Oppositeness in buildings and representations of finite groups of Lie type

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Outline

Introduction

Notation and terminology

Oppositeness

Permutation modules

Carter-Lusztig Theorem

Relation to representations of algebraic groups

An $E_6$ Example
The oppositeness graph of the Tits building of a finite group $G = G(q)$ of Lie type is a $q$-analog of the classical Kneser graph.
In this talk we consider oppositeness from the point of view of representation theory of $G$. 
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Groups with BN-pairs

- $G = G(q)$ group with a split BN-pair $(B = UH, N)$, characteristic $p$, rank $\ell$
- $I = \{1, \ldots, \ell\}$
- $W$, Weyl group euclidean reflection group in a real vector space $V$,
- root system $R$, positive roots $R^+$, simple roots $S = \{\alpha_i \mid i \in I\}$
- $w_i$ reflection in hyperplane perpendicular to $\alpha_i$.
- $W = \langle w_i \mid i \in I \rangle$ Coxeter group.
- $\ell(w)$, is the length of the shortest expression for $w$ as a word in the generators $w_i$.
- $\ell(w) = $ the number of positive roots which $w$ transforms to negative roots.
- $w_0$ unique longest element of $W$, sends all positive roots to negative roots.
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Parabolic subgroups

- $J \subseteq I$
- $W_J := \langle w_i \mid i \in J \rangle$ standard parabolic subgroup of $W$
- $P_J = BW_JB$ is a standard parabolic subgroup of $G$. 
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Types and objects of the building

- A type and its cotype are simply a subset of $I$ and its complement.
- An object of cotype $J$ is a right coset of $P_J$ in $G$. 
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Opposite types

Definition
Two types \( J \) and \( K \) are *opposite* if

\[
\{-w_0(\alpha_i) \mid i \in J\} = \{\alpha_j \mid j \in K\},
\]
or, equivalently, if

\[
\{w_0 w_i w_0 \mid i \in J\} = \{w_i \mid i \in K\}.
\]
$A_3$, skew lines in $PG(3, q)$
$D_5$, flags in oriflamme geometry
$E_6$
Let $J$ and $K$ be fixed opposite types.

**Definition**

An object $P_Jg$ of cotype $J$ is *opposite* an object $P_Kh$ of cotype $K$ iff

\[ P_Kh g^{-1} P_J = P_K w_0 P_J \]

\[ \iff P_Kh \subseteq P_K w_0 P_J g \]

\[ \iff P_J g \subseteq P_J w_0 P_K h \]
The oppositeness matrix

Let $A = A(J, K)$ be the oppositeness matrix for objects of cotype $J$ and $K$.

**Theorem**
(Brouwer, 2009) The square of every eigenvalue $\lambda$ of $A$ is a power of $q$.

We will show that the $p$-rank of $A$ is the degree of an irreducible representation of $G$. 
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Let $k$ be a field of characteristic $p$. Let $\mathcal{F}_J$ denote the space of functions from the set $P_J \setminus G$ of objects of cotype $J$ to $k$. Then $\mathcal{F}_J$ is a left $kG$-module by the rule

$$(xf)(P_Jg) = f(P_Jgx), \quad f \in \mathcal{F}_J, \quad g, x \in G.$$ 

Let $\delta_{P_Jg}$ denote the characteristic function of the object $P_Jg \in P_J \setminus G$. Then $\mathcal{F}_J$ is generated as a $kG$-module by $\delta_{P_Jg}$.
The oppositeness homomorphism

The relation of oppositeness defines a $kG$-homomorphism $\eta : \mathcal{F}_J \to \mathcal{F}_K$ given by

$$\eta(f)(P_K h) = \sum_{P_J g \subseteq P_J w_0 P_K h} f(P_J g).$$

We have

$$\eta(\delta_{P_J g}) = \sum_{P_K h \subseteq P_K w_0 P_J g} \delta_{P_K h}.$$

so the characteristic function of an object of cotype $J$ is sent to the sum of the characteristic functions of all objects opposite to it.
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Theorem
The image of $\eta$ is a simple module, uniquely characterized by the property that its one-dimensional $U$-invariant subspace has full stabilizer equal to $P_J$, which acts trivially on it.

This result is essentially a corollary of a more general result of Carter and Lusztig (1976) on the Iwahori-Hecke Algebra $\text{End}_{kG}(F_\emptyset)$. We next describe their result.
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An $E_6$ Example
The Iwahori-Hecke Algebra

- $\mathcal{F} = \mathcal{F}_{\emptyset}$.
- For $w \in W$ define $T_w \in \text{End}_k(\mathcal{F})$ by
  \[ T_w(f)(Bg) = \sum_{Bg' \subseteq Bw^{-1}Bg} f(Bg'). \]
- Then
  \[ T_w \in \text{End}_{kG}(\mathcal{F}), \quad \text{for all } w \in W. \]
- One can show that
  \[ T_{ww'} = T_w T_{w'} \quad \text{if } \ell(ww') = \ell(w) + \ell(w'). \]
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Let \( w \in W \) have reduced expression

\[ w_{j_n} \cdots w_{j_1}. \]

We consider the partial products \( w_{j_1}, w_{j_2} w_{j_1}, \ldots w_{j_n} \cdots w_{j_1} \).

Each partial product sends exactly one more positive root to a negative root than its predecessors, namely

\[ w_{j_1} \cdots w_{j_{i-1}} (r_{j_i}). \]

Let \( J \) be a subset of \( I \).

\( V_J := \) subspace of \( V \) spanned by \( S_J = \{ \alpha_i \mid i \in J \} \).
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Let $J$ be a subset of $I$.

$V_J :=$ subspace of $V$ spanned by $S_J = \{\alpha_i \mid i \in J\}$. 
For any reduced expression

\[ w_0 = w_{j_k} \cdots w_{j_1} \]

define

\[ \Theta_{j_i} = \begin{cases} T_{w_{j_i}} & \text{if } w_{j_1} \cdots w_{j_{i-1}}(r_{j_i}) \notin V_J \\ I + T_{w_{j_i}} & \text{if } w_{j_1} \cdots w_{j_{i-1}}(r_{j_i}) \in V_J \end{cases} \]

and set

\[ \Theta^J_{w_0} = \Theta_{j_k} \Theta_{j_{k-1}} \cdots \Theta_{j_1}. \]

The definition depends on the choice of reduced expression but it can be seen that different expressions give the same endomorphism up to a nonzero scalar multiple.
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Theorem
(Carter,Lusztig) The image $\Theta^J_{w_0}(F)$ is a simple $kG$-module. The full stabilizer of the one-dimensional subspace of $U$-fixed points in this module is $P_J$ and the action of $P_J$ on this one-dimensional subspace is trivial.
We choose a particular expression for $w_0$ to define $\Theta^J_{w_0}(\mathcal{F})$.

$R_J = R \cap V_J$ is a root system in $V_J$ with simple system $S_J$ and Weyl group $W_J$.

Let $w_J$ be the longest element in $W_J$.

Let

$$w_J = w_{i_m} \cdots w_{i_2} w_{i_1}$$

be a reduced expression for $w_J$. The above expression can be extended to a reduced expression

$$w_0 = w_{i_k} \cdots w_{i_{m+1}} w_{i_m} \cdots w_{i_1}$$

of $w_0$. Then

$$w^* = w_{i_k} \cdots w_{i_{m+1}}$$

is a reduced expression for $w^*$.

Write $\Theta^J_{w_0}$ using the above expression for $w_0$. 

Application of Carter-Lusztig to oppositeness

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Write \( \Theta^J_{w_0} \) using the above expression for \( w_0 \).
The expression $w_0 = w^* w_J$ is chosen so that for the first $m$ partial products the new positive root sent to a negative root belongs to $V_J$, and the new positive roots for the remaining partial products do not belong to $V_J$.

$$\Theta^J_{w_0} = T_{w^*}(1 + T_{w_{i_m}}) \cdots (1 + T_{w_{i_1}}),$$

Since $\ell(w^* w) = \ell(w^*) + \ell(w)$ for all $w \in W_J$, we see that $\Theta^J_{w_0}$ is a sum of endomorphisms of the form $T_{w^* w}$, for certain elements $w \in W_J$, with exactly one term of this sum equal to $T_{w^*}$. 
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The projections $\pi_J$ and $\pi_K$

Let $\pi_J : \mathcal{F} \to \mathcal{F}_J$ be defined by

$$(\pi_J(f))(P_J g) = \sum_{Bh \subseteq P_J g} f(Bh)$$

$\pi_J(\delta_B) = \delta_{P_J}$, $\pi_J$ is a surjective $kG$-module homomorphisms

$\pi_K$ defined similarly.
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$$(\pi_J(f))(P_J g) = \sum_{Bh \subseteq P_J g} f(Bh)$$

$\pi_J(\delta_B) = \delta_{P_J}$, $\pi_J$ is a surjective $kG$-module homomorphisms

$\pi_K$ defined similarly.
A computation

Compare $\eta \pi_J$ with $\pi_K T_{w^* w}$ for $w \in W_J$. For $f \in \mathcal{F}$,

$$[\eta(\pi_J(f))] (P_K g) = \sum_{P_J h \subseteq P_J w^*} \sum_{Bx \subseteq P_J h} f(Bx)$$

and

$$[\pi_K(T_{w^* w}(f))] (P_K h) = \sum_{Bx \subseteq P_K h} (T_{w^* w} f)(Bg)$$

for $w \in W_J$. For $f \in \mathcal{F}$,
Thus, we have for each $w \in \mathcal{W}_J$ a commutative diagram

$\begin{array}{c}
\mathcal{F}_J \\
\downarrow \pi_J \\
\mathcal{F}
\end{array} \xrightarrow{T_{ww^*}} \begin{array}{c}
\mathcal{F}_K \\
\downarrow \pi_K \\
\mathcal{F}
\end{array}$

If $w \neq 1$ we have $\pi_K T_{ww^*} = 0$.

Hence $\pi_K \Theta^J_{w_0} = \pi_K T_{w^*} = \eta \pi_J$.

Finally, $\Theta^J_{w_0}(\mathcal{F})$ is simple and $\eta \pi_J(\mathcal{F}) \neq 0$, so $\eta(\mathcal{F}_J) = \eta \pi_J(\mathcal{F}) \cong \Theta^J_{w_0}(\mathcal{F})$. 

$\square$
Thus, we have for each $w \in \mathcal{W}_J$ a commutative diagram

\[
\begin{array}{ccc}
\mathcal{F}_J & \xrightarrow{q^\ell(w)\eta} & \mathcal{F}_K \\
\pi_J & & \pi_K \\
\mathcal{F} & \xrightarrow{T_{ww^*}} & \mathcal{F},
\end{array}
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Simple modules are restrictions of certain simple rational modules $L(\lambda)$ of the ambient algebraic group, so we want to identify the highest weight $\lambda$ of the oppositeness modules.

If $G$ is an untwisted group, then the fundamental weights $\omega_i$ for the ambient algebraic group are indexed by $I$, and the highest weight of the simple module in the theorem is $\sum_{i \in I \setminus J} (q - 1) \omega_i$. 
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If $G$ is a twisted group, inside untwisted $G^*$. There are two cases.

Suppose that all roots of $G^*$ have the same length ($2A_\ell$, $2D_\ell$, $3D_4$, $2E_6$). Let $I^* = \{1, \ldots, \ell^*\}$ index the fundamental roots. Then $G$ arises from a symmetry $\rho$ of the Dynkin diagram of $G^*$ and the index set $I$ for $G$ labels the $\rho$-orbits on $I^*$. Let $\omega_i, i \in I^*$ be the fundamental weights of the ambient algebraic group. For $J \subseteq I$, let $J^* \subset I^*$ be the union of the orbits in $J$. Then the highest weight of the $kG^*$-module in the theorem is $\sum_{i \in I^* \setminus J^*} (q - 1)\omega_i$.

Suppose that there are roots of different lengths for $G^*$ (Suzuki and Ree groups). Then the set $I$ for $G$ indexes the subset of fundamental weights of the ambient algebraic group which are orthogonal to the long simple roots. and for $J \subset I$, the simple module of the theorem has highest weight $\sum_{i \in I \setminus J} (q - 1)\omega_i$. 
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Example

- $G = E_6(q)$, group of isometries of a certain 3-form on a 27-dimensional vector space $V$. Consider singular points and singular hyperplanes.
Point-hyperplane incidence for $E_6(q)$

- $\text{rank}_p A = \dim L((q - 1)\omega_1)) = \dim L((p - 1)\omega_1))^t$, where $q = p^t$. (Steinberg’s tensor product theorem)

- In this case we can work out $\dim L((p - 1)\omega_1))$ using representation theory. (Weyl modules, Weyl Character formula, Jantzen sum formula).
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The Weyl module $V(\lambda)$ has a descending filtration, of submodules $V(\lambda)^i$, $i > 0$, such that

$$V(\lambda)^1 = \text{rad } V(\lambda), \quad \text{so} \quad V(\lambda)/V(\lambda)^1 \cong L(\lambda).$$

and

$$\sum_{i>0} \text{Ch}(V(\lambda)^i) = - \sum_{\alpha>0} \sum_{\{m: 0 < m\rho < \langle \lambda+\rho, \alpha \rangle \}} v_p(mp) \chi(\lambda - mp\alpha)$$
Using the sum formula, one can show that there is an exact sequence

\[
0 \to V((p - 11)\omega_1 + 2\omega_2) \to V((p - 10)\omega_1 + \omega_2 + \omega_5) \\
\quad \to V((p - 9)\omega_1 + \omega_3 + \omega_6) \to V((p - 8)\omega_1 + \omega_4 + 2\omega_6) \\
\quad \to V((p - 7)\omega_1 + 3\omega_6) \to V((p - 1)\omega_1) \to L((p - 1)\omega_1) \to 0
\]

The dimensions of the \( V(\mu) \) are given by Weyl’s formula. Hence

\[
\dim L((p - 1)\omega_1) = \frac{1}{2^7 \cdot 3 \cdot 5 \cdot 11} p(p + 1)(p + 3) \\
\times (3p^8 - 12p^7 + 39p^6 + 320p^5 \\
- 550p^4 + 1240p^3 + 2080p^2 - 1920p + 1440)
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Thank you for your attention!