Oppositeness in buildings and representations of finite groups of Lie type

Peter Sin
University of Florida

Conference on Designs, Codes and Geometries
Lewes, Delaware, March 30th, 2010.

Outline

Introduction

The oppositeness graph of the Tits building of a finite group $G = G(q)$ of Lie type is a q-analog of the classical Kneser graph.

In this talk we consider oppositeness from the point of view of representation theory of G.

Outline

Groups with BN-pairs

- $G = G(q)$ group with a split BN-pair ($B = UH$,N), characteristic p, rank ℓ
- $I = \{1, \ldots, \ell\}$
- W, Weyl group euclidean reflection group in a real vector space V,
- root system R, positive roots R^+, simple roots $S = \{\alpha_i \mid i \in I\}$
- w_i reflection in hyperplane perpendicular to α_i.
- $W = \langle w_i \mid i \in I \rangle$ Coxeter group.
- $\ell(w)$, is the length of the shortest expression for w as a word in the generators w_i.
- $\ell(w) =$ the number of positive roots which w transforms to negative roots.
- w_0 unique longest element of W, sends all positive roots to negative roots.

Groups with BN-pairs

- $G = G(q)$ group with a split BN-pair ($B = UH$,N), characteristic p, rank ℓ
- $I = \{1, \ldots, \ell\}$
- W, Weyl group euclidean reflection group in a real vector space V,
- root system R, positive roots R^+, simple roots $S = \{\alpha_i \mid i \in I\}$
- w_i reflection in hyperplane perpendicular to α_i.
- $W = \langle w_i \mid i \in I \rangle$ Coxeter group.
- $\ell(w)$, is the length of the shortest expression for w as a word in the generators w_i.
- $\ell(w) =$ the number of positive roots which w transforms to negative roots.
- w_0 unique longest element of W, sends all positive roots to negative roots.

Groups with BN-pairs

- $G = G(q)$ group with a split BN-pair ($B = UH$,N), characteristic p, rank ℓ
- $I = \{1, \ldots, \ell\}$
- W, Weyl group euclidean reflection group in a real vector space V,
- root system R, positive roots R^+, simple roots $S = \{\alpha_i \mid i \in I\}$
- w_i reflection in hyperplane perpendicular to α_i.
- $W = \langle w_i \mid i \in I \rangle$ Coxeter group.
- $\ell(w)$, is the length of the shortest expression for w as a word in the generators w_i.
- $\ell(w) =$ the number of positive roots which w transforms to negative roots.
- w_0 unique longest element of W, sends all positive roots to negative roots.

Groups with BN-pairs

- $G = G(q)$ group with a split BN-pair ($B = UH$,N), characteristic p, rank ℓ
- $I = \{1, \ldots, \ell\}$
- W, Weyl group euclidean reflection group in a real vector space V,
- root system R, positive roots R^+, simple roots $S = \{\alpha_i \mid i \in I\}$
- w_i reflection in hyperplane perpendicular to α_i.
- $W = \langle w_i \mid i \in I \rangle$ Coxeter group.
- $\ell(w)$, is the length of the shortest expression for w as a word in the generators w_i.
- $\ell(w) =$ the number of positive roots which w transforms to negative roots.
- w_0 unique longest element of W, sends all positive roots to negative roots.

Groups with BN-pairs

- $G = G(q)$ group with a split BN-pair ($B = UH$,N), characteristic p, rank ℓ
- $I = \{1, \ldots, \ell\}$
- W, Weyl group euclidean reflection group in a real vector space V,
- root system R, positive roots R^+, simple roots $S = \{\alpha_i \mid i \in I\}$
- w_i reflection in hyperplane perpendicular to α_i.
- $W = \langle w_i \mid i \in I \rangle$ Coxeter group.
- $\ell(w)$, is the length of the shortest expression for w as a word in the generators w_i.
- $\ell(w) =$ the number of positive roots which w transforms to negative roots.
- w_0 unique longest element of W, sends all positive roots to negative roots.

Groups with BN-pairs

- $G = G(q)$ group with a split BN-pair ($B = UH$,N), characteristic p, rank ℓ
- $I = \{1, \ldots, \ell\}$
- W, Weyl group euclidean reflection group in a real vector space V,
- root system R, positive roots R^+, simple roots $S = \{\alpha_i \mid i \in I\}$
- w_i reflection in hyperplane perpendicular to α_i.
- $W = \langle w_i \mid i \in I \rangle$ Coxeter group.
- $\ell(w)$, is the length of the shortest expression for w as a word in the generators w_i.
- $\ell(w) =$ the number of positive roots which w transforms to negative roots.
- w_0 unique longest element of W, sends all positive roots to negative roots.

Groups with BN-pairs

- $G = G(q)$ group with a split BN-pair ($B = UH$,N), characteristic p, rank ℓ
- $I = \{1, \ldots, \ell\}$
- W, Weyl group euclidean reflection group in a real vector space V,
- root system R, positive roots R^+, simple roots $S = \{\alpha_i \mid i \in I\}$
- w_i reflection in hyperplane perpendicular to α_i.
- $W = \langle w_i \mid i \in I \rangle$ Coxeter group.
- $\ell(w)$, is the length of the shortest expression for w as a word in the generators w_i.
- $\ell(w) =$ the number of positive roots which w transforms to negative roots.
- w_0 unique longest element of W, sends all positive roots to negative roots.

Groups with BN-pairs

- $G = G(q)$ group with a split BN-pair ($B = UH$,N), characteristic p, rank ℓ
- $I = \{1, \ldots, \ell\}$
- W, Weyl group euclidean reflection group in a real vector space V,
- root system R, positive roots R^+, simple roots $S = \{\alpha_i \mid i \in I\}$
- w_i reflection in hyperplane perpendicular to α_i.
- $W = \langle w_i \mid i \in I \rangle$ Coxeter group.
- $\ell(w)$, is the length of the shortest expression for w as a word in the generators w_i.
- $\ell(w) =$ the number of positive roots which w transforms to negative roots.
- w_0 unique longest element of W, sends all positive roots to negative roots.

Groups with BN-pairs

- $G = G(q)$ group with a split BN-pair ($B = UH$,N), characteristic p, rank ℓ
- $I = \{1, \ldots, \ell\}$
- W, Weyl group euclidean reflection group in a real vector space V,
- root system R, positive roots R^+, simple roots $S = \{\alpha_i \mid i \in I\}$
- w_i reflection in hyperplane perpendicular to α_i.
- $W = \langle w_i \mid i \in I \rangle$ Coxeter group.
- $\ell(w)$, is the length of the shortest expression for w as a word in the generators w_i.
- $\ell(w) =$ the number of positive roots which w transforms to negative roots.
- w_0 unique longest element of W, sends all positive roots to negative roots.

Parabolic subgroups

- $J \subseteq I$
- $W_J := \langle w_i \mid i \in J \rangle$ *standard parabolic subgroup* of W
- $P_J = BW_JB$ is a *standard parabolic subgroup* of G.

Parabolic subgroups

- $J \subseteq I$
- $W_J := \langle w_i \mid i \in J \rangle$ *standard parabolic subgroup* of W
- $P_J = BW_JB$ is a *standard parabolic subgroup* of G.

Parabolic subgroups

- $J \subseteq I$
- $W_J := \langle w_i \mid i \in J \rangle$ *standard parabolic subgroup* of W
- $P_J = BW_JB$ is a *standard parabolic subgroup* of G.

Types and objects of the building

- A *type* and its *cotype* are simply a subset of I and its complement.
- An object of cotype J is a right coset of P_J in G.

Types and objects of the building

- A *type* and its *cotype* are simply a subset of I and its complement.
- An object of cotype J is a right coset of P_J in G.

Outline

Opposite types

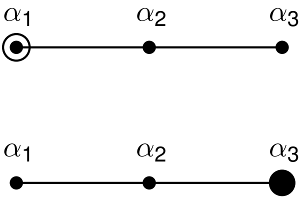
Definition

Two types J and K are *opposite* if

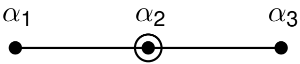
$$\{-w_0(\alpha_i) \mid i \in J\} = \{\alpha_j \mid j \in K\},$$

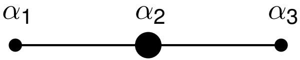
or, equivalently, if

$$\{w_0 w_i w_0 \mid i \in J\} = \{w_i \mid i \in K\}.$$

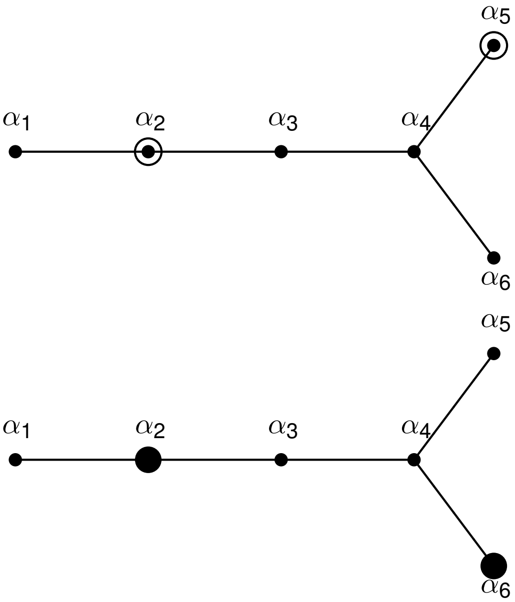


A_3, skew lines in $PG(3, q)$

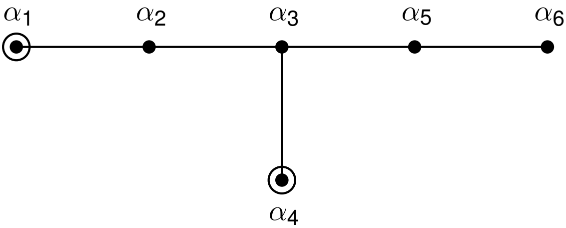




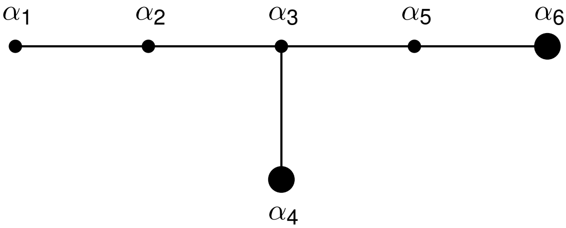
D_5, flags in oriflamme geometry



E_6



α_1
α_2
α_3
α_5
α_6
α_4



α_1
α_2
α_3
α_5
α_6
α_4

Opposite objects

Let J and K be fixed opposite types.

Definition

An object $P_J g$ of cotype J is *opposite* an object $P_K h$ of cotype K iff

$$\begin{aligned} P_K h g^{-1} P_J &= P_K w_0 P_J \\ (\iff P_K h &\subseteq P_K w_0 P_J g \\ \iff P_J g &\subseteq P_J w_0 P_K h). \end{aligned}$$

The oppositeness matrix

- Let $A = A(J, K)$ be the oppositeness matrix for objects of cotype J and K.

Theorem

(Brouwer, 2009) The square of every eigenvalue λ of A is a power of q.

- We will show that the p-rank of A is the degree of an irreducible representation of G.

The oppositeness matrix

- Let $A = A(J, K)$ be the oppositeness matrix for objects of cotype J and K.

Theorem

(Brouwer, 2009) The square of every eigenvalue λ of A is a power of q.

- We will show that the p-rank of A is the degree of an irreducible representation of G.

The oppositeness matrix

- Let $A = A(J, K)$ be the oppositeness matrix for objects of cotype J and K.

Theorem

(Brouwer, 2009) The square of every eigenvalue λ of A is a power of q.

- We will show that the p-rank of A is the degree of an irreducible representation of G.

Outline

Permutation modules on flags

- Let k be a field of characteristic p. Let $\mathcal{F}_J$ denote the space of functions from the set $P_J \backslash G$ of objects of cotype J to k. Then $\mathcal{F}_J$ is a left kG-module by the rule

$$(xf)(P_J g) = f(P_J gx), \quad f \in \mathcal{F}_J, \quad g, x \in G.$$

  Let $\delta_{P_J g}$ denote the characteristic function of the object $P_J g \in P_J \backslash G$. Then $\mathcal{F}_J$ is generated as a kG-module by $\delta_{P_J g}$

The oppositeness homomorphism

- The relation of oppositeness defines a kG-homomorphism $\eta : \mathcal{F}_J \to \mathcal{F}_K$ given by

$$\eta(f)(P_K h) = \sum_{P_J g \subseteq P_J w_0 P_K h} f(P_J g).$$

- We have

$$\eta(\delta_{P_J g}) = \sum_{P_K h \subseteq P_K w_0 PJg} \delta_{P_K h}.$$

so the characteristic function of an object of cotype J is sent to the sum of the characteristic functions of all objects opposite to it.

The oppositeness homomorphism

- The relation of oppositeness defines a kG-homomorphism $\eta : \mathcal{F}_J \to \mathcal{F}_K$ given by

$$\eta(f)(P_K h) = \sum_{P_J g \subseteq P_J w_0 P_K h} f(P_J g).$$

- We have

$$\eta(\delta_{P_J g}) = \sum_{P_K h \subseteq P_K w_0 P J g} \delta_{P_K h}.$$

  so the characteristic function of an object of cotype J is sent to the sum of the characteristic functions of all objects opposite to it.

Simplicity of oppositeness modules

Theorem

The image of η is a simple module, uniquely characterized by the property that its one-dimensional U-invariant subspace has full stablizer equal to P_J, which acts trivially on it.

This result is essentially a corollary of a more general result of Carter and Lusztig (1976) on the *Iwahori-Hecke Algebra* $\mathrm{End}_{kG}(\mathcal{F}_\emptyset)$. We next describe their result.

Outline

The Iwahori-Hecke Algebra

- $\mathcal{F} = \mathcal{F}_\emptyset$.
- For $w \in W$ define $T_w \in \mathrm{End}_k(\mathcal{F})$ by

$$T_w(f)(Bg) = \sum_{Bg' \subseteq Bw^{-1}Bg} f(Bg').$$

- Then

$$T_w \in \mathrm{End}_{kG}(\mathcal{F}), \quad \text{for all } w \in W.$$

- One can show that

$$T_{ww'} = T_w T_{w'} \quad \text{if } \ell(ww') = \ell(w) + \ell(w').$$

The Iwahori-Hecke Algebra

- $\mathcal{F} = \mathcal{F}_{\emptyset}$.
- For $w \in W$ define $T_w \in \mathrm{End}_k(\mathcal{F})$ by

$$T_w(f)(Bg) = \sum_{Bg' \subseteq Bw^{-1}Bg} f(Bg').$$

- Then

$$T_w \in \mathrm{End}_{kG}(\mathcal{F}), \quad \text{for all } w \in W.$$

- One can show that

$$T_{ww'} = T_w T_{w'} \quad \text{if } \ell(ww') = \ell(w) + \ell(w').$$

The Iwahori-Hecke Algebra

- $\mathcal{F} = \mathcal{F}_\emptyset$.
- For $w \in W$ define $T_w \in \mathrm{End}_k(\mathcal{F})$ by

$$T_w(f)(Bg) = \sum_{Bg' \subseteq Bw^{-1}Bg} f(Bg').$$

- Then

$$T_w \in \mathrm{End}_{kG}(\mathcal{F}), \quad \text{for all } w \in W.$$

- One can show that

$$T_{ww'} = T_w T_{w'} \quad \text{if } \ell(ww') = \ell(w) + \ell(w').$$

The Iwahori-Hecke Algebra

- $\mathcal{F} = \mathcal{F}_\emptyset$.
- For $w \in W$ define $T_w \in \mathrm{End}_k(\mathcal{F})$ by

$$T_w(f)(Bg) = \sum_{Bg' \subseteq Bw^{-1}Bg} f(Bg').$$

- Then

$$T_w \in \mathrm{End}_{kG}(\mathcal{F}), \quad \text{for all } w \in W.$$

- One can show that

$$T_{ww'} = T_w T_{w'} \quad \text{if } \ell(ww') = \ell(w) + \ell(w').$$

- Let $w \in W$ have reduced expression

$$w_{j_n} \cdots w_{j_1}.$$

- We consider the partial products $w_{j_1}, w_{j_2} w_{j_1}, \ldots w_{j_n} \cdots w_{j_1}$.
- Each partial product sends exactly one more positive root to a negative root than its predecessors, namely $w_{j_1} \cdots w_{j_{i-1}}(r_{j_i})$.
- Let J be a subset of I.
- $V_J :=$ subspace of V spanned by $S_J = \{\alpha_i \mid i \in J\}$.

- Let $w \in W$ have reduced expression

$$w_{j_n} \cdots w_{j_1}.$$

- We consider the partial products $w_{j_1}, w_{j_2} w_{j_1}, \ldots w_{j_n} \cdots w_{j_1}$.
- Each partial product sends exactly one more positive root to a negative root than its predecessors, namely $w_{j_1} \cdots w_{j_{i-1}}(r_{j_i})$.
- Let J be a subset of I.
- $V_J :=$ subspace of V spanned by $S_J = \{\alpha_i \mid i \in J\}$.

- Let $w \in W$ have reduced expression

$$w_{j_n} \cdots w_{j_1}.$$

- We consider the partial products $w_{j_1}, w_{j_2} w_{j_1}, \ldots w_{j_n} \cdots w_{j_1}$.
- Each partial product sends exactly one more positive root to a negative root than its predecessors, namely $w_{j_1} \cdots w_{j_{i-1}}(r_{j_i})$.
- Let J be a subset of I.
- $V_J :=$ subspace of V spanned by $S_J = \{\alpha_i \mid i \in J\}$.

- Let $w \in W$ have reduced expression

$$w_{j_n} \cdots w_{j_1}.$$

- We consider the partial products $w_{j_1}, w_{j_2} w_{j_1}, \ldots w_{j_n} \cdots w_{j_1}$.
- Each partial product sends exactly one more positive root to a negative root than its predecessors, namely $w_{j_1} \cdots w_{j_{i-1}}(r_{j_i})$.
- Let J be a subset of I.
- $V_J :=$ subspace of V spanned by $S_J = \{\alpha_i \mid i \in J\}$.

- Let $w \in W$ have reduced expression

$$w_{j_n} \cdots w_{j_1}.$$

- We consider the partial products $w_{j_1}, w_{j_2} w_{j_1}, \ldots w_{j_n} \cdots w_{j_1}$.
- Each partial product sends exactly one more positive root to a negative root than its predecessors, namely $w_{j_1} \cdots w_{j_{i-1}}(r_{j_i})$.
- Let J be a subset of I.
- $V_J :=$ subspace of V spanned by $S_J = \{\alpha_i \mid i \in J\}$.

- For any reduced expression

$$w_0 = w_{j_k} \cdots w_{j_1}$$

define

$$\Theta_{j_i} = \begin{cases} T_{w_{j_i}} & \text{if } w_{j_1} \cdots w_{j_{i-1}}(r_{j_i}) \notin V_J \\ I + T_{w_{j_i}} & \text{if } w_{j_1} \cdots w_{j_{i-1}}(r_{j_i}) \in V_J \end{cases}$$

and set

$$\Theta^J_{w_0} = \Theta_{j_k} \Theta_{j_{k-1}} \cdots \Theta_{j_k}.$$

- The definition depends on the choice of reduced expression but it can be seen that different expressions give the same endomorphism up to a nonzero scalar multiple.

- For any reduced expression

$$w_0 = w_{j_k} \cdots w_{j_1}$$

define

$$\Theta_{j_i} = \begin{cases} T_{w_{j_i}} & \text{if } w_{j_1} \cdots w_{j_{i-1}}(r_{j_i}) \notin V_J \\ I + T_{w_{j_i}} & \text{if } w_{j_1} \cdots w_{j_{i-1}}(r_{j_i}) \in V_J \end{cases}$$

and set

$$\Theta^J_{w_0} = \Theta_{j_k}\Theta_{j_{k-1}} \cdots \Theta_{j_k}.$$

- The definition depends on the choice of reduced expression but it can be seen that different expressions give the same endomorphism up to a nonzero scalar multiple.

Theorem

(Carter,Lusztig) The image $\Theta^{J}_{w_0}(\mathcal{F})$ is a simple kG-module. The full stablizer of the one-dimensional subspace of U-fixed points in this module is P_J and the action of P_J on this one-dimensional subspace is trivial.

Application of Carter-Lusztig to oppositeness

- We choose a particular expression for w_0 to define $\Theta^J_{w_0}(\mathcal{F})$.
- $R_J = R \cap V_J$ is a root system in V_J with simple system S_J and Weyl group W_J.
- w_J be the longest element in W_J.
- Let

$$w_J = w_{i_m} \cdots w_{i_2} w_{i_1}$$

be a reduced expression for w_J. The above expression can be extended to a reduced expresion

$$w_0 = w_{i_k} \cdots w_{i_{m+1}} w_{i_m} \cdots w_{i_1}$$

of w_0. Then

$$w^* = w_{i_k} \cdots w_{i_{m+1}}.$$

is a reduced expression for w^*.
- Write $\Theta^J_{w_0}$ using the above expression for w_0.

Application of Carter-Lusztig to oppositeness

- We choose a particular expression for w_0 to define $\Theta^J_{w_0}(\mathcal{F})$.
- $R_J = R \cap V_J$ is a root system in V_J with simple system S_J and Weyl group W_J.
- w_J be the longest element in W_J.
- Let

$$w_J = w_{i_m} \cdots w_{i_2} w_{i_1}$$

be a reduced expression for w_J. The above expression can be extended to a reduced expresion

$$w_0 = w_{i_k} \cdots w_{i_{m+1}} w_{i_m} \cdots w_{i_1}$$

of w_0. Then

$$w^* = w_{i_k} \cdots w_{i_{m+1}}.$$

is a reduced expression for w^*.
- Write $\Theta^J_{w_0}$ using the above expression for w_0.

Application of Carter-Lusztig to oppositeness

- We choose a particular expression for w_0 to define $\Theta^J_{w_0}(\mathcal{F})$.
- $R_J = R \cap V_J$ is a root system in V_J with simple system S_J and Weyl group W_J.
- w_J be the longest element in W_J.
- Let
$$w_J = w_{i_m} \cdots w_{i_2} w_{i_1}$$
be a reduced expression for w_J. The above expression can be extended to a reduced expresion
$$w_0 = w_{i_k} \cdots w_{i_{m+1}} w_{i_m} \cdots w_{i_1}$$
of w_0. Then
$$w^* = w_{i_k} \cdots w_{i_{m+1}}.$$
is a reduced expression for w^*.
- Write $\Theta^J_{w_0}$ using the above expression for w_0.

Application of Carter-Lusztig to oppositeness

- We choose a particular expression for w_0 to define $\Theta^J_{w_0}(\mathcal{F})$.
- $R_J = R \cap V_J$ is a root system in V_J with simple system S_J and Weyl group W_J.
- w_J be the longest element in W_J.
- Let

$$w_J = w_{i_m} \cdots w_{i_2} w_{i_1}$$

be a reduced expression for w_J. The above expression can be extended to a reduced expresion

$$w_0 = w_{i_k} \cdots w_{i_{m+1}} w_{i_m} \cdots w_{i_1}$$

of w_0. Then

$$w^* = w_{i_k} \cdots w_{i_{m+1}}.$$

is a reduced expression for w^*.

- Write $\Theta^J_{w_0}$ using the above expression for w_0.

Application of Carter-Lusztig to oppositeness

- We choose a particular expression for w_0 to define $\Theta^J_{w_0}(\mathcal{F})$.
- $R_J = R \cap V_J$ is a root system in V_J with simple system S_J and Weyl group W_J.
- w_J be the longest element in W_J.
- Let
$$w_J = w_{i_m} \cdots w_{i_2} w_{i_1}$$
be a reduced expression for w_J. The above expression can be extended to a reduced expresion
$$w_0 = w_{i_k} \cdots w_{i_{m+1}} w_{i_m} \cdots w_{i_1}$$
of w_0. Then
$$w^* = w_{i_k} \cdots w_{i_{m+1}}.$$
is a reduced expression for w^*.
- Write $\Theta^J_{w_0}$ using the above expression for w_0.

- The expression $w_0 = w^* w_J$ is chosen so that for the first m partial products the new positive root sent to a negative root belongs to V_J, and the new positive roots for the remaining partial products do not belong to V_J.
- $$\Theta^J_{w_0} = T_{w^*}(1 + T_{w_{i_m}}) \cdots (1 + T_{w_{i_1}}),$$
- Since $\ell(w^* w) = \ell(w^*) + \ell(w)$ for all $w \in W_J$, we see that $\Theta^J_{w_0}$ is a sum of endomorphisms of the form $T_{w^* w}$, for certain elements $w \in W_J$, with exactly one term of this sum equal to T_{w^*}.

- The expression $w_0 = w^* w_J$ is chosen so that for the first m partial products the new positive root sent to a negative root belongs to V_J, and the new positive roots for the remaining partial products do not belong to V_J.
- $$\Theta^J_{w_0} = T_{w^*}(1 + T_{w_{i_m}}) \cdots (1 + T_{w_{i_1}}),$$
- Since $\ell(w^* w) = \ell(w^*) + \ell(w)$ for all $w \in W_J$, we see that $\Theta^J_{w_0}$ is a sum of endomorphisms of the form $T_{w^* w}$, for certain elements $w \in W_J$, with exactly one term of this sum equal to T_{w^*}.

- The expression $w_0 = w^* w_J$ is chosen so that for the first m partial products the new positive root sent to a negative root belongs to V_J, and the new positive roots for the remaining partial products do not belong to V_J.
- $$\Theta^J_{w_0} = T_{w^*}(1 + T_{w_{i_m}}) \cdots (1 + T_{w_{i_1}}),$$
- Since $\ell(w^* w) = \ell(w^*) + \ell(w)$ for all $w \in W_J$, we see that $\Theta^J_{w_0}$ is a sum of endomorphisms of the form $T_{w^* w}$, for certain elements $w \in W_J$, with exactly one term of this sum equal to T_{w^*}.

The projections π_J and π_K

- Let $\pi_J : \mathcal{F} \to \mathcal{F}_J$ be defined by

$$(\pi_J(f))(P_J g) = \sum_{Bh \subseteq P_J g} f(Bh)$$

- $\pi_J(\delta_B) = \delta_{P_J}$, π_J is a surjective kG-module homomorphisms
- π_K defined similarly.

The projections π_J and π_K

- Let $\pi_J : \mathcal{F} \to \mathcal{F}_J$ be defined by

$$(\pi_J(f))(P_J g) = \sum_{Bh \subseteq P_J g} f(Bh)$$

- $\pi_J(\delta_B) = \delta_{P_J}$, π_J is a surjective kG-module homomorphisms
- π_K defined similarly.

The projections π_J and π_K

- Let $\pi_J : \mathcal{F} \to \mathcal{F}_J$ be defined by

$$(\pi_J(f))(P_J g) = \sum_{Bh \subseteq P_J g} f(Bh)$$

- $\pi_J(\delta_B) = \delta_{P_J}$, π_J is a surjective kG-module homomorphisms
- π_K defined similarly.

A computation

Compare $\eta\pi_J$ with $\pi_K T_{w^* w}$ for $w \in W_J$. For $f \in \mathcal{F}$,

$$\begin{aligned}[\eta(\pi_J(f))](P_K g) &= \sum_{P_J h \subseteq P_J w^{*-1} P_K g} \sum_{Bx \subseteq P_J h} f(Bx) \\ &= \sum_{Bx \subseteq P_J w^{*-1} P_K g} f(Bx).\end{aligned}$$

and

$$\begin{aligned}[\pi_K(T_{w^* w}(f))](P_K h) &= \sum_{Bg \subseteq P_K h} (T_{w^* w} f)(Bg) \\ &= \sum_{Bg \subseteq P_K h} \sum_{Bx \subseteq B(w^* w)^{-1} Bg} f(Bx) \\ &= \sum_{Bg \subseteq P_K h} \sum_{Bg \subseteq B(w^* w) Bx} f(Bx) \\ &= q^{\ell(w)} \sum_{Bx \subseteq P_J w^{*-1} P_K g} f(Bx).\end{aligned}$$

- Thus, we have for each $w \in W_J$ a commutative diagram

$$\begin{tikzcd} \mathcal{F}_J \arrow[r, "q^{\ell(w)}\eta"] & \mathcal{F}_K \\ \mathcal{F} \arrow[u, "\pi_J"] \arrow[r, "T_{ww^*}"] & \mathcal{F}, \arrow[u, "\pi_K"'] \end{tikzcd}$$

- If $w \neq 1$ we have $\pi_K T_{ww^*} = 0$.
- Hence $\pi_K \Theta^J_{w_0} = \pi_K T_{w^*} = \eta\pi_J$.
- Finally, $\Theta^J_{w_0}(\mathcal{F})$ is simple and $\eta\pi_J(\mathcal{F}) \neq 0$, so $\eta(\mathcal{F}_J) = \eta\pi_J(\mathcal{F}) \cong \Theta^J_{w_0}(\mathcal{F})$.

□

- Thus, we have for each $w \in W_J$ a commutative diagram

$$\begin{tikzcd} \mathcal{F}_J \arrow[r, "q^{\ell(w)}\eta"] & \mathcal{F}_K \\ \mathcal{F} \arrow[u, "\pi_J"] \arrow[r, "T_{ww^*}"] & \mathcal{F}, \arrow[u, "\pi_K"'] \end{tikzcd}$$

- If $w \neq 1$ we have $\pi_K T_{ww^*} = 0$.
- Hence $\pi_K \Theta^J_{w_0} = \pi_K T_{w^*} = \eta\pi_J$.
- Finally, $\Theta^J_{w_0}(\mathcal{F})$ is simple and $\eta\pi_J(\mathcal{F}) \neq 0$, so $\eta(\mathcal{F}_J) = \eta\pi_J(\mathcal{F}) \cong \Theta^J_{w_0}(\mathcal{F})$.

$\square$

- Thus, we have for each $w \in W_J$ a commutative diagram

$$\begin{tikzcd} \mathcal{F}_J \arrow[r, "q^{\ell(w)}\eta"] & \mathcal{F}_K \\ \mathcal{F} \arrow[u, "\pi_J"] \arrow[r, "T_{ww^*}"] & \mathcal{F}, \arrow[u, "\pi_K"'] \end{tikzcd}$$

- If $w \neq 1$ we have $\pi_K T_{ww^*} = 0$.
- Hence $\pi_K \Theta^J_{w_0} = \pi_K T_{w^*} = \eta\pi_J$.
- Finally, $\Theta^J_{w_0}(\mathcal{F})$ is simple and $\eta\pi_J(\mathcal{F}) \neq 0$, so $\eta(\mathcal{F}_J) = \eta\pi_J(\mathcal{F}) \cong \Theta^J_{w_0}(\mathcal{F})$.

$\square$

- Thus, we have for each $w \in W_J$ a commutative diagram

$$\begin{tikzcd} \mathcal{F}_J \arrow[r, "q^{\ell(w)}\eta"] & \mathcal{F}_K \\ \mathcal{F} \arrow[u, "\pi_J"] \arrow[r, "T_{ww^*}"] & \mathcal{F}, \arrow[u, "\pi_K"'] \end{tikzcd}$$

- If $w \neq 1$ we have $\pi_K T_{ww^*} = 0$.
- Hence $\pi_K \Theta^J_{w_0} = \pi_K T_{w^*} = \eta\pi_J$.
- Finally, $\Theta^J_{w_0}(\mathcal{F})$ is simple and $\eta\pi_J(\mathcal{F}) \neq 0$, so $\eta(\mathcal{F}_J) = \eta\pi_J(\mathcal{F}) \cong \Theta^J_{w_0}(\mathcal{F})$.

$\square$

Outline

Highest weights of oppositeness modules

- $G = G(q)$ is a Chevalley group of universal type or a twisted subgroup.
- Simple modules are restrictions of certain simple rational modules $L(\lambda)$ of the ambient algebraic group, so we want to identify the highest weight λ of the oppositeness modules.
- If G is an untwisted group, then the fundamental weights ω_i for the ambient algebraic group are indexed by I, and the highest weight of the simple module in the theorem is $\sum_{i \in I \setminus J}(q-1)\omega_i$.

Highest weights of oppositeness modules

- $G = G(q)$ is a Chevalley group of universal type or a twisted subgroup.
- Simple modules are restrictions of certain simple rational modules $L(\lambda)$ of the ambient algebraic group, so we want to identify the highest weight λ of the oppositeness modules.
- If G is an untwisted group, then the fundamental weights ω_i for the ambient algebraic group are indexed by I, and the highest weight of the simple module in the theorem is $\sum_{i \in I \setminus J}(q-1)\omega_i$.

Highest weights of oppositeness modules

- $G = G(q)$ is a Chevalley group of universal type or a twisted subgroup.
- Simple modules are restrictions of certain simple rational modules $L(\lambda)$ of the ambient algebraic group, so we want to identify the highest weight λ of the oppositeness modules.
- If G is an untwisted group, then the fundamental weights ω_i for the ambient algebraic group are indexed by I, and the highest weight of the simple module in the theorem is $\sum_{i \in I \setminus J} (q-1)\omega_i$.

- If G is a twisted group, inside untwisted G^*. There are two cases.
- Suppose that all roots of G^* have the same length (${}^2A_\ell$, ${}^2D_\ell$, ${}^3D_4$, ${}^2E_6$). Let $I^* = \{1, \ldots, \ell^*\}$ index the fundamental roots. Then G arises from a symmetry ρ of the Dynkin diagram of G^* and the index set I for G labels the ρ-orbits on I^*. Let ω_i, $i \in I^*$ be the fundamental weights of the ambient algebraic group. For $J \subseteq I$, let $J^* \subset I^*$ be the union of the orbits in J. Then the highest weight of the kG^*-module in the theorem is $\sum_{i \in I^* \setminus J^*}(q-1)\omega_i$.
- Suppose that there are roots of different lengths for G^* (Suzuki and Ree groups). Then the set I for G indexes the subset of fundamental weights of the ambient algebraic group which are orthogonal to the long simple roots. and for $J \subset I$, the simple module of the theorem has highest weight $\sum_{i \in I \setminus J}(q-1)\omega_i$.

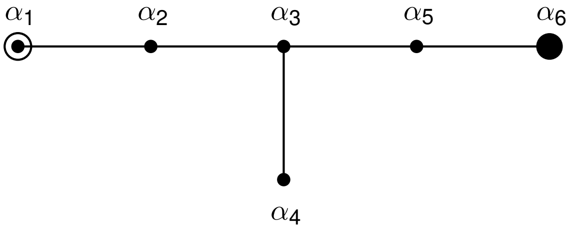
- If G is a twisted group, inside untwisted G^*. There are two cases.
- Suppose that all roots of G^* have the same length (${}^2A_\ell$, ${}^2D_\ell$, ${}^3D_4$, ${}^2E_6$). Let $I^* = \{1, \ldots, \ell^*\}$ index the fundamental roots. Then G arises from a symmetry ρ of the Dynkin diagram of G^* and the index set I for G labels the ρ-orbits on I^*. Let ω_i, $i \in I^*$ be the fundamental weights of the ambient algebraic group. For $J \subseteq I$, let $J^* \subset I^*$ be the union of the orbits in J. Then the highest weight of the kG^*-module in the theorem is $\sum_{i \in I^* \setminus J^*}(q-1)\omega_i$.
- Suppose that there are roots of different lengths for G^* (Suzuki and Ree groups). Then the set I for G indexes the subset of fundamental weights of the ambient algebraic group which are orthogonal to the long simple roots. and for $J \subset I$, the simple module of the theorem has highest weight $\sum_{i \in I \setminus J}(q-1)\omega_i$.

- If G is a twisted group, inside untwisted G^*. There are two cases.
- Suppose that all roots of G^* have the same length (${}^2A_\ell$, ${}^2D_\ell$, ${}^3D_4$, ${}^2E_6$). Let $I^* = \{1, \ldots, \ell^*\}$ index the fundamental roots. Then G arises from a symmetry ρ of the Dynkin diagram of G^* and the index set I for G labels the ρ-orbits on I^*. Let ω_i, $i \in I^*$ be the fundamental weights of the ambient algebraic group. For $J \subseteq I$, let $J^* \subset I^*$ be the union of the orbits in J. Then the highest weight of the kG^*-module in the theorem is $\sum_{i \in I^* \setminus J^*}(q-1)\omega_i$.
- Suppose that there are roots of different lengths for G^* (Suzuki and Ree groups). Then the set I for G indexes the subset of fundamental weights of the ambient algebraic group which are orthogonal to the long simple roots. and for $J \subset I$, the simple module of the theorem has highest weight $\sum_{i \in I \setminus J}(q-1)\omega_i$.

Outline

Example

- $G = E_6(q)$, group of isometries of a certain 3-form on a 27-dimensional vector space V. Consider singular points and singular hyperplanes.



Point-hyperplane incidence for $E_6(q)$

- $rank_p A = \dim L((q-1)\omega_1)) = \dim L((p-1)\omega_1))^t$, where $q = p^t$. (Steinberg's tensor product theorem)
- In this case we can work out $\dim L((p-1)\omega_1))$ using representation theory. (Weyl modules, Weyl Character formula, Jantzen sum formula).

Point-hyperplane incidence for $E_6(q)$

- $rank_p A = \dim L((q-1)\omega_1)) = \dim L((p-1)\omega_1))^t$, where $q = p^t$. (Steinberg's tensor product theorem)
- In this case we can work out $\dim L((p-1)\omega_1))$ using representation theory. (Weyl modules, Weyl Character formula, Jantzen sum formula).

Jantzen Sum Formula

The Weyl module $V(\lambda)$ has a descending filtration, of submodules $V(\lambda)^i$, $i > 0$, such that

$$V(\lambda)^1 = \mathrm{rad}\, V(\lambda), \quad \text{so} \quad V(\lambda)/V(\lambda)^1 \cong L(\lambda).$$

and

$$\sum_{i>0} \mathrm{Ch}(V(\lambda)^i) = -\sum_{\alpha>0} \sum_{\{m: 0<mp<\langle\lambda+\rho,\alpha^\vee\rangle\}} v_p(mp)\chi(\lambda - mp\alpha)$$

Using the sum formula, one can show that there is an exact sequence

$$
\begin{aligned}
0 &\to V((p-11)\omega_1 + 2\omega_2) \to V((p-10)\omega_1 + \omega_2 + \omega_5) \\
&\to V((p-9)\omega_1 + \omega_3 + \omega_6) \to V((p-8)\omega_1 + \omega_4 + 2\omega_6) \\
&\to V((p-7)\omega_1 + 3\omega_6) \to V((p-1)\omega_1) \to L((p-1)\omega_1) \to 0
\end{aligned}
$$

The dimensions of the $V(\mu)$ are given by Weyl's formula. Hence

$$
\begin{aligned}
\dim L((p-1)\omega_1) = \frac{1}{2^7.3.5.11} & p(p+1)(p+3) \\
& \times (3p^8 - 12p^7 + 39p^6 + 320p^5 \\
& - 550p^4 + 1240p^3 + 2080p^2 - 1920p + 1440)
\end{aligned}
$$

2,	27
3,	351
5,	19305
7,	439439
11,	45822672
13,	274187550
17,	5030354043
19,	16937278357
23,	137112098409
29,	1744146121068
31,	3628038332724
37,	25349391871621
41,	78345931447980
43,	132256396016732
47,	351675426454470
53,	1317968719988571
59,	4286665842359706
61,	6185074367788952
67,	17356733399472663
71,	32843689463427543
73,	44580694495895104
79,	106281498207828698
83,	182978611275724173
89,	394284508288312914
97,	1016219651834875565

- Thank you for your attention!