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# $G$-invariant quadratic forms

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### Introduction

The Frobenius-Schur indicator ([4], Chap. XI, § 8) tells us whether a self-dual complex representation of a finite group is an orthogonal or a symplectic one. In the $p$-modular theory, there is an algorithm derived from this criterion for determining the type of a $G$-invariant form on a self-dual, simple module as long as $p$ is odd (see [5]). In characteristic two, the problem appears to be subtle and has not yet found a satisfactory answer. We therefore aim in this paper to investigate systematically modules with $G$-invariant quadratic forms, paying particular attention to fields of characteristic two. Our main result gives a simple way to compute the Witt index of a $G$-invariant quadratic form when $G$ is a finite solvable group and the field is finite of characteristic two. Our methods allow us to simplify and unify some known results along the way (2.3, 2.4, 3.4).

Several of the results (Proposition 4.9(b), Proposition 4.10 and Theorem 5.4) have been obtained first by M. Aschbacher in [2] ((7.3)(1), (7.3)(2) and (7.6) respectively), but we have included the proofs for the convenience of the reader and because our argument in Theorem 5.4 differs from his proof of [2], (7.6). We wish to thank the referee for drawing our attention to this work.

### § 1. Quadratic modules in characteristic two

Let $k$ be a perfect field of characteristic two and $V$ a finite-dimensional $k$-vector space. Since $k$ is perfect, the Frobenius map

$$\sigma : x \rightarrow x^2$$

is an automorphism of $k$. We denote by $V^{(2)}$ the $k$-module with the same underlying group as $V$ but with scalars operating by

$$\lambda \circ v := \sigma^{-1}(\lambda) v,$$

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where the right hand side is the original multiplication. Let $S^2(V^*)$ and $\Lambda^2(V^*)$ denote respectively the spaces of quadratic and symplectic forms on $V$. We recall that a quadratic form is defined to be a map

$$Q : V \to k$$

satisfying the following conditions (1.1):

(a) $Q(\lambda v) = \lambda^2 Q(v)$, $(\lambda \in k, v \in V)$;

(b) The map $B : V \times V \to k$ defined by $B(v, w) = Q(v + w) - Q(v) - Q(w)$ is a symmetric bilinear form.

Of course, in characteristic two, the associated symmetric bilinear forms are symplectic; condition (b) gives a natural map (*polarization*)

$$\theta : S^2(V^*) \to \Lambda^2(V^*)$$

taking a quadratic form to its associated symplectic form. Now a quadratic form $Q$ lies in the kernel of $\theta$ if and only if it satisfies the following (1.2):

(a) $Q(\lambda v) = \lambda^2 Q(v)$, $(\lambda \in k, v \in V)$;

(b) $Q(v + w) = Q(v) + Q(w)$, $(v, w \in V)$.

Since all field elements are squares, these conditions simply say that $Q \in V^{(2)*}$. By considering dimensions, we obtain the exact sequence of natural maps

$$(1.3) \qquad 0 \to V^{(2)*} \to S^2(V^*) \xrightarrow{\theta} \Lambda^2(V^*) \to 0.$$

Next, we consider the space $V^* \otimes_k V^*$ of bilinear forms on $V$. Each bilinear form $B$ on $V$ defines a quadratic form $Q$ by "restriction to the diagonal": $Q(v) := B(v, v)$, for $v \in V$. Thus, we obtain a natural map

$$\zeta : V^* \otimes_k V^* \to S^2(V^*)$$

whose kernel is obviously $\Lambda^2(V^*)$. Since $\zeta$ is also surjective, we see that the following sequence of natural maps is exact:

$$(1.4) \qquad 0 \to \Lambda^2(V^*) \to V^* \otimes_k V^* \xrightarrow{\zeta} S^2(V^*) \to 0.$$

We note that (1.4) holds over any field, and splits naturally if the characteristic of $k$ is not two.

We end this section with some further basic definitions and facts which we shall use freely without further reference. First, the (*left*) *radical* of $B \in V^* \otimes_k V^*$ is defined by $\operatorname{Rad} B = \{v \in V \mid B(v, w) = 0 \text{ for all } w \in V\}$, and $B$ is *nonsingular* if $\operatorname{Rad} B = 0$. Under the natural isomorphism $V^* \otimes_k V^* \cong \operatorname{Hom}_k(V, V^*)$, nonsingular forms correspond to isomorphisms. Finally, we have natural isomorphisms

$$S^2(V^* \oplus W^*) \cong S^2(V^*) \oplus S^2(W^*) \oplus V^* \otimes_k W^*$$

and

$$\Lambda^2(V^* \oplus W^*) \cong \Lambda^2(V^*) \oplus \Lambda^2(W^*) \oplus V^* \otimes_k W^*.$$

## § 2. Quadratic $kG$-modules

We keep the notation of the previous section. Let $G$ be a group, acting as automorphisms of $V$. Then the induced automorphisms define $G$-module structures on $S^2(V^*)$, $\Lambda^2(V^*)$ and $V^* \otimes_k V^*$ and the subspaces $S^2(V^*)^G$, $\Lambda^2(V^*)^G$ and $(V^* \otimes_k V^*)^G$ of $G$-fixed points are precisely the $G$-invariant forms of the various kinds. Since the space of $G$-invariant bilinear forms on $V$ is isomorphic to the space $\mathrm{Hom}_{kG}(V, V^*)$ of $G$-equivariant linear maps from $V$ to its dual, there is a nonsingular $G$-invariant bilinear form on $V$ if and only if $V \cong_{kG} V^*$. If $V$ is simple, then any nonzero $G$-invariant form is nonsingular because its radical is a $G$-submodule.

Combining (1.3) and (1.4) we obtain the following diagram of natural maps:

$$\text{(2.1)} \qquad \begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \uparrow & & & & \\ 0 & \to & V^{(2)*} & \to & S^2(V^*) & \xrightarrow{\theta} & \Lambda^2(V^*) & \to & 0 \\ & & & & \zeta\uparrow & & & & \\ & & & & V^* \otimes_k V^* & & & & \\ & & & & \uparrow & & & & \\ & & & & \Lambda^2(V^*) & & & & \\ & & & & \uparrow & & & & \\ & & & & 0 & & & & \end{array}$$

Taking cohomology, we obtain the diagram

$$\text{(2.2)} \qquad \begin{array}{ccccccccc} & & & & H^1(G, \Lambda^2(V^*)) & & & & \\ & & & & \uparrow & & & & \\ 0 & \to & (V^{(2)*})^G & \to & S^2(V^*)^G & \xrightarrow{\theta} & \Lambda^2(V^*)^G & \to & H^1(G, V^{(2)*}) \\ & & & & \zeta\uparrow & & & & \\ & & & & (V^* \otimes_k V^*)^G & & & & \\ & & & & \uparrow & & & & \\ & & & & \Lambda^2(V^*)^G & & & & \\ & & & & \uparrow & & & & \\ & & & & 0 & & & & \end{array}$$

in which both the vertical and horizontal sequences are exact.

Now we may read off various results from (2.2).

**Proposition 2.3** (Fong), (cf. [4], Chap. VII, 8.13). *Let $V \neq 0$ be a self-dual G-module over k with $V^G = 0$. Then V has a nonzero G-invariant symplectic form.*

*Proof.* Since $V \cong V^*$ we have $(V^* \otimes_k V^*)^G \neq 0$. Since $V^G = 0$, it follows that $(V^{(2)*})^G = 0$. Therefore the map $\theta$ in (2.2) is injective. Thus, the map $\theta\zeta$ in (2.2) has either a nonzero image or a nonzero kernel, so the proposition is proved.

**Proposition 2.4** (cf. [6], Satz 2.5). *Let V be a self-dual G-module over k and suppose that $H^1(G, V) = 0$. Then every G-invariant symplectic form on V is the polarization of a G-invariant quadratic form.*

*Proof.* As $V \cong V^*$, $H^1(G, V) = 0$ implies $H^1(G, V^{(2)*}) = 0$, and the result follows from the horizontal sequence of (2.2).

The hypothesis on the cohomology in 2.4 is obviously satisfied if $V$ is a projective indecomposable $kG$-module for a finite group $G$. If in addition, $V$ is self-dual and not equal to the projective cover of the trivial module, then $\Lambda^2(V^*)^G \neq 0$ by 2.3, since $V^G = 0$. However, there may not be a nonsingular $G$-invariant symplectic form, as shown by the following easy example with a group of order 12.

**Example 2.5.** Let $G = \langle g, t \mid g^3 = t^4 = 1, g^t = g^{-1} \rangle$, and $k = \mathbb{F}_2$. The projective cover $V$ of the simple 2-dimensional $kG$-module $M$ is an extension of $M$ with itself, and is self-dual since $M$ is. One easily computes that $\Lambda^2(V)$ is isomorphic to the direct sum of $M$ and the projective cover of the trivial module, hence that $\Lambda^2(V)^G \cong k$. Since we already know of a $G$-invariant symplectic form on $V$, namely the one lifted from $M \cong V/M$, and since this is singular, all $G$-invariant symplectic forms on $V$ are singular.

**Proposition 2.6.** *Let G be a finite group. Suppose that $\mathcal{O}_{2'}(G) \neq 1$ and that V is a faithful, self-dual, indecomposable kG-module. Then there exist (non-zero) G-invariant symplectic forms on V and every one is the polarization of some G-invariant quadratic form.*

*Proof.* We first note that $V$ does not lie in the principal block, for if so, we would have $V = fV$, where

$$f = \frac{1}{|\mathcal{O}_{2'}(G)|} \sum_{g \in \mathcal{O}_{2'}(G)} g$$

is the principal block indempotent. Then $\mathcal{O}_{2'}(G)$ would act trivially on $V$, contrary to our hypothesis. Thus, $V$ is an indecomposable module not in the principal block, whence both $V^G$ and $H^1(G, V)$ are zero. The result now follows from 2.3 and 2.4.

We note that 2.6 applies to self-dual simple nontrivial modules for solvable groups. Here, $\mathcal{O}_{2'}(G/\mathrm{Ker}\, V) \neq 1$. This special case is Satz 2.8 of [6].

We close this section with a criterion for the nonexistence of $G$-invariant quadratic forms.

**Proposition 2.7.** *Suppose $V \not\cong k$ is a self-dual, absolutely simple G-module over k. Then if $H^1(G, \Lambda^2(V)) = 0$ there is no G-invariant quadratic form on V.*

*Proof.* By 2.3, we have $\Lambda^2(V^*)^G \neq 0$, and by absolute irreducibility, we have $(V^* \otimes_k V^*)^G \cong k$. Now the assertion follows from the vertical sequence of (2.2).

## § 3. Tensor products

In this section, $k$ is still a perfect field of characteristic two. The basic lemma for dealing with tensor products is the following.

**Lemma 3.1.** *Let V and W be finite dimensional vector spaces over k. Then in the pull-back diagram*

$$\begin{array}{ccccccccc} 0 & \to & (V \otimes_k W)^{(2)*} & \xrightarrow{i} & S^2(V^* \otimes_k W^*) & \xrightarrow{\theta} & \Lambda^2(V^* \otimes_k W^*) & \to & 0 \\ & & \| & & \uparrow\cup & & \uparrow\cup & & \\ 0 & \to & (V \otimes_k W)^{(2)*} & \xrightarrow{i} & E(V, W) & \xrightarrow{\theta} & \Lambda^2(V^*) \otimes_k \Lambda^2(W^*) & \to & 0 \end{array}$$

*the bottom row splits naturally.*

*Proof.* $E(V, W)$ is the space of quadratic forms on $V \otimes_k W$ whose associated symplectic forms lie in $\Lambda^2(V^*) \otimes_k \Lambda^2(W^*)$. The latter vanish on pairs $(v \otimes w, v' \otimes w')$ whenever $v = v'$ or $w = w'$. Let $Q \in E(V, W)$. Define

$$\phi_Q : V \times W \to k \quad \text{by} \quad \phi_Q(v, w) = Q(v \otimes w).$$

for all $v \in V$ and $w \in W$. Then

$$\begin{aligned} \phi_Q(v + v', w) &= Q(v \otimes w + v' \otimes w) \\ &= Q(v \otimes w) + Q(v' \otimes w) \quad (\text{since } Q \in E(V, W)) \\ &= \phi_Q(v, w) + \phi_Q(v', w). \end{aligned}$$

Similarly, $\phi_Q$ is additive on $W$. Furthermore,

$$\phi_Q(\lambda v, w) = Q(\lambda v \otimes w) = \lambda^2 Q(v \otimes w) = \lambda^2 \phi_Q(v, w).$$

Thus $\phi_Q$ defines an element of $(V \otimes_k W)^{(2)*}$, and we obtain a natural map

$$\phi : E(V, W) \to (V \otimes_k W)^{(2)*},$$

mapping $Q$ to $\phi_Q$, which clearly splits the inclusion $i$.

As an application, we obtain the next result which was first proved by R. Gow and the second author using different methods. Via Steinberg's Tensor Product Theorem it has applications to the representation theory of Chevalley groups in characteristic two. We mention here only that one consequence is that for finite Chevalley groups in characteristic two, all of the self-dual projective indecomposable modules carry an invariant nonsingular quadratic form, which may be surprising in view of Example 2.5.

For the sequel, we recall that the *Witt index* of a nonsingular quadratic module $(U, Q)$ is the dimension of a maximal isotropic subspace of $U$ and is bounded by $\frac{1}{2}\dim_k U$ (see [1]).

**Proposition 3.4.** *Let $V$ and $W$ be finite-dimensional modules over $k$ for groups $G$ and $H$ respectively. Then every symplectic form in $\Lambda^2(V^*)^G \otimes_k \Lambda^2(W^*)^H$ is the polarization of a $G \times H$-invariant quadratic form on $V \otimes_k W$. Moreover, a form in $\Lambda^2(V^*)^G \otimes_k \Lambda^2(W^*)^H$ arising from a nonsingular $G$-invariant symplectic form on $V$ and a nonsingular $H$-invariant one on $W$ is the polarization of a nonsingular $G \times H$-invariant quadratic form on $V \otimes_k W$ of maximal Witt index.*

*Proof.* The first statement is an immediate consequence of 3.1. Let $B_V$ and $B_W$ be forms as in the second statement. Then $B_V \otimes B_W$ is also nonsingular. By 3.1, we may choose $Q \in \operatorname{Ker}\phi$ with $\theta(Q) = B_V \otimes B_W$. Then $Q(v \otimes w) = 0$ for all $v \in V$ and $w \in W$, so that

$$Q(v \otimes w + v' \otimes w') = (B_V \otimes B_W)(v \otimes w, v' \otimes w') = B_V(v, v')B_W(w, w')$$

for all $v, v' \in V$ and $w, w' \in W$. It follows that if $I$ is a maximal isotropic subspace of $V$ with respect to $B_V$ then $Q(I \otimes_k W) = 0$. Hence $Q$ is of maximal Witt index, as

$$\dim_k(I \otimes_k W) = \frac{1}{2}\dim_k(V \otimes_k W).$$

This gives rise to a well known isomorphism. Suppose $4 \leqq |k| < \infty$. By 3.4 and consideration of determinants we have

$$\mathrm{SL}(2, k) \times \mathrm{SL}(2, k) \cong \mathrm{Sp}(2, k) \times \mathrm{Sp}(2, k) \subseteqq \mathrm{SO}^+(4, k).$$

From the orders, we see that in fact the inclusion on the right is an equality.

We end this section with another corollary of the exact sequence (2.2).

**Proposition 3.5.** *Let $G$ and $H$ be groups, $V$ a $kG$-module with $V^{*G} = 0$ and $W$ a $kH$-module with $W^{*H} = 0$. Then we have an isomorphism*

$$S^2(V^* \otimes_k W^*)^{G \times H} \cong \Lambda^2(V^* \otimes_k W^*)^{G \times H}.$$

*Proof.* We apply the horizontal sequence of (2.2) to $G \times H$ and $V \otimes_k W$. The term $((V \otimes_k W)^{(2)*})^{G \times H}$ is zero by hypothesis, and the term $H^1(G \times H, (V \otimes_k W)^{(2)*})$ vanishes by the Künneth formula.

## § 4. Field extensions

Throughout this section, unless otherwise stated explicitly, $k$ is an arbitrary field and $K$ a finite Galois extension of $k$. $V$ and $W$ will always denote a (finite-dimensional) $K$-module and $k$-module respectively. Then $V$ is also a $k$-module by restriction, and we shall indicate this by the notation $V_k$, with corresponding notations $V_k^* = \mathrm{Hom}_k(V_k, k)$, $S^2(V_k^*)$, etc. If necessary, we shall also write $V_K$ for $V$ in its original role as $K$-module. This should not be confused with $V_k \otimes_k K$. Similar conventions apply to intermediate fields.

Galois theory provides an isomorphism

$$(4.1)\qquad V_k \otimes_k K \xrightarrow{\cong} \bigoplus_{\sigma\in\mathrm{Gal}(K/k)} V^\sigma, \quad v\otimes\lambda \mapsto \left(\sigma^{-1}(\lambda)v\right)_{\sigma\in\mathrm{Gal}(K/k)}.$$

Here, $V^\sigma$ denotes the $K$-module obtained from $V$ by letting $K$ act via $\sigma^{-1}$ (cf. § 1). The action of $\mathrm{Gal}(K/k)$ on $V_k \otimes_k K$ given by $\sigma(v\otimes\lambda) = v\otimes\sigma(\lambda)$ for $v \in V$, $\sigma\in\mathrm{Gal}(K/k)$ becomes, under the isomorphism (4.1), the coordinate permutation

$$(4.2)\qquad \sigma\left((v_\tau)_{\tau\in\mathrm{Gal}(K/k)}\right) = (v_{\sigma^{-1}\tau})_{\tau\in\mathrm{Gal}(K/k)}$$

for $v_\tau \in V^\tau$, and the $k$-submodule $V\otimes_k 1_K$ becomes identified with the diagonal

$$\Delta = \{(v_\tau) \mid v_\tau = v \text{ for all } \tau\in\mathrm{Gal}(K/k)\}.$$

Suppose $V$ is a $G$-module over $K$ for a group $G$. Then $(V^\sigma)^* \cong (V^*)^\sigma$ as $G$-modules over $K$, and (4.1) is an isomorphism of $G$-modules over the smaller field $k$.

For $Q\in S^2(V^*)$ we define $Q^\sigma \in S^2\left((V^\sigma)^*\right)$ by

$$(4.3)\qquad Q^\sigma(v) = Q(v)^\sigma \quad \text{for} \quad v\in V = V^\sigma$$

and the Galois trace,

$$\mathrm{Tr}_{K/k}: S^2(V^*) \to S^2(V_k^*)$$

by

$$(4.4)\qquad (\mathrm{Tr}_{K/k}Q)(v) = \sum_{\sigma\in\mathrm{Gal}(K/k)} Q^\sigma(v).$$

We may also define an exterior Galois trace

$$\widehat{\mathrm{Tr}}_{K/k}: S^2(V^*) \to S^2\left((V\otimes_k K)^*\right)$$

using (4.1) by

$$(4.5)\qquad (\widehat{\mathrm{Tr}}_{K/k}Q)\left((v_\sigma)_{\sigma\in\mathrm{Gal}(K/k)}\right) = \sum_{\sigma\in\mathrm{Gal}(K/k)} Q^\sigma(v_\sigma).$$

In other words, the quadratic $K$-module $(V\otimes_k K, \widehat{\mathrm{Tr}}_{K/k}Q)$ is the orthogonal sum of the quadratic modules $(V^\sigma, Q^\sigma)$. In particular, $\widehat{\mathrm{Tr}}_{K/k}Q$ is nonsingular if and only if $Q$ is. If $V$ is a $G$-module, then both trace maps are $G$-module maps over $k$.

Finally, for a $k$-module $W$ and $Q \in S^2(W^*)$, we define $Q \otimes K \in S^2((W \otimes_k K)^*)$ by

$$(4.6) \qquad (Q \otimes K)(w \otimes \lambda) = \lambda^2 Q(w) \quad \text{for} \quad w \in W \quad \text{and} \quad \lambda \in K.$$

If $B$ is the associated symmetric form of $Q$, then the symmetric form $B \otimes K$ associated to $Q \otimes K$ is given by

$$(4.7) \quad (B \otimes K)(w \otimes \lambda, w' \otimes \mu) = \lambda\mu B(w, w'), \quad \text{for} \quad w, w' \in W \quad \text{and} \quad \lambda, \mu \in K.$$

In particular, $Q \otimes K$ is nonsingular if and only if $Q$ is. Again, the map

$$- \otimes K : S^2(W^*) \to S^2((W \otimes_k K)^*)$$

commutes with $k$-linear group actions. With these notations we have

**Proposition 4.8.** (a) *The following diagram commutes*:

$$\begin{array}{ccc} S^2(V_k^*) & \xrightarrow{-\otimes K} & S^2((V \otimes_k K)^*) \\ {\scriptstyle \mathrm{Tr}_{K/k}} \nwarrow & & \nearrow {\scriptstyle \widehat{\mathrm{Tr}}_{K/k}} \\ & S^2(V^*). & \end{array}$$

(b) *All maps in* (a) *are injective and preserve nonsingular forms.*

*Proof.* (a) For $Q \in S^2(V^*)$, $v \in V$ and $\lambda \in K$ we have

$$\begin{aligned} \widehat{\mathrm{Tr}}_{K/k} Q(v \otimes \lambda) &= \sum_{\sigma \in \mathrm{Gal}(K/k)} Q^\sigma(\sigma^{-1}(\lambda)v) \\ &= \lambda^2 \sum_{\sigma \in \mathrm{Gal}(K/k)} Q^\sigma(v) \\ &= (\mathrm{Tr}_{K/k} Q \otimes K)(v \otimes \lambda). \end{aligned}$$

(b) We have already seen that $\widehat{\mathrm{Tr}}_{K/k}$ and $- \otimes K$ preserve nonsingular forms. They are also clearly injective. Since the diagram commutes, $\mathrm{Tr}_{K/k}$ also has these properties.

**Proposition 4.9** ([2], (7.3)(1)). *Suppose $K$ is finite and that* $\dim_K V$ *and* $\dim_k W$ *are even. Then the following hold*:

(a) $\widehat{\mathrm{Tr}}_{K/k}$ *and* $- \otimes K$ *preserve nonsingular quadratic forms of maximal Witt index.*

(b) *If $Q \in S^2(V^*)$ is nonsingular, then $Q$ is of maximal Witt index if and only if* $\mathrm{Tr}_{K/k} Q$ *is of maximal Witt index.*

(c) $\widehat{\mathrm{Tr}}_{K/k}$ *carries nonsingular quadratic forms of non-maximal Witt index on $V$ to forms of maximal Witt index on $V \otimes_k K$ if and only if $|K : k|$ is even.*

(d) $- \otimes K$ *carries nonsingular forms of non-maximal index on $W$ to forms of maximal index on $W \otimes_k K$ if and only if $|K : k|$ is even.*

*Proof.* (a) Both maps send isotropic subspaces to isotropic subspaces.

(b) In view of (a) it remains to prove that if $Q$ has non-maximal index, then so does $\mathrm{Tr}_{K/k}Q$. This is a consequence of the fact that a quadratic module $(V, Q)$ is of non-maximal index if and only if there is an irreducible automorphism (Singer cycle) of $V$ preserving $Q$ (see [3], in particular "Bemerkung" on page 148). Such an automorphism also acts irreducibly on $V_k$ and preserves $\mathrm{Tr}_{K/k}Q$.

(c) For $Q \in S^2(V^*)$, the quadratic module $(V \otimes_k K, \widehat{\mathrm{Tr}}_{K/k}Q)$ is the orthogonal sum of the $|K:k|$ quadratic modules $(V^\sigma, Q^\sigma)$, all of which clearly have the same index. Witt's theorem implies that the orthogonal sum of two even dimensional nonsingular quadratic modules with the same Witt index has maximal Witt index.

(d) Suppose $Q \in S^2(W^*)$ is nonsingular of non-maximal index. By part (a) and the theory of normal forms for quadratic forms, we may assume that $(W, Q)$ is 2-dimensional and anisotropic. Thus, $Q$ is given in suitable coordinates by

$$Q(x, y) = x^2 + xy + \alpha y^2, \quad \text{with} \quad \alpha \notin \{\mu^2 - \mu \mid \mu \in k\}$$

if $k$ is of characteristic two, and

$$Q(x, y) = x^2 - \alpha y^2, \quad \text{with} \quad \alpha \notin k^2$$

if $k$ has odd characteristic. By (4.6), $Q \otimes K$ is given by the same formula, and it is clear that the conditions $\alpha \in \{v^2 - v \mid v \in K\}$, respectively $\alpha \in K^2$ will hold if and only if $|K:k|$ is even.

If $K$ is finite and $|K:k|$ is odd then the Witt index cannot change under $-\otimes_k K$ or trace maps. Thus only the case where $\dim_K V$ is odd and $|K:k|$ is even is left. In this case the characteristic has to be odd, because a nonsingular symplectic space is even dimensional.

**Proposition 4.10** ([2], (7.3)(2)). *Suppose $|k| = q$ and $|K| = q^{2m}$. Suppose that $\dim_K V$ is odd and that $Q \in S^2(V^*)$ is a nonsingular form. Write $V$ as the orthogonal sum*

$$W \perp \langle v \rangle$$

*with respect to $Q$, where the restriction of $Q$ to $W$ is nonsingular of maximal index.*

(a) *$\widehat{\mathrm{Tr}}_{K/k}Q$ has maximal index.*

(b) *If $q^m \equiv 1 \pmod 4$ then $\mathrm{Tr}_{K/k}Q$ has maximal index if and only if $Q(v)$ is not a square in $K$.*

(c) *If $q^m \equiv 3 \pmod 4$ then $\mathrm{Tr}_{K/k}Q$ has maximal index if and only if $Q(v)$ is a square in $K$.*

*Proof.* Let $Q(v) = a$. (a) Since $(V \otimes_k K, \widehat{\mathrm{Tr}}_{K/k}Q)$ is the orthogonal sum of the modules $(V^\sigma, Q^\sigma)$, $\sigma \in \mathrm{Gal}(K/k)$, the result will follow once we show that the restriction of $\widehat{\mathrm{Tr}}_{K/k}Q$ to the 2-dimensional space $\langle v \rangle \perp \langle v \rangle^\sigma$ $(\sigma \neq 1)$ has a nonzero isotropic vector. Since both $-1$ and $a^\sigma a^{-1}$ are squares, the equation

$$ax^2 + a^\sigma y^2 = 0$$

has a solution $(x, y) \neq (0, 0)$.

(b), (c) Let $k \leqq L \leqq K$ with $|L| = q^m$. By 4.9 (b), it is sufficient to determine the type of $\mathrm{Tr}_{K/L}Q$. By 4.9 (b) again, the type of $\mathrm{Tr}_{K/L}Q$ is the same as that of its restriction to the 2-dimensional space $\langle v\rangle_L$. For $\lambda \in K^\times$, we have $\mathrm{Tr}_{K/L}Q(\lambda v) = \lambda^2 a + (\lambda^2 a)^{q^m} = 0$ if and only if $(\lambda^2 a)^{q^m-1} = -1$. Now $|K^{\times 2}| = \frac{1}{2}(q^m-1)(q^m+1)$, so the elements of multiplicative order $2(q^m-1)$ lie in $K^{\times 2}$ in case (c) and outside $K^{\times 2}$ in case (b). Thus, the vector $\lambda v$ is isotropic precisely when $\lambda^2 a$ is not a square in case (b), and when it is not a square in (c).

## § 5. The Witt index of simple G-modules

Let $k$ be any finite field and $G$ a finite group. Let $V$ be a simple $kG$-module and $K = \mathrm{End}_{kG}(V)$. Then $K$ is a finite Galois extension of $k$ and $V$ has the structure of a $K$-module which we denote by $V_K$, with similar notations for intermediate fields. $K$ is a splitting field for $V$ and $(V_K)_k = V$.

**Lemma 5.1.** *Suppose $V$ is self-dual but that $V_K$ is not. Then $|K:k|$ is even and for the element $\tau \in \mathrm{Gal}(K/k)$ of order 2 and its fixed subfield $L = K^\tau$ the following hold*:

(a) *$V_L$ is self-dual as an $LG$-module.*

(b) *Under the isomorphism*

$$(V^* \otimes_k V^*)^G \cong \mathrm{End}_{kG}(V) = K$$

*the subfield $L$ becomes identified with $\Lambda^2(V^*)^G$ on the left and on the right, for any nonzero $B \in \Lambda^2(V^*)^G$, with the set of self-adjoint $kG$-maps with respect to $B$.*

*Proof.* We have $V \otimes_k K \cong \bigoplus_{\sigma\in\mathrm{Gal}(K/k)} V_K^\sigma$, in which the modules $V_K^\sigma$ are absolutely simple and mutually nonisomorphic. Since $(V_K^\sigma)^* \cong (V_K^*)^\sigma$, none of the $V_K^\sigma$ is self-dual. But $V \otimes_k K$ is self-dual, hence $V_K^* \cong V_K^\zeta$ for some $\zeta \in \mathrm{Gal}(K/k)$. Since $V_K \cong (V_K^*)^*$, it follows that $\zeta$ has order 2, so $\zeta = \tau$. Now $V_L \otimes_L K \cong V_K \oplus V_K^\tau \cong V_K \oplus V_K^*$ is self-dual, therefore $V_L$ is too, proving (a).

To prove (b), we write $V \otimes_k K \cong X \oplus X^*$, with $\mathrm{Hom}_{KG}(X, X^*) = 0$. Then

$$\begin{aligned}\Lambda^2(V^*)^G \otimes_k K &\cong \Lambda^2((V \otimes_k K)^*)^G \cong (X^* \otimes_K X)^G \cong \mathrm{End}_{KG}(X) \subseteqq \mathrm{End}_{KG}(V \otimes_k K)\\ &\cong \mathrm{End}_{kG}(V) \otimes_k K \cong K \otimes_k K.\end{aligned}$$

Thus, $\Lambda^2(V^*)^G \subseteqq K$. Now fix $0 \neq B \in \Lambda^2(V^*)^G$. Then all $G$-invariant $k$-bilinear forms on $V$ are of the form $B_\alpha$, $\alpha \in \mathrm{End}_{kG}(V)$, given by $B_\alpha(v, w) = B(v, \alpha w)$, for $v, v' \in V$. It is clear that taking adjoints is an automorphism of the field $K$, and all statements will be proved if we show that the fixed subfield is $\Lambda^2(V^*)^G$. Let $\alpha^{\mathrm{ad}}$ be the adjoint of $\alpha$ with respect to $B$. If $K$ has odd characteristic then we have:

$$B(\alpha^{\mathrm{ad}} v, v') = B(v, \alpha v') = B_\alpha(v, v')$$

and

$$B(\alpha v, v') = -B(v', \alpha v) = -B_\alpha(v', v).$$

Since $B$ is nonsingular, these equations show that $B_\alpha \in \Lambda^2(V^*)^G$ if and only if $\alpha = \alpha^{\mathrm{ad}}$. If $K$ is of characteristic two, and $\alpha = \alpha^{\mathrm{ad}}$, there exists $\beta = \beta^{\mathrm{ad}}$ such that $\beta^2 = \alpha$. Then

$$B_\alpha(v, v) = B(v, \alpha v) = B(v, \beta^{\mathrm{ad}} \beta v) = B(\beta v, \beta v) = 0.$$

This shows that $B_\alpha \in \Lambda^2(V^*)^G$. Conversely, if $B_\alpha \in \Lambda^2(V^*)^G$, then

$$0 = B_\alpha(v + w, v + w) = B(v, \alpha w) + B(\alpha v, w),$$

which shows that $\alpha = \alpha^{\mathrm{ad}}$.

**Lemma 5.2.** *Suppose $V$ is a self-dual simple $kG$-module and that $V_L$ is an even-dimensional self-dual $LG$-module for some field $L$ with $k \subseteq L \subseteq K = \mathrm{End}_{kG}(V)$. Then for any natural number $e$,*

$$\mathrm{Tr}_{L/k} : S^2(eV_L^*)^G \to S^2(eV^*)^G$$

*is a $k$-isomorphism which preserves nonsingular forms and maximality or non-maximality of Witt indices. (Here $eV_L$ denotes the direct sum of $e$ isomorphic copies of the $LG$-module $V_L$ etc.)*

*Proof.* By 4.8(b) and 4.9(b), we know that $\mathrm{Tr}_{L/k}$ is an injective $kG$-map which preserves nonsingular forms and the type of Witt index. Thus, $G$-fixed points are mapped to $G$-fixed points and only surjectivity remains to be shown. We have

$$S^2(eV_L^*)^G \cong eS(V_L^*)^G \oplus \frac{e(e-1)}{2}(V_L^* \otimes_L V_L^*)^G$$

and

$$S^2(eV^*)^G \cong eS^2(V^*)^G \oplus \frac{e(e-1)}{2}(V^* \otimes_k V^*)^G.$$

Since $(V^* \otimes_k V^*)^G \cong \mathrm{End}_{kG}(V) = K \cong \mathrm{End}_{LG}(V_L) \cong (V_L^* \otimes_L V_L^*)^G$, we will be done if we prove

$$\dim_k S^2(V^*)^G = \dim_k S^2(V_L^*)^G.$$

But this is true since

$$S^2(V^*)^G \otimes_k L \cong S^2(V^* \otimes_k L)^G \cong \bigoplus_{\sigma \in \mathrm{Gal}(L/k)} S^2((V_L^\sigma)^*)^G,$$

so

$$\dim_k S^2(V^*)^G = |L:k| \dim_L S^2(V_L^*)^G = \dim_k S^2(V_L^*)^G.$$

**Lemma 5.3.** *Let $(V, B)$ be a nonsingular symplectic or symmetric $k$-space. Let $\beta \in \mathrm{End}_k(V)$ and let $\beta^{\mathrm{ad}}$ be its adjoint with respect to $B$.*

(a) *If $\beta \neq \pm\beta^{\mathrm{ad}}$, then there is some $v \in V$ such that $B(v, \beta v) \neq 0$.*

(b) *If, in addition the minimal polynomial of $\beta$ is irreducible of degree two and $\beta + \beta^{\mathrm{ad}} = \mu \in k^\times$, then $(V, B)$ is the orthogonal direct sum of $\beta$-invariant 2-dimensional subspaces.*

(c) *In* (b), *if $B$ is symplectic then restriction to each of these 2-dimensional spaces of the quadratic form $Q(v) = B(v, \beta v)$ is anisotropic.*

*Proof.* (a) Suppose $B(v, \beta v) = 0$ for all $v \in V$. Then

$$\begin{aligned} 0 &= B(v + v', \beta(v + v')) = B(v, \beta v') + B(v', \beta v) \\ &= B(\beta^{\mathrm{ad}} v, v') \pm B(\beta v, v') = B((\beta \pm \beta^{\mathrm{ad}})v, v') \end{aligned}$$

for all $v, v' \in V$, contradicting the nonsingularity of $B$.

(b) By (a), we may choose $v \in V$ such that $B(v, \beta v) \neq 0$. Set $H = \langle v, \beta v\rangle$. Then $H$ is a $\beta$-invariant subspace. Moreover, for $h \in H$ and $h^\perp \in H^\perp$, we have

$$B(\beta h^\perp, h) = B(h^\perp, \beta^{\mathrm{ad}} h) = B(h^\perp, (\mu - \beta)h) = 0,$$

so $H^\perp$ is $\beta$-invariant. The hypothesis on the minimum polynomial is valid for $\beta|_{H^\perp}$, so the result follows by induction on dimension.

(c) Let $H$ be as above and suppose $B(v, \beta v) = \xi \in k^\times$. Let

$$m_\beta(x) = x^2 + \lambda x + \alpha$$

be the minimal polynomial of $\beta$. Then the matrix of $Q|_H$ with respect to the basis $v, \beta v$ is

$$\begin{pmatrix} 0 & \xi \\ -\xi & 0 \end{pmatrix} \begin{pmatrix} 0 & -\alpha \\ 1 & -\lambda \end{pmatrix} = \xi \begin{pmatrix} 1 & -\lambda \\ 0 & \alpha \end{pmatrix},$$

and $Q|_H$ is anisotropic because $x^2 - \lambda x + \alpha = m_\beta(-x)$ is irreducible over $k$.

**Theorem 5.4** (see [2], (7.6)). *Let $V$ be a self-dual, simple, even-dimensional $kG$-module and let $K = \mathrm{End}_{kG}(V)$. Then one of the following holds:*

(a) *$V_K \cong_{KG} V_K^*$ and $\dim_K V_K$ is even. Then $V$ has a nonsingular $G$-invariant quadratic form if and only if $V_K$ does. All nonsingular forms on $V$ and $V_K$ have the same type of Witt index.*

(b) *$V_K \ncong_{KG} V_K^*$. Then $V$ has a nonsingular $G$-invariant quadratic form and all nonsingular $G$-invariant forms have the same type of Witt index, which is maximal if and only if $\dim_K V_K$ is even.*

(c) $V_K \cong_{KG} V_K^*$ *and* $\dim_K V_K$ *is odd. Then V has a nonsingular G-invariant quadratic form if and only if* $V_K$ *does. If so, then V has G-invariant quadratic forms of both maximal and non-maximal Witt index.*

*Proof.* (a) If there is a nonzero $G$-invariant quadratic form on $V_K$ then it is unique up to $K$-scalars. Therefore, (a) is an immediate corollary of 5.2.

(b) Let $k \subseteqq L \subseteqq K$ with $|K:L| = 2$ as in 5.1. Then by 5.2 we may assume that $L = k$, and so by 5.1 (b), $\Lambda^2(V^*)^G \cong k$. Now taking $G$-fixed points in the two exact sequences (1.3) and (1.4) yields that $S^2(V^*)^G \cong k$, so all nonzero $G$-invariant quadratic forms on $V$ have the same Witt index and it remains to determine this.

Pick $0 \neq B \in \Lambda^2(V^*)^G$ and choose $\beta \in K$ with minimal polynomial $x^2 + x + \alpha$ over $k$. Then by 5.1 (b), the adjoint $\beta^{\mathrm{ad}}$ of $\beta$ with respect to $B$ is the other root of this polynomial, so $\beta^{\mathrm{ad}} = -(\beta + 1)$. We may now apply 5.3 (c). The orthogonal sum of two 2-dimensional anisotropic quadratic modules over a finite field has maximal Witt index (see [1]). Thus, $Q$ is of maximal index if and only if the number of 2-dimensional subspaces in the decomposition of 5.3 (b), namely $\frac{1}{2}\dim_k V = \dim_K V_K$, is even. This completes the proof.

(c) Since $V$ has even dimension, $|K:k|$ must be even. Let $k \subseteqq L \subseteqq K$ with $|K:L| = 2$. Then $V_L$ is self dual and of even dimension, so by 5.2 we may assume $k = L$. The first assertion in (c) now follows from the isomorphism

$$S^2(V^*)^G \otimes_k K \cong S^2(V_K^*)^G \oplus S^2(V_K^{\tau *})^G, \quad 1 \neq \tau \in \mathrm{Gal}(K/L).$$

The last part is an immediate consequence of 4.10.

**Remark.** In 5.4 (b), it can be shown that the module $V_K$ carries a nonsingular $G$-invariant Hermitian form (see [2], (7.6)(3)).

If $G$ has odd order, then the absolutely simple module $V_K$ in the theorem is not self-dual. Also it has odd dimension; for characteristic two this is an elementary fact, but for odd characteristic it is a consequence of the Feit-Thompson theorem on the solvability of groups of odd order and the Fong-Swan theorem on the liftability of absolutely simple modules for solvable groups. Thus:

**Corollary 5.5.** *The orthogonal group* $\mathrm{O}^+(Q)$ *of an even-dimensional quadratic module of maximal index over a finite field has no irreducible subgroups of odd order.*

**Lemma 5.6.** *Let V be a kG-module carrying a nonsingular G-invariant quadratic form Q. Suppose* $V = U \oplus U^*$ *where U is a kG-submodule with* $\mathrm{Hom}_{kG}(U, U^*) = 0$. *Then Q is of maximal index.*

*Proof.* Since

$$S^2(U \oplus U^*)^G \cong (U^* \otimes_k U)^G \cong \mathrm{End}_{kG}(U),$$

every $G$-invariant quadratic form on $U$ is of the form $Q_\alpha$ for $\alpha \in \mathrm{End}_{kG}(U)$, where

$$Q_\alpha(u,f) = f(\alpha u), \quad \text{for} \quad u \in U, \quad f \in U^*.$$

Thus, for all $Q_\alpha$ the submodule $U$ of $U \oplus U^*$ is isotropic of half the dimension of $U \oplus U^*$, so in particular $Q$ is of maximal index.

**Lemma 5.7.** *Let $Z = \langle z \rangle$ be a cyclic group of odd order and suppose $V$ is the direct sum of $e$ isomorphic copies of a nontrivial, simple, self-dual $kZ$-module $W$. Then there exist $Z$-invariant nonsingular quadratic forms on $V$, and all of them have the same Witt index, which is maximal if and only if $e$ is even.*

*Proof.* Since the nontrivial absolutely simple modules for $Z$ are not self-dual, $W$ must have even dimension and so 5.4 (b) shows that there is a nonsingular $G$-invariant quadratic form on $W$. By forming an orthogonal sum, we obtain one on $V$, proving existence.

Let $K = \mathrm{End}_{kZ}(W)$. Since $W_K$ is not self-dual, 5.1 yields an intermediate field $L$ with $|K:L| = 2$ and such that $W_L$ is self-dual, and two-dimensional since $W_K$ is one-dimensional. By 5.2, it suffices to prove that every nonsingular $Z$-invariant form on $V_L \cong eW_L$ has the claimed Witt index. Thus we may assume $k = L$. Then the minimal polynomial of $z$ is irreducible of degree 2, and $z$ has determinant 1 on $W$ because it preserves a nonsingular symplectic or symmetric form, and $z$ has odd order. Therefore the minimum polynomial is $x^2 + \lambda x + 1$, for some $\lambda \neq 0$. Let $Q$ be any nonsingular $Z$-invariant quadratic form on $V$, and let $z^{\mathrm{ad}}$ be the adjoint of $z$ with respect to the associated bilinear form. Then since $Z \subseteq K = \mathrm{End}_{kZ}(W)$, 5.1 implies that $z^{\mathrm{ad}} = -(z + \lambda)$. Thus, by 5.3 (b), the direct sum decomposition of $V$ into $e$ copies of $W$ can be chosen to be an orthogonal decomposition with respect to $Q$. Since the restriction of $Q$ to one of the $e$ components in this decomposition is anisotropic, by 5.3 (b), the proof is complete.

Combining the last two lemmas yields the following result.

**Theorem 5.8.** *Let $V$ be a $kG$-module carrying a nonsingular $G$-invariant quadratic form $Q$. Suppose that $G$ has an elementary abelian $r$-subgroup $A$ for some odd prime $r$ for which $V^A = 0$. Let $m$ be the multiplicative order of $|k|$ (mod $r$). Then $Q$ is of maximal index if and only if $m$ divides $\frac{1}{2}\dim_k V$.*

*Proof.* Write

$$V_{|A} = U \oplus U^* \oplus \bigoplus_{j=1}^{s} e_j W_j,$$

where the $W_j$ are self-dual, simple and pairwise nonisomorphic $kA$-modules and

$$\mathrm{Hom}_{kA}(U, U^*) = 0.$$

Each $W_j$, being a self-dual module for a group of odd order, has even dimension, so $V$ does too. Setting $V_j = e_j W_j$, we have an orthogonal decomposition

$$V_{|A} = (U \oplus U^*) \perp V_1 \perp \ldots \perp V_s$$

with respect to $Q$. If $U \neq 0$, the restriction of $Q$ to $U \oplus U^*$ is of maximal index by 5.6. Next, suppose $V_j \neq 0$ and consider $Q_{|V_j}$. Since $A/\mathrm{Ker}(W_j)$ is cyclic of order $r$, 5.7 applies. Thus, $Q$ is of maximal index if and only if $\sum_j e_j$ is even. Now all nontrivial simple $kA$-modules have dimension $m$, so if $\dim_k U = tm$, we have $\dim_k V = 2tm + m \sum_j e_j$, and the theorem follows.

**Corollary 5.9.** *Let $V$ be a faithful, simple kG-module admitting a nonzero G-invariant quadratic form. Let $\dim_k V = 2n$. Suppose G has a nontrivial normal r-subgroup for some odd prime r. Let m be the multiplicative order of $|k|$ (mod r). Then all nonzero G-invariant quadratic forms on V have the same Witt index, which is maximal if and only if m divides n.*

*Proof.* Let $A$ be a nontrivial, elementary abelian, normal $r$-subgroup of $G$. Then since $V$ is faithful, $r$ is different from the characteristic of $k$ and $V^A = 0$.

Finally, we summarize our results for 2-modular representations.

**Corollary 5.10.** *Let $1 \neq G$ be a finite solvable group, k a finite field of characteristic two and V a self-dual, faithful, simple kG-module. Let r be a prime divisor of the order of the Fitting subgroup and m the multiplicative order of $|k|$ (mod r). Then there exist nonsingular G-invariant quadratic forms on V, and they all have the same Witt index, which is maximal if and only if m divides $\frac{1}{2}\dim_k V$.*

*Proof.* This is immediate by 2.6 and 5.9.

**Remark.** If $k$ is a finite field of characteristic two and $G$ is solvable, our results show that $G$ cannot be embedded as an irreducible subgroup of both $O^+(2n, k)$ and $O^-(2n, k)$ for any $n$. We know of no example of a group $G$ for which this may happen.

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