Quantum walks on normal Cayley graphs

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Quantum Information on Graphs,
CanaDAM 2025, May 23rd

Overview

Continuous-time quantum walks on normal Cayley graphs

- G, a group,
- S, an inverse-closed, union of conjugacy classes, not containing 1.
- $\mathrm{Cay}(G, S)$ normal Cayley graph, adjacency matrix A.
- Eigenvalues of $\mathrm{Cay}(G, S)$ are $\theta_\chi = \frac{\chi(S)}{\chi(1)}$, for $\chi \in Irr(G)$.

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- Perfect state transfer (PST): $\mathrm{Cay}(G, S)$ has PST from g to h at time τ iff $|U(\tau)_{h,g}| = 1$.
- Other phenomena of interest include pretty good state transfer, fractional revival, uniform mixing.

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- There has been a lot of work on cyclic, abelian, dihedral groups and on 2-groups of nilpotency class 2 (e.g. extraspecial, Heisenberg, Suzuki 2-groups). See References.
- Today, I will focus on the other end of the spectrum, nonsolvable groups.

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PST in normal Cayley graphs is covered by PST characterization for association schemes (Coutinho et.al).

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Theorem

Let z be central involution in G. In $\mathrm{Cay}(G, S)$ we have PST between vertices g and $h = gz$ at some time if and only if the following hold.

(a) *The eigenvalues are integers.*

(b) *Let $\Phi^+ = \{\theta_\chi | \chi(z) > 0\}$ and $\Phi^- = \{\theta_\chi | \chi(z) < 0\}$. There is an integer N such that*

   (i) *for all $\theta_\chi \in \Phi^-$, $v_2(\theta_\chi - \theta_1) = N$; and*

   (ii) *for all $\theta_\chi \in \Phi^+$, $v_2(\theta_\chi - \theta_1) > N$.*

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  and $\theta_\chi - \theta_1$ is the sum over $C \subseteq S$ of such terms.
- Conjugacy class sizes and character degrees can be divisible by large powers of 2

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- As an example, for the groups $\mathrm{SL}(2, q)$, $q = p^a$, a possible connection set consists of all elements of order p and $2p$, together with central involution $-I$.
- Generalization to higher dimensions may be difficult, as the irreducible characters fall into families, whose number grows fast with respect to dimension.

Quasisimple groups

Let G be a quasisimple group (a perfect central extension of a simple group) with a central involution. G is generated by any noncentral conjugacy class C, so if C is real (closed under inverses), then $\mathrm{Cay}(G, C)$ is a connected normal Cayley graph.

Lemma

Let C be a real conjugacy class of G and $x \in C$. Suppose, for all $\chi \in \mathrm{Irr}(G)$, that $\chi(x)$ is an integer and $\frac{|C|\chi(x)}{\chi(1)} - |C| \equiv 0 \pmod 4$. Then we have PST at $t = \frac{\pi}{2}$ in $\mathrm{Cay}(G, S)$, where $S = C \cup \{z\}$

Proof.

$$\begin{aligned}\theta_\chi - \theta_1 &= \left(\frac{|C|(\chi(x)}{\chi(1)} + \frac{\chi(z)}{\chi(1)}\right) - (|C| + 1) \\ &= \left(\frac{|C|(\chi(x))}{\chi(1)} - |C|\right) + \left(\frac{\chi(z)}{\chi(1)} - 1\right).\end{aligned}$$

Thus

$$\theta_\chi - \theta_1 \cong \begin{cases} 2 \pmod 4, & \text{if } \chi(z) = -1; \\ 0 \pmod 4, & \text{if } \chi(z) = 1. \end{cases}$$

$\square$

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- For example let $G = 2.B$, the double cover of the Baby Monster, and let C be the unique conjugacy class of elements of order 110.
- If we don't restrict to single classes, we can search all possible connection sets and find lots of examples.

Multiple state transfer in oriented normal Cayley graphs

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- For example, we can find Cayley graphs for 3.A7 generated by a single class (of order 15) and a central element that exhibits MST of order 3.
- We can search for other connection sets and find more examples, eg. in 3.M22 and 3.J3.

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- Other phenomena, PGST, IUM, FR, MST for nonabelian groups? What groups are most promising for providing examples?
- MST on set of size 6?

Overview

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