building of $G(q)$. When $q = 2$, we find that ω is one of the fundamental weights, and the corresponding simple module has dimension 26.

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- For $\mathrm{H}(3, p^2)$, we use its duality to $\mathrm{Q}^-(5, p)$, so the oppositeness matrix of lines of of $\mathrm{H}(3, p^2)$ is the oppositeness matrix of points in $\mathrm{Q}^-(5, p)$. The dimension of the oppositeness module was calculated by Arslan-S.(2011) using algebraic group methods.

p-ranks from representation theory

Lemma

(a) *The* 2*-rank of the oppositeness matrix of* $\mathrm{O}(2)$ *is equal to 26.*

(b) *The* p*-rank of the oppositeness matrix of lines of* $\mathrm{H}(3, p^2)$ *is* $\frac{2p^3+p}{3}$.

Theorem 1 and the Theorem 2(a)-(b) now follow. For Theorem 2(c), the corresponding dimension of the oppositeness module is not known, and the p-rank of the oppositeness matrix is bounded using the representation theory of *association schemes*.

Definition

Let X be a finite set of size n. An *association scheme* with $d+1$ classes is a pair $(X, \mathcal{R})$, where $\mathcal{R} = \{R_0, \ldots, R_d\}$ is a set of symmetric binary relations on X with the following properties:

(a) $\mathcal{R}$ is a partition of $X \times X$.

(b) R_0 is the identity relation.

(c) There are numbers p_{ij}^k such that for $x, y \in X$ with xR_ky there are exactly p_{ij}^k elements z with xR_iz and zR_jy.

The relations R_i are described by their *adjacency matrices* $A_i \in \mathbb{C}^{n,n}$ defined by

$$(A_i)_{xy} = \begin{cases} 1, & \text{if } xR_iy, \\ 0, & \text{otherwise.} \end{cases}$$

A_d is the oppositeness matrix.

idempotents

Denote the all-ones matrix by J. There exist idempotent Hermitian matrices $E_j \in \mathbb{C}^{n,n}$ with the properties

$$\sum_{j=0}^{d} E_j = I, \qquad E_0 = n^{-1}J,$$

$$A_j = \sum_{i=0}^{d} P_{ij} E_i, \qquad E_j = \frac{1}{n} \sum_{i=0}^{d} Q_{ij} A_i,$$

where $P = (P_{ij}) \in \mathbb{C}^{d+1,d+1}$ and $Q = (Q_{ij}) \in \mathbb{C}^{d+1,d+1}$ are the so-called eigenmatrices of the association scheme. The P_{ij} are the eigenvalues of A_j. The multiplicity f_i of P_{ij} satisfies

$$f_i = \operatorname{rank}(E_i) = \operatorname{tr}(E_i) = Q_{0i}.$$

Association scheme for $\mathrm{H}(2d-1, q^2)$

From $\mathrm{H}(2d-1, q^2)$ we get the following association scheme. Let X be the set of generators of $\mathrm{H}(2d-1, q^2)$ and two generators a, b are in relation R_i, where $0 \leq i \leq d$, if and only if a and b intersect in codimension i. For this scheme it is known that

$$f_d = q^{2d}\frac{q^{1-2d}+1}{q+1} = q^{2d-1} - q\frac{q^{2d-2}-1}{q+1},$$

and

$$Q_{id} = \frac{P_{di}}{P_{0i}}Q_{0d} = \frac{(-1)^i f_d}{q^i}$$

E_d has rank

$$f_d = p^{2d-1} - p\frac{p^{2d-2}-1}{p+1}.$$

When d even the matrix $np^{d-1}E_d$ has only integer entries and we have $A_d \equiv np^{d-1}E_d \mod p$. Hence, $\mathrm{rank}_p(A_d) = \mathrm{rank}_p(np^{d-1}E_d) \leq \mathrm{rank}(np^{d-1}E_d) = \mathrm{rank}(E_d) = p^{2d-1} - p^{\frac{p^{2d-2}-1}{p+1}}$.

Lemma

The p-rank of the oppositeness matrix of generators of $\mathrm{H}(2d-1, p^2)$, d even, is at most $p^{2d-1} - p^{\frac{p^{2d-2}-1}{p+1}}$.

Theorem 2(c) now follows.

Thank you for your attention!

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