Spreads, Ovoids, Opposites and Irreducible Group Representations

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Ovoids and Spreads

New bounds

Oppositeness and simple modules

Association Schemes
Collaborators

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A *partial spread* is a set of generators where no two generators have a point in common. It is a *spread* if it covers every point.
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Ovoids and spreads in polar spaces

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- A *partial spread* is a set generators where no two generators have a point in common. It is a *spread* if it covers every point.
The *Hermitian polar space* \( H(2d - 1, q^2) \), for \( q = p^t \) a prime power, is given by a non-degenerate Hermitian form \( f \) of \( \mathbb{F}_{q^2}^{2d} \). The generators of \( \mathbb{F}_{q^2}^{2d} \) have dimension \( d \). A simple counting argument shows a partial spread of \( H(2d - 1, q^2) \) has size at most \( q^{2d-1} + 1 \). (No spreads exist, as shown by Segre \( (d = 2) \) and Thas \( (d > 2) \).)

When \( d \) is odd, Vanhove found a better upper bound of \( q^d + 1 \) for partial spreads, and both Aguglia and Luyckx showed that partial spreads of that size exist. So we are interested in the case when \( d \) is even.
The Hermitian polar space $H(2d - 1, q^2)$, for $q = p^t$ a prime power, is given by a non-degenerate Hermitian form $f$ of $\mathbb{F}_{q^2}^{2d}$. The generators of $\mathbb{F}_{q^2}^{2d}$ have dimension $d$. A simple counting argument shows a partial spread of $H(2d - 1, q^2)$ has size at most $q^{2d-1} + 1$. (No spreads exist, as shown by Segre ($d = 2$) and Thas ($d > 2$).)

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A generalized \textit{n-gon} of order \((s, r)\) is a triple \(\Gamma = (\mathcal{P}, \mathcal{L}, I)\), where elements of \(\mathcal{P}\) are called \textit{points}, elements of \(\mathcal{L}\) are called \textit{lines}, and \(I \subseteq \mathcal{P} \times \mathcal{L}\) is an \textit{incidence relation} between the points and lines, which satisfies the following axioms:

1. Each line is incident with \(s + 1\) points.
2. Each point is incident with \(r + 1\) lines.
3. The \textit{incidence graph} has diameter \(n\) and girth \(2n\).

Here the incidence graph is the bipartite graph with \(\mathcal{P} \cup \mathcal{L}\) as vertices, \(p \in \mathcal{P}\) and \(\ell \in \mathcal{L}\) are adjacent if \((p, \ell) \in I\).
A partial ovoid of a generalized $n$-gon $\Gamma$ is a set of points pairwise at distance $n$ in the incidence graph. An easy counting argument shows that the size of a partial ovoid of a generalized octagon of order $(s, r)$ is at most $(sr)^2 + 1$. A partial ovoid of a generalized octagon of order $(s, r)$ is called an ovoid if it has the maximum possible size $(sr)^2 + 1$. The Ree-Tits octagon $O(2^t)$ is a generalized octagon of order $(2^t, 4^t)$, so the size of an ovoid is $64^t + 1$. The only known thick finite generalized octagons are the Ree-Tits octagons $O(2^t)$, $t$ odd, and their duals.
Oppositeness

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An partial ovoid is a clique in the oppositeness graph on points. A *partial spread* in a polar space is a clique in the oppositeness graph on the set of generators.
Lemma

Let $(X, \sim)$ be a graph. Let $A$ be the adjacency matrix of $X$. Let $Y$ be a clique of $X$. Then

$$|Y| \leq \begin{cases} \text{rank}_p(A) + 1, & \text{if } p \text{ divides } |Y| - 1, \\ \text{rank}_p(A), & \text{otherwise}. \end{cases}$$

Proof.

Let $J$ be the all-ones matrix of size $|Y| \times |Y|$. Let $I$ be the identity matrix of size $|Y| \times |Y|$. As $Y$ is a clique, the submatrix $A'$ of $A$ indexed by $Y$ is $J - I$. Hence, the submatrix has $p$-rank $|Y| - 1$ if $p$ divides $|Y| - 1$, and it has $p$-rank $|Y|$ if $p$ does not divide $|Y| - 1$. As $\text{rank}_p(A') \leq \text{rank}_p(A)$, the assertion follows.
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A new bound for partial ovoids in the *Ree-Tits octagons*

Theorem 1

The size of a partial ovoid of the Ree-Tits octagon $O(2^t)$, $t$ odd, is at most $26^t + 1$. In particular, no ovoids exist.

Coolsaet and Van Maldeghem (2000) showed that in $O(2)$ a partial ovoid has at most 27 points. They conjectured nonexistence of ovoids.
Theorem 2
Let $q = p^t$ with $p$ prime and $t \geq 1$. Let $Y$ be a partial spread of $H(2d - 1, q^2)$, where $d$ is even.

(a) If $d = 2$, then $|Y| \leq \left( \frac{2p^3+p}{3} \right)^t + 1$.

(b) If $d = 2$ and $p = 3$, then $|Y| \leq 19^t$.

(c) If $d > 2$, then $|Y| \leq \left( p^{2d-1} - p^{2d-2} \frac{p^2-1}{p+1} \right)^t + 1$.

- For $d = 2$ the previous best known bound is $(q^3 + q + 2)/2$ by DeBeule (2008). For fixed $p$ (and let $q = p^t$), the bound in part (a) is $o(q^3)$, which is asymptotically better than the bound of $(q^3 + q + 2)/2$.

- For $d > 2$ the new bound improves all previous bounds if $t > 1$. 
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- **Proposition (S, 2012)**

  Let \( G(q) \), \( q = p^t \) a prime power, be a finite group of Lie type and let \( A(q) \) denote the oppositeness matrix for objects of a fixed self-opposite type in the building of \( G(q) \).
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Let $G(q)$, $q = p^t$ a prime power, be a finite group of Lie type and let $A(q)$ denote the oppositeness matrix for objects of a fixed self-opposite type in the building of $G(q)$.

1. The column space of $A(q)$ over $\mathbb{F}_q$ is a simple $\mathbb{F}_q G(q)$-module with highest weight $(q - 1)\omega$, where $\omega$ is a (explicity known) sum of fundamental weights.
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     $$\text{rank}_p(A(q)) = \text{rank}_p(A(p))^t.$$
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  $$\text{rank}_p(A(q)) = \text{rank}_p(A(p))^t.$$ 

- This result reduces the $p$-rank problem to the case $q = p$. 
Let \( q = 2^t, \) t odd. There is a Steinberg endomorphism \( \tau \) of the algebraic group \( F_4 \) (over an algebraic closure of \( F_2 \)) such that the Ree group \( G(q) \) is the subgroup of fixed points of \( \tau \), and the subgroup of fixed points of \( \tau^2 \) is the Chevalley group \( F_4(q) \). The octagon \( O(2^t) \) is the building of \( G(q) \). When \( q = 2 \), we find that \( \omega \) is one of the fundamental weights, and the corresponding simple module has dimension 26.
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For $H(3, p^2)$, we use its duality to $Q^-(5, p)$, so the oppositeness matrix of lines of $H(3, p^2)$ is the oppositeness matrix of points in $Q^-(5, p)$. The dimension of the oppositeness module was calculated by Arslan-S. (2011) using algebraic group methods.
Lemma

(a) The 2-rank of the oppositeness matrix of $O(2)$ is equal to 26.

(b) The $p$-rank of the oppositeness matrix of lines of $H(3, p^2)$ is $\frac{2p^3 + p}{3}$.

Theorem 1 and the Theorem 2(a)-(b) now follow. For Theorem 2(c), the corresponding dimension of the oppositeness module is not known, and the $p$-rank of the oppositeness matrix is bounded using the representation theory of association schemes.
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Definition
Let $X$ be a finite set of size $n$. An association scheme with $d + 1$ classes is a pair $(X, \mathcal{R})$, where $\mathcal{R} = \{R_0, \ldots, R_d\}$ is a set of symmetric binary relations on $X$ with the following properties:

(a) $\mathcal{R}$ is a partition of $X \times X$.

(b) $R_0$ is the identity relation.

(c) There are numbers $p_{ij}^k$ such that for $x, y \in X$ with $xR_ky$ there are exactly $p_{ij}^k$ elements $z$ with $xR_i z$ and $zR_j y$. 


The relations $R_i$ are described by their *adjacency matrices* $A_i \in \mathbb{C}^{n,n}$ defined by

$$(A_i)_{xy} = \begin{cases} 1, & \text{if } xR_iy, \\ 0, & \text{otherwise}. \end{cases}$$

$A_d$ is the oppositeness matrix.
Denote the all-ones matrix by $J$. There exist idempotent Hermitian matrices $E_j \in \mathbb{C}^{n,n}$ with the properties

$$
\sum_{j=0}^{d} E_j = I, \quad E_0 = n^{-1}J,
$$

$$
A_j = \sum_{i=0}^{d} P_{ij} E_i, \quad E_j = \frac{1}{n} \sum_{i=0}^{d} Q_{ij} A_i,
$$

where $P = (P_{ij}) \in \mathbb{C}^{d+1,d+1}$ and $Q = (Q_{ij}) \in \mathbb{C}^{d+1,d+1}$ are the so-called eigenmatrices of the association scheme. The $P_{ij}$ are the eigenvalues of $A_j$. The multiplicity $f_i$ of $P_{ij}$ satisfies

$$
f_i = \text{rank}(E_i) = \text{tr}(E_i) = Q_{0i}.
$$
Association scheme for $H(2d - 1, q^2)$

From $H(2d - 1, q^2)$ we get the following association scheme. Let $X$ be the set of generators of $H(2d - 1, q^2)$ and two generators $a, b$ are in relation $R_i$, where $0 \leq i \leq d$, if and only if $a$ and $b$ intersect in codimension $i$. For this scheme it is known that

$$f_d = q^{2d} \frac{q^{1-2d} + 1}{q + 1} = q^{2d-1} - q \frac{q^{2d-2} - 1}{q + 1},$$

and

$$Q_{id} = \frac{P_{di}}{P_{0i}} Q_{0d} = \frac{(-1)^i f_d}{q^i}.$$

$E_d$ has rank

$$f_d = p^{2d-1} - p \frac{p^{2d-2} - 1}{p + 1}.$$
When $d$ even the matrix $np^{d-1}E_d$ has only integer entries and we have $A_d \equiv np^{d-1}E_d \mod p$. Hence, 
\[ \text{rank}_p(A_d) = \text{rank}_p(np^{d-1}E_d) \leq \text{rank}(np^{d-1}E_d) = \text{rank}(E_d) = p^{2d-1} - p^{p^{2d-2}-1} \frac{1}{p+1}. \]

**Lemma**

The $p$-rank of the oppositeness matrix of generators of $H(2d - 1, p^2)$, $d$ even, is at most $p^{2d-1} - p^{p^{2d-2}-1} \frac{1}{p+1}$. Theorem 2(c) now follows.
Thank you for your attention!


References


