This exam is **open book**. You may use your textbook and your own notes. Write your proofs using *complete English sentences* as well as mathematical formulae. You may refer to results from the text by number or by briefly stating them.
Work which the grader cannot follow may receive a grade of zero.
Extra credit may be awarded for particularly well-written answers.
In this exam F denotes a field and $\mathbb{R}$ denotes the field of real numbers.

Name: ______________________________

1. (8 points) In $P_3(\mathbb{R})$, consider the set

$$X = \{1 + 2x - x^3, x + x^2, 2 + 4x^2 - 2x^3, 1 + x - x^2 - x^3\}.$$

of four polynomials. Find a subset of X which is a basis for the span of X.

Solution:

We must find a linearly independent subset Y of X such that the span of Y is the span of X. Let $f_1 = 1+2x-x^3$, $f_2 = x+x^2$, $f_3 = 2+4x^2-2x^3$, and $f_4 = 1+x-x^2-x^3$, and fix the standard basis $\beta = \{1, x, x^2, x^3\}$. Then these are linearly independent in $P_3(\mathbb{R})$ if and only if their coordinate vectors are linearly independent in $\mathbb{R}^4$. Let

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 2 & 1 & 0 & 1 \\ 0 & 1 & 4 & -1 \\ -1 & 0 & -2 & -1 \end{pmatrix}$$

be the matrix with the coordinate of f_i as the i-th column. Note that this has reduced row echelon form

$$B = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which has rank 3, and the first, second and third rows are e_1, e_2, and e_3, respectively. By Theorem 3.16 of the text, we have that these columns are linearly independent, and have the span equal to that of the span of the whole column set. Hence, $\{f_1, f_2, f_3\}$ are linearly independent.

2. Let the linear map $T : P_3(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be given by $T(f(x)) = \begin{pmatrix} f(1) & f'(1) \\ f''(1) & f'''(1) \end{pmatrix}$.

   (a) (4 points) By computing with the matrix of T or otherwise, prove that T is invertible.

(b) (4 points) Compute $T^{-1}(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix})$.

Solution: (a) Let $\alpha = \{1, x, x^2, x^3\}$ and $\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$. Note then that T is invertible if and only if

$$[T]_\alpha^\beta = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 6 \end{pmatrix}$$

is invertible. As it is an upper triangular matrix, we get that it has determinant $(1)(1)(2)(6) = 12 \neq 0$. Hence, $[T]_\alpha^\beta$ is invertible.

(b) As T is invertible, it suffices to find $f \in P_3(\mathbb{R})$ such that $T(f) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Suppose that $f = a+bx+cx^2+dx^3$ if such a polynomial, then $6d = f'''(1) = 1$ so $d = 1/6$. Then $1 = f''(1) = 2c+6d = 2c+1$ implies $c = 0$. With $1 = f'(1) = b+2cx+3d = b+(1/2)$ we get $b = 1/2$, and so $1 = f(1) = a+b+c+d = a+(2/3)$ we get that $a = 1/3$. Thus, $T(\frac{1}{3} + \frac{1}{2}x + \frac{1}{6}x^3) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

3. (8 points) Determine whether the linear map $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by

$$T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} -a & b-2a \\ -c-2b+2a & -d-2b+2a \end{pmatrix}$$

is diagonalizable or not. is diagonalizable or not. If so, find a basis of $M_{2\times 2}(\mathbb{R})$ consisting of eigenvectors of T.

Solution: Let $\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$, the standard basis for $M_{2\times 2}(\mathbb{R})$. Computing the values under T, we get that

$$[T]_\beta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 2 & -2 & -1 & 0 \\ 2 & -2 & 0 & -1 \end{pmatrix}$$

so that the characteristic polynomial is

$$f(t) = \det\left(\begin{pmatrix} -1-t & 0 & 0 & 0 \\ -2 & 1-t & 0 & 0 \\ 2 & -2 & -1-t & 0 \\ 2 & -2 & 0 & -1-t \end{pmatrix}\right) = (t-1)(t+1)^3$$

which has roots 1 and -1 of multiplicity 1 and 3, respectively. For diagonalizability, we need only check -1. Consider the eigenspace E_{-1} and the matrix

$$B_{-1} = [T]_\beta + I = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 2 & -2 & 0 & 0 \end{pmatrix}$$

which has rank 1. Now, $\dim(E_{-1}) = \dim(N(L_{B_{-1}})) = 4 - \text{rank}(B_{-1}) = 4 - 1 = 3$ so T is indeed diagonalizable. Solving

$$B_{-1} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

yields the solution set $\left\{ u \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} : u, t, s \in \mathbb{R} \right\}$ so that

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is a basis for the B_{-1}-eigenspace of -1. Converting, we get that

$$\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

is a basis for the T-eigenspace of -1. Now for

$$B_1 = [T]_\beta - I = \begin{pmatrix} -2 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 2 & -2 & -2 & 0 \\ 2 & -2 & 0 & -2 \end{pmatrix},$$

solving

$$B_1 = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

yields the solution set

$$\left\{ t \begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\}$$

so that $\left\{\begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \end{pmatrix}\right\}$ is a basis for the B_1-eigenspace of 1, so $\left\{\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}\right\}$ is a basis for the T-eigenspace of 1. Thus,

$$\left\{\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}\right\}$$

is a basis for $M_{2\times 2}(\mathbb{R})$ consisting of eigenvectors of T.

4. (8 points) Use the Gram-Schmidt procedure to find an orthonormal basis of the subspace of $\mathbb{R}^4$ spanned by $(1, 0, 0, -1)$, $(1, 0, -1, 0)$, and $(1, -1, 0, 0)$.

Solution: Recall that the Gram-Schmidt procedure requires a linearly independent set, though a quick check will verify that these are indeed linearly independent. Let $v_1 = (1, 0, 0, -1)$. We have that

$$\frac{\langle w_2, v_1\rangle}{\|v_1\|^2} v_1 = (1/2, 0, 0, -1/2)$$

so

$$v_2 = w_2 - \frac{\langle w_2, v_1\rangle}{\|v_1\|^2} v_1 = (1, 0, -1, 0) - (1/2, 0, 0, -1/2) = (1/2, 0, -1, 1/2).$$

Now, we have

$$\frac{\langle w_3, v_1\rangle}{\|v_1\|^2} v_1 = (1/2, 0, 0, -1/2)$$

and

$$\frac{\langle w_3, v_2\rangle}{\|v_2\|^2} v_2 = (1/6, 0, -1/3, 1/6)$$

so that

$$\sum_{i=1}^{2} \frac{\langle w_3, v_i\rangle}{\|v_i\|^2} v_i = (2/3, 0, -1/3, -1/3).$$

Thus,

$$v_3 = w_3 - \sum_{i=1}^{2} \frac{\langle w_3, v_i\rangle}{\|v_i\|^2} v_i = (1/3, -1, 1/3, 1/3)$$

and $\beta = \{v_1, v_2, v_3\}$ is an orthogonal set with span equal to that of the original set. We may obtain an orthonormal basis by normalizing β. Let

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}}(1, 0, 0, -1),$$

$$u_2 = \frac{v_2}{\|v_2\|} = \sqrt{\frac{2}{3}}(-1/2, 0, -1, 1/2)$$

and

$$u_3 = \frac{v_3}{\|v_3\|} = \sqrt{\frac{3}{4}}(1/3, -1, 1/3, 1/3)$$

so that $\gamma = \{u_1, u_2, u_3\}$ is an orthonormal basis for the span of the original set.

5. Let V be the space of infinitely differentiable functions of the real line.
   (a) (2 points) Show that differentiation is a linear map from V to itself.
   (b) (2 points) Show that for each $a \in \mathbb{R}$ the function $f(t) = e^{at}$ is an eigenvector of T, with eigenvalue a.
   (c) (4 points) Using the above, or otherwise, show that the set of functions $\{e^{at} | a \in \mathbb{R}\}$ is linearly independent.
   (d) (4 points) (Extra credit) Prove that the set of functions $\{\sin at | a \in \mathbb{R}, a > 0\}$ is linearly independent.

Solution: (a) Let $f, g \in V$, let $c \in \mathbb{R}$. If $T : V \to V$ denotes differentiation, then

$$T(f + cg) = (f + cg)' = f' + cg' = T(f) + cT(g)$$

so that T is linear.

(b) We have that

$$T(f(t)) = \frac{d}{dt}(e^{at}) = ae^{at} = af(t)$$

so that $f(t)$ is an eigenvector of T with eigenvalue a.

(c) Recall (**Theorem 5.5** of the text) that if T is a linear operator on a vector space and $\lambda_1, \ldots, \lambda_k$ are distinct eigenvalues of T, with v_i an eigenvector of λ_i, then $\{v_1, \ldots, v_k\}$ is linearly independent. To show that $A = \{e^{at} : a \in \mathbb{R}\}$ is linearly independent, we must show that any finite subset of A is linearly independent. Consider $B = \{e^{a_1 t}, \ldots, e^{a_k t}\}$, so that $e^{a_i t}$ has eigenvalue a_i by part (b). By the beginning remark, we get that B is linearly independent.

(d) Note that $S : V \to V$, $S = T \circ T$ is also a linear transformation by composition, and S just maps a function f to its second derivative. Then for $A = \{\sin at : a \in \mathbb{R}, a > 0\}$, we have that for each $a > 0$, $S(\sin at) = -a^2 \sin at$ so that $\sin at$ is an eigenvector with eigenvalue $-a^2$. If $B = \{\sin a_1 t, \ldots, \sin a_k t\}$ is a finite subset of A, then it will be linearly independent as in part (c).