On the dimensions of some error-correcting codes

Peter Sin

Gainesville, September 13, 2007.
Overview

- LDPC (low density parity check) codes have attracted much attention recently, due to their good performance in theory and practice.

- A main division is into random and structured types.

- One structured family, constructed using certain bipartite graphs was studied by: J.-L. Kim, U. Peled, I. Perepelitsa, V. Pless, and S. Friedland (2004)

- They conjectured the dimensions of the codes.

- We’ll describe the conjecture and its proof (with Q. Xiang).

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The codes $LU(3, q)$

- $q$, any prime power
- $P^*$, $L^*$ be two sets in bijection with $\mathbb{F}_q^3$
- $(a, b, c) \in P^*$ is incident with $[x, y, z] \in L^*$ if and only if
  \[ y = ax + b \quad \text{and} \quad z = ay + c. \] (1)

- The binary incidence matrix $M_2(P^*, L^*)$ and its transpose can be taken as parity check matrices of two codes.
- These codes are designated $LU(3, q)$. We have:
  \[ \dim LU(3, q) = q^3 - \text{rank}M_2(P^*, L^*). \]
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Conjecture: If $q$ is odd, the dimension of $\text{LU}(3, q)$ is $(q^3 - 2q^2 + 3q - 2)/2$.

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The symplectic generalized quadrangle

- \( q \), any prime power
- \((V, (.,.))\), a 4-dimensional \( \mathbb{F}_q \)-vector space with a nonsingular alternating bilinear form
- \( e_0, e_1, e_2, e_3 \), a symplectic basis such that
  \( (e_0, e_3) = (e_1, e_2) = 1 \)
- \( x_0, x_1, x_2, x_3 \), coordinates for basis
- \( P = \mathbb{P}(V) \), the set of points of the projective space of \( V \)
- \( L \), the set of totally isotropic 2-dimensional subspaces of \( V \), considered as lines in \( P \)
- \( (P, L) \) is called the *symplectic generalized quadrangle*. 
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Quadrangle property
Given any line and any point not on the line, there is a unique line which passes though the given point and meets the given line.
- $p_0 = \langle e_0 \rangle$ and $\ell_0 = \langle e_0, e_1 \rangle$.
- $p^\perp$, the set of points on lines through the point $p$
- $P_1 = P \setminus p_0^\perp$
- $L_1$, the set of lines in $L$ which do not meet $\ell_0$
- We have new incidence systems $(P_1, L_1)$, $(P, L_1)$, $(P_1, L)$. 
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• We have new incidence systems \((P_1, L_1), (P, L_1), (P_1, L)\).
We will see below that \((P_1, L_1)\) is equivalent to the system \((P^*, L^*)\).

So we want to prove:

**Theorem**
Assume \(q\) is odd. The rank of \(M_2(P_1, L_1)\) equals 
\[
\frac{q^3 + 2q^2 - 3q + 2}{2}.
\]

A known result is:

**Theorem**
\(\text{(Bagchi-Brouwer-Wilbrink)}\) Assume \(q\) is a power of an odd prime. Then the rank of \(M_2(P, L)\) is 
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Note that the difference in ranks is \(2q\).
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Note that the difference in ranks is \(2q\).
Next, see $(P_1, L_1) \cong (P^*, L^*)$, for $q$ any prime power.
Coordinates of $P_1$

- $x_0, x_1, x_2, x_3$ be homogeneous coordinates of $P$
- $p_0 = \langle e_0 \rangle$
- $P_1 = \{(x_0 : x_1 : x_2 : x_3) \mid x_3 \neq 0\}$
  
  $= \{(a : b : c : 1) \mid a, b, c \in F_q\} \cong F_q^3$.  

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Coordinates of lines in $P(V)$

- $e_i \wedge e_j$, $0 \leq i < j \leq 3$, basis of the exterior square $\wedge^2(V)$
- $p_{01}, p_{02}, p_{03}, p_{12}, p_{13}, p_{23}$, homogeneous coordinates for $P(\wedge^2(V))$

If $W$ is a 2-dimensional subspace of $V$ then $\wedge^2(W) \in P(\wedge^2(V))$.

If $W = \langle (a_0 : a_1 : a_2 : a_3), (b_0 : b_1 : b_2 : b_3) \rangle$ then $\wedge^2(W)$ has coordinates $p_{ij} = a_i b_j - a_j b_i$, its Grassmann-Plücker coordinates.

The totality of points of $P(\wedge^2(V))$ obtained from all $W$ forms the set with equation $p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 0$, called the Klein Quadric.
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Coordinates of $L$ and $L_1$

- $L$ corresponds to the subset of points of the Klein quadric which satisfy the additional linear equation $p_{03} = -p_{12}$.
- $\ell_0 = \langle (1 : 0 : 0 : 0), (0 : 1 : 0 : 0) \rangle$
- $L_1$ is the subset of $L$ given by $p_{23} \neq 0$.
- The quadratic relation yields $L_1 \cong \{ (z^2 + xy : x : z : -z : y : 1) \mid x, y, z \in \mathbb{F}_q \}$ (3)

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Incidence equations

- When is \((a : b : c : 1) \in P_1\) on \((z^2 + xy : x : z : -z : y : 1) \in L_1\)?

- If the line is spanned by points with homogeneous coordinates \((a_0 : a_1 : a_2 : a_3)\) and \((b_0 : b_1 : b_2 : b_3)\). The given point and line are incident if and only if all 3 \(\times\) 3 minors of the matrix

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a & b & c & 1 \\
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are zero.
The four equations which result reduce to the two equations

\[ z = -cy + b, \quad x = cz - a. \]  

Hence \((P_1, L_1)\) and \((P^*, L^*)\) are equivalent.
The four equations which result reduce to the two equations
\[ z = -cy + b, \quad x = cz - a. \] (5)

Hence \((P_1, L_1)\) and \((P^*, L^*)\) are equivalent.
Relative dimensions and a bound $q$ is any prime power.

- $\mathbf{F}_2[P]$, the vector space of all $\mathbf{F}_2$-valued functions on $P$
- Abuse notation slightly, identify points and lines with their characteristic functions in $\mathbf{F}_2[P]$.
- $C(P, L)$, the subspace of $\mathbf{F}_2[P]$ spanned by the $\ell \in L$.
- $C(P, L_1)$, the subspace generated by lines in $L_1$
- $\pi_{P_1} : \mathbf{F}_2[P] \to \mathbf{F}_2[P_1]$, natural projection map
- $C(P_1, L) = \pi_{P_1}(C(P, L)), C(P_1, L_1) = \pi_{P_1}(C(P, L_1))$
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$Z \subset C(P, L_1)$, a set of lines in $L_1$ which maps bijectively under $\pi_{P_1}$ to a basis of $C(P_1, L_1)$.

* $X$, the set of the lines through $p_0$ and let $X_0 = X \setminus \{\ell_0\}$
* $Y$ be any $q$ lines which meet $\ell_0$ in the $q$ distinct points other than $p_0$
* $|X_0 \cup Y| = 2q$ (cf. Theorem 1).
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\( |X_0 \cup Y| = 2q \) (cf. Theorem 1).
Lemma

\( Z \cup X_0 \cup Y \) is linearly independent over \( F_2 \).

Corollary

\[
\dim_{F_2} LU(3, q) \geq q^3 - \dim_{F_2} C(P, L) + 2q.
\] (6)
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Corollary

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Proof of Theorem 1
Assume that $q$ is odd. By Corollary 4 the proof of Theorem 1 will be completed if we can show that $Z \cup X_0 \cup Y$ spans $C(P, L)$ as a vector space over $F_2$. 

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Geometric arguments

**Lemma**

Let $\ell \in L$. Then the sum of all lines which meet $\ell$ (excluding $\ell$ itself) is the constant function 1.

**Proof.**

The function given by the sum takes the value $q \equiv 1$ at any point of $\ell$ and value 1 at any point off $\ell$, by the quadrangle property.
Geometric arguments

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Similarly:

**Lemma**

Let \( \ell \neq \ell_0 \) be a line which meets \( \ell_0 \) at a point \( p \). Let \( \Phi_\ell \) be the sum of all lines in \( L_1 \) which meet \( \ell \). Then

\[
\Phi_\ell(p') = \begin{cases} 
0, & \text{if } p' = p; \\
q, & \text{if } p' \in \ell \setminus \{p\}; \\
0, & \text{if } p' \in p^\perp \setminus \ell; \\
1, & \text{if } p' \in P \setminus p^\perp.
\end{cases}
\] (7)

**Corollary**

Let \( p \in \ell_0 \) and let \( \ell, \ell' \) be two lines through \( p \), neither equal to \( \ell_0 \). Then \( \ell - \ell' \in C(P, L_1) \).
Similarly:

**Lemma**

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Some representation theory

Lemma

\(\ker \pi_{P_1} \cap C(P, L)\) has dimension \(q + 1\), with basis \(X\).

Proof:

- Let \(G_{p_0}\) be the stabilizer in \(Sp(V)\) of \(p_0\).
- \(\ker \pi_{P_1} = F_2[p_0^\perp] = F_2[\{p_0\}] \oplus F_2[p_0^\perp \setminus \{p_0\}]\) (8) as an \(F_2 G_{p_0}\)-module. Clearly \(F_2[\{p_0\}]\) is a one-dimensional trivial \(F_2 G_{p_0}\)-module.
Some representation theory

**Lemma**

\[ \ker \pi_{P_1} \cap C(P, L) \text{ has dimension } q + 1, \text{ with basis } X. \]

**Proof:**

- Let \( G_{p_0} \) be the stabilizer in \( S_p(V) \) of \( p_0 \).

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Some representation theory

Lemma

ker \pi_{P_1} \cap C(P, L) has dimension q + 1, with basis X.

Proof:

- Let \( G_{\rho_0} \) be the stabilizer in \( Sp(V) \) of \( \rho_0 \).

\[
\ker \pi_{P_1} = F_2[\rho_0^\perp] = F_2[\{\rho_0\}] \oplus F_2[\rho_0^\perp \setminus \{\rho_0\}] \quad (8)
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as an \( F_2 G_{\rho_0} \)-module. Clearly \( F_2[\{\rho_0\}] \) is a one-dimensional trivial \( F_2 G_{\rho_0} \)-module.
We consider the following subgroups of $G_{p_0}$.

\[ Q = \left\{ \begin{pmatrix} 1 & a & b & c \\ 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 1 \end{pmatrix} \left| a, b, c \in F_q \right. \right\}, \quad Z(Q) = \left\{ \begin{pmatrix} 1 & 0 & 0 & c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \left| c \in F_q \right. \right\} \]

(9)

$Q \triangleleft G_{p_0}$, $Q/Z(Q)$ is elementary abelian of order $q^2$ and $Z(Q)$ acts trivially on $p_{0}^\perp$.

Since $Q$ has odd order, it acts semisimply on $F_2[p_{0}^\perp]$ and we can compute the decomposition.
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Since $Q$ has odd order, it acts semisimply on $\mathbb{F}_2[p_{0}^\perp]$ and we can compute the decomposition.
Applying Clifford’s Theorem, we have a \( \mathbb{F}_2 G_{\rho_0} \)-module decomposition

\[
\mathbb{F}_2[\rho_0^\perp] = T \oplus W,
\]

where \( T \) is the \( q + 2 \)-dimensional space of \( Q \)-fixed points and \( W \) is simple of dimension \( q^2 - 1 \).

The intersection

\[
\ker \pi_{P_1} \cap C(P, L) = \mathbb{F}_2[\rho_0^\perp] \cap C(P, L),
\]

is an \( \mathbb{F}_2 G_{\rho_0} \)-submodule of \( \mathbb{F}_2[\rho_0^\perp] \).

The \( q + 1 \) lines through \( \rho_0 \) lie in the intersection, accounting for \( q + 1 \) dimensions of \( T \).

We must argue that the intersection is no bigger than their span. If it were, then by (10), \( \mathbb{F}_2[\rho_0^\perp] \cap C(P, L) \) must contain either \( W \) or all the \( Q \)-fixed points on \( \mathbb{F}_2[\rho_0^\perp] \).

Both possibilities lead immediately to contradictions.
Applying Clifford’s Theorem, we have a $F_2G_{P_0}$-module decomposition

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The $q + 1$ lines through $p_0$ lie in the intersection, accounting for $q + 1$ dimensions of $T$.

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$$\ker \pi_{P_1} \cap C(P, L) = \mathbb{F}_2[\rho_0^\perp] \cap C(P, L), \quad (11)$$

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Both possibilities lead immediately to contradictions.
Lemma

\( \ker \pi_{P_1} \cap C(P, L_1) \) has dimension \( q - 1 \), and basis the set of functions \( \ell - \ell' \), where \( \ell \neq \ell_0 \) is an arbitrary but fixed line through \( p_0 \) and \( \ell' \) varies over the \( q - 1 \) lines through \( p_0 \) different from \( \ell_0 \) and \( \ell \).
Lemma

$Z \cup X_0 \cup Y$ spans $C(P, L)$ as a vector space over $F_2$.

Proof:

- By Lemma 9, the span of $X_0$ and $Z$ is equal to the span of $X_0$ and $L_1$, since $\ker \pi_{P_1} \cap C(P, L_1)$ is contained in the span of $X_0$.

- We must show that the span of $X_0 \cup L_1 \cup Y$ contains all lines through $\ell_0$, including $\ell_0$.

- First, consider a line $\ell \neq \ell_0$ through $\ell_0$. We can assume that $\ell$ meets $\ell_0$ at a point other than $p_0$, since otherwise $\ell \in X_0$. Therefore $\ell$ meets $\ell_0$ in the same point $p$ as some element $\ell' \in Y$. Then Corollary 7 shows that $\ell$ lies in the span of $Y$ and $L_1$. 
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On the dimensions of some error-correcting codes
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The only line still missing is $\ell_0$.

By Lemma 5 applied to $\ell_0$, we see that the constant function $1$ is in the span.

Finally, we see from Lemma 6 that

$$\sum_{\ell \in \mathcal{X}_0} \Phi_\ell = 1 - \ell_0,$$

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Further research

- Consider the binary code $\text{LU}(3, q)$ when $q = 2^t$, $t \geq 1$.
- Corollary 4 provides a lower bound for the dimension.
- Note, however, that $\dim_{F_2} C(P, L)$ is quite different:

**Theorem**

(Sastry-Sin) Assume $q = 2^t$. Then the rank of $M_2(P, L)$ is

\[ 1 + \left( \frac{1 + \sqrt{17}}{2} \right)^{2t} + \left( \frac{1 - \sqrt{17}}{2} \right)^{2t}. \]  

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Nevertheless:

- Computer calculations of J.-L. Kim (up to $q = 16$) suggested that the inequality (6) is equality for even $q$ as well.
- Ogul Arslan has found a proof (2007).
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**Theorem**

(Sastry-Sin) Assume $q = 2^t$. Then the rank of $M_2(P, L)$ is

$$1 + \left(\frac{1 + \sqrt{17}}{2}\right)^{2t} + \left(\frac{1 - \sqrt{17}}{2}\right)^{2t}.$$  \hspace{1cm} (13)

Nevertheless:

- Computer calculations of J.-L. Kim (up to $q = 16$) suggested that the inequality (6) is equality for even $q$ as well.
- Ogul Arslan has found a proof (2007).