Please write your proofs carefully and in complete English sentences. If you wish to use theorems from the text, make it clear which theorem you are using, by stating or describing it. Be careful to avoid using mathematical notation incorrectly. When in doubt, use English. Anything that the grader cannot understand may receive no credit.

Name: ______________________________

1. Let $\phi : G \to H$ be a homomorphism of groups.

   (a) (5 points) Show that if K is a subgroup of H, then the set

   $$X = \{g \in G \mid \phi(g) \in K\}$$

   is a subgroup of G.

   Solution: We check that X is nonemepty and closed under inversion and under products. Since $\phi(e_G) = e_H \in K$, we have $e_G \in X$. Let $g \in X$. Then $\phi(g^{-1}) = \phi(g)^{-1} \in K$, since $\phi(g) \in K$ and K is a subgroup. Therefore $g^{-1} \in X$. Finally if g, $g' \in X$, then $\phi(gg') = \phi(g)\phi(g') \in K$, whence $gg' \in X$. This completes the proof that X is a subgroup of G.

   (b) (5 points) Show that if K is a normal subgroup of H then X is a normal subgroup of G.

   Solution: Let $x \in X$ and $g \in G$. Then $\phi(gxg^{-1}) = \phi(g)\phi(x)\phi(g)^{-1}$ and the latter element belongs to K since $\phi(x) \in K$ and K is a normal subgroup of H. Therefore, by definition of X, we have $gxg^{-1} \in X$. This completes the proof that X is a normal subgroup of G.

2. (10 points) Find, up to isomorphism, all Abelian groups of order 600.

Solution: First, we have $600 = 2^3 \cdot 3 \cdot 5^2$. so we must consider partitions of 3, 1 and 2. The partitions of 3 are : 3, $2+1$, $1+1+1$. There is only one partition 1 of 1. The partitions of 2 are 2 and $1+1$. It follows from the fundamental theorem of finite abelian groups that there are six abelian groups of order 600 up to isomorphism, namely

$$Z_8 \oplus Z_3 \oplus Z_{25}, \quad Z_8 \oplus Z_3 \oplus Z_5 \oplus Z_5,$$
$$Z_4 \oplus Z_2 \oplus Z_3 \oplus Z_{25}, \quad Z_4 \oplus Z_2 \oplus Z_3 \oplus Z_5 \oplus Z_5,$$
$$Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_3 \oplus Z_{25}, \quad Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_3 \oplus Z_5 \oplus Z_5.$$

3. True or false? If you think the statement is true, give a proof, stating any theorems you need. If false, provide a concrete counterexample.

   (a) (3 points) If a factor group G/N has an element of order n then G has an element of order n.

   Solution: Some of you took G to be finite, while others did not make that assumption. I gave full credit for correct complete answers in either case.

   If we do not assume that G is finite, then the answer is "false", counterexample being $G = \mathbb{Z}$, and $N = 2\mathbb{Z}$, in which case G/N has an element of order 2 but G does not.

   If we assume that G is finite, then the answer is "true". Suppose $gN \in G/N$ has order n. Then n divides the order of g (a hwk problem). Hence g has order nk for some positive integer k, whence g^k is the required element of order n.

   A common mistake was to argue that n must divide $|G|$ and so G has an element of order n. While it is true that n must divide $|G|$, it is not true that for each positive integer d dividing $|G|$ there is an element of order d. For example, consider any noncyclic group.

   (b) (3 points) If G is a group of permutations of a set S and $s \in S$, then the stabilizer $Stab_G(s)$ is a normal subgroup of G.

   Solution: False. Let $G = S_3$, $S = \{1, 2, 3\}$, $s = 3$. Then $Stab_G(s) = \{id, (12)\}$ which is not a normal subgroup of G since $(13)(12)(13)^{-1} = (13)(12)(13) = (23) \notin \{id, (12)\}$

   (c) (4 points) The number of elements of order 4 in $\mathbf{Z}_4 \oplus \mathbf{Z}_8$ is 12.

   Solution: True. An element (a, b) has order 4 if and only of the least common multiple of the orders of a and b is 4. The elements of order 4 are therefore:

$$(1,0), (1,2), (1,4), (1,6), (3,0), (3,2), (3,4), (3,6), (0,2), (0,6), (2,2), (2,6).$$

   There are 12 in all.

4. (5 points) (Extra credit) Determine the positive integers n for which every Abelian group of order n is cyclic.

Solution: All abelian groups of order n are cyclic if and only if n is not divisible by the square of any prime. To see this, suppose first that n satisfies this condition. Then by Cauchy's theorem for abelian groups, any abelian group G of order n has an element g_p for each prime p dividing n. Then the product of all the elemnts g_p will have order n so G is cyclic. Conversely, if p^2 divides n, then let G be the abelian group $Z_p \oplus Z_p \oplus Z_m$, where $m = n/p^2$. Then G has a subgroup isomorphic to $Z_p \oplus Z_p$, and this subgroup has more than one subgroup of order p. Hence G also has more than one subgroup of order p, so G is not cyclic.