Some Weyl Modules for Classical Groups

Peter Sin

University of Florida

Southwestern Group Theory Day
November 7th, 2009
University of Arizona
Outline

Introduction

Statement of results

Jantzen Sum Formula

Applications

Further Research

Conclusion
Joint work with Ogul Arslan.
Let $G$ be a semisimple algebraic group in characteristic $p > 0$.

An important class of modules are the Weyl modules $V(\lambda)$.

The characters of Weyl modules are given by Weyl’s Character Formula.

But their precise submodule structure is not fully understood.

This talk is about a uniform description of the submodule structure of some infinite families of Weyl modules for classical groups.
Let $G$ be a semisimple algebraic group in characteristic $p > 0$.

An important class of modules are the Weyl modules $V(\lambda)$.

The characters of Weyl modules are given by Weyl’s Character Formula.

But their precise submodule structure is not fully understood.

This talk is about a uniform description of the submodule structure of some infinite families of Weyl modules for classical groups.
Let $G$ be a semisimple algebraic group in characteristic $p > 0$.

An important class of modules are the Weyl modules $V(\lambda)$.

The characters of Weyl modules are given by Weyl’s Character Formula.

But their precise submodule structure is not fully understood.

This talk is about a uniform description of the submodule structure of some infinite families of Weyl modules for classical groups.
Let $G$ be a semisimple algebraic group in characteristic $p > 0$.

An important class of modules are the Weyl modules $V(\lambda)$.

The characters of Weyl modules are given by Weyl’s Character Formula.

But their precise submodule structure is not fully understood.

This talk is about a uniform description of the submodule structure of some infinite families of Weyl modules for classical groups.
Let $G$ be a semisimple algebraic group in characteristic $p > 0$.
An important class of modules are the Weyl modules $V(\lambda)$.
The characters of Weyl modules are given by Weyl’s Character Formula.
But their precise submodule structure is not fully understood.
This talk is about a uniform description of the submodule structure of some infinite families of Weyl modules for classical groups.
Outline

Introduction

Statement of results

Jantzen Sum Formula

Applications

Further Research

Conclusion
Groups and weights considered

(B) $G$ of type $B_\ell$, $(\ell \geq 2)$ $\lambda = r(\omega_1)$, $0 \leq r \leq p - 1$;

(D) $G$ of type $D_\ell$, $(\ell \geq 3)$ $\lambda = r(\omega_1)$, $0 \leq r \leq p - 1$;

(A) $G$ of type $A_\ell$, $(\ell \geq 3)$ $\lambda = r(\omega_1 + \omega_\ell)$, $0 \leq r \leq p - 1$;

(A') $G$ of type $A_\ell$, $(\ell \geq 4)$ $\lambda = (\omega_2 + \omega_{\ell-1})$; and

(A'') $G$ of type $A_4$, $\lambda = (p - 2)(\omega_2 + \omega_{\ell-1})$ or $(p - 1)(\omega_2 + \omega_{\ell-1})$. 
For each of the weights considered we obtain:

- The character of the simple module $L(\lambda)$
- The characters (and multiplicities) of the simple composition factors of $V(\lambda)$
- The submodule lattice of $V(\lambda)$
Theorem
Let $G$ be of type $B_{\ell}$, $\ell \geq 2$. Let $\omega_1$ be the highest weight of the standard orthogonal module of dimension $2\ell + 1$. Assume $0 \leq r \leq p - 1$. Then the following hold.

(a) $H^0(r\omega_1)$ is simple unless (i) $p = 2$ and $r = 1$ or (ii) $p > 2$ and there exists a positive odd integer $m$ such that

$$r + 2\ell - 1 \leq mp \leq 2r + 2\ell - 2.$$

(b) If (i) holds then the quotient $H^0(\omega_1)/L(\omega_1)$ is the one-dimensional trivial module.

(c) If (ii) holds then $m$ is unique and

$$H^0(r\omega_1)/L(r\omega_1) \cong H^0(r_1\omega_1),$$

where $r_1 = mp - 2\ell + 1 - r$. Furthermore the module $H^0(r_1\omega_1)$ is simple.
Theorem
Let $G$ be of type $D_\ell$, $\ell \geq 3$. Let $\omega_1$ be the highest weight of the standard orthogonal module of dimension $2\ell$. Assume $0 \leq r \leq p - 1$. Then the following hold.

(a) Suppose that there exists a positive even integer $m$ such that

$$r + 2\ell - 2 \leq mp \leq 2r + 2\ell - 3.$$ 

Then $m$ is unique and

$$H^0(r\omega_1)/L(r\omega_1) \cong H^0(r_1\omega_1),$$

where $r_1 = mp - 2\ell + 2 - r$. Furthermore the module $H^0(r_1\omega_1)$ is simple.

(b) Otherwise, $H^0(r\omega_1)$ is simple.
Theorem
Let \( G \) be of type \( A_\ell, \ell \geq 3 \). Assume \( 0 \leq r \leq p - 1 \). Then the following hold.

(a) Suppose that there exists a positive integer \( m \) such that

\[
r + \ell \leq mp \leq 2r + \ell - 1.
\]

Then \( m \) is unique and

\[
H^0\left( r(\omega_1 + \omega_\ell) \right) / L\left( r(\omega_1 + \omega_\ell) \right) \cong H^0\left( r_1(\omega_1 + \omega_\ell) \right),
\]

where \( r_1 = mp - \ell - r \). Furthermore the module \( H^0\left( r_1(\omega_1 + \omega_\ell) \right) \) is simple.

(b) Otherwise, \( H^0\left( r(\omega_1 + \omega_\ell) \right) \) is simple.
Theorem

Let $G$ be of type $A_\ell$, $\ell \geq 4$. If $p > 2$ then the following hold.

(a) If $\ell \equiv 0 \pmod{p}$ then $H^0(\omega_2 + \omega_{\ell-1})/L(\omega_2 + \omega_{\ell-1}) \cong k$.

(b) If $\ell \equiv 1 \pmod{p}$ then

$$H^0(\omega_2 + \omega_{\ell-1})/L(\omega_2 + \omega_{\ell-1}) \cong H^0(\omega_1 + \omega_{\ell})$$

and this module is simple.

(c) In all other cases $H^0(\omega_2 + \omega_{\ell-1})$ is simple.
If $p = 2$ then the following hold.

(d) If $\ell \equiv 0 \pmod{4}$ then

$$H^0(\omega_2 + \omega_{\ell-1})/L(\omega_2 + \omega_{\ell-1}) \cong k.$$

(e) If $\ell \equiv 1 \pmod{4}$ then

$$H^0(\omega_2 + \omega_{\ell-1})/L(\omega_2 + \omega_{\ell-1}) \cong V(\omega_1 + \omega_\ell).$$

(f) If $\ell \equiv 2 \pmod{4}$ then $H^0(\omega_2 + \omega_{\ell-1})$ is simple.

(g) If $\ell \equiv 3 \pmod{4}$ then

$$H^0(\omega_2 + \omega_{\ell-1})/L(\omega_2 + \omega_{\ell-1}) \cong L(\omega_1 + \omega_\ell).$$
Theorem
Let $G$ be of type $A_4$.

(a) If $p = 2$, then $H^0((p-1)(\omega_2 + \omega_3))/L((p-1)(\omega_2 + \omega_3)) \cong k$.

If $p > 2$, then the following hold.

(b) $H^0((p-1)(\omega_2 + \omega_3))/L((p-1)(\omega_2 + \omega_3)) \cong L((p-2)(\omega_2 + \omega_3))$.

(c) $H^0((p-2)(\omega_2 + \omega_3))/L((p-2)(\omega_2 + \omega_3)) \cong H^0((p-2)(\omega_1 + \omega_4))$, which is simple.
Outline

- Introduction
- Statement of results
- Jantzen Sum Formula
- Applications
- Further Research
- Conclusion
$V(\lambda)$ has a descending filtration, of submodules $V(\lambda)^i, i > 0$, such that

$$V(\lambda)^1 = \text{rad } V(\lambda), \quad \text{so} \quad V(\lambda)/V(\lambda)^1 \cong L(\lambda).$$

and

$$\sum_{i>0} \text{Ch}(V(\lambda)^i) = - \sum_{\alpha>0} \sum_{\{m: 0 < mp < \langle \lambda + \rho, \alpha \vee \rangle\}} v_p(mp) \chi(\lambda - mp\alpha)$$
The RHS of the Sum Formula can be computed by the following algorithm: For each positive root $\alpha$,

(i) Compute $\langle \lambda + \rho, \alpha^\vee \rangle$

(ii) Compute $\lambda + \rho - mp\alpha$ for $0 < m < \langle \lambda + \rho, \alpha^\vee \rangle$

(iii) Find the Weyl group conjugate $w(\lambda + \rho - mp\alpha)$ in $X_+$ and note the sign of a Weyl group element $w$.

(iv) Compute $w(\lambda + \rho - mp\alpha) - \rho$.

(v) The contribution to the sum is $- \text{sign}(w) v_p(mp) \chi(w(\lambda + \rho - mp\alpha) - \rho)$. 
The main challenge lies in trying to do infinitely many Sum Formula computations at once. For fixed type the parameters of the problem are $\rho$, $r$ and the rank $\ell$.

**Lemma**

(a) If $R$ is of type $B_\ell$ or $D_\ell$ and $\lambda + \rho - mp\alpha$ has two coordinates with the same absolute value then the pair $(\alpha, m)$ contributes nothing to the final sum.

(b) If $R$ is of type $A_\ell$ and $\lambda + \rho - mp\alpha$ has two equal coordinates, then the pair $(\alpha, m)$ contributes nothing to the final sum.
Eliminating multiplicities

- The Sum Formula *overestimates* the character of rad $V(\lambda)$.
- Example. Type $D_\ell$.
- $S^r(V^*)$ has a *good filtration* (Andersen-Jantzen). with subquotients of the form $H^0(s\omega_1)$, $s < r$.
- Then for $r_1 < r$,

$$\dim \text{Hom}_G(V(r_1\omega_1), V(r\omega_1)) = \dim \text{Hom}_G(H^0(r\omega_1), H^0(r_1\omega_1))$$

$$\leq \dim \text{Hom}_G(S^r(V^*), H^0(r_1\omega_1))$$

$$= \dim \text{Hom}_G(V(r_1\omega_1), S^r(V^*))$$

(by self-duality of $S^r(V^*)$)

$$= \text{multiplicity of } H^0(r_1\omega_1)$$

in a *good filtration* of $S^r(V^*)$

$$\leq 1.$$
Eliminating multiplicities

▶ The Sum Formula *overestimates* the character of $\text{rad } V(\lambda)$.
▶ Example. Type $D_\ell$.
▶ $S^r(V^*)$ has a *good filtration* (Andersen-Jantzen) with subquotients of the form $H^0(s\omega_1)$, $s < r$.
▶ Then for $r_1 < r$,

$$\dim \text{Hom}_G(V(r_1\omega_1), V(r\omega_1)) = \dim \text{Hom}_G(H^0(r\omega_1), H^0(r_1\omega_1))$$

$$\leq \dim \text{Hom}_G(S^r(V^*), H^0(r_1\omega_1))$$

$$= \dim \text{Hom}_G(V(r_1\omega_1), S^r(V^*))$$

(by self-duality of $S^r(V^*)$)

$$= \text{multiplicity of } H^0(r_1\omega_1)$$

in a *good filtration* of $S^r(V^*)$

$$\leq 1.$$
Eliminating multiplicities

- The Sum Formula \textit{overestimates} the character of \(\text{rad } V(\lambda)\).
- Example. Type \(D_\ell\).
- \(S^r(V^*)\) has a \textit{good filtration} (Andersen-Jantzen). with subquotients of the form \(H^0(s\omega_1), s < r\).
- Then for \(r_1 < r\),

\[
\dim \text{Hom}_G(V(r_1\omega_1), V(r\omega_1)) = \dim \text{Hom}_G(H^0(r\omega_1), H^0(r_1\omega_1)) \\
\leq \dim \text{Hom}_G(S^r(V^*), H^0(r_1\omega_1)) \\
= \dim \text{Hom}_G(V(r_1\omega_1), S^r(V^*)) \\
(\text{by self-duality of } S^r(V^*)) \\
= \text{multiplicity of } H^0(r_1\omega_1) \\
\text{in a good filtration of } S^r(V^*) \leq 1.
\]
The Sum Formula overestimates the character of \( \text{rad} \ V(\lambda) \).

Example. Type \( D_\ell \).

\( S^r(V^*) \) has a good filtration (Andersen-Jantzen). with subquotients of the form \( H^0(s\omega_1), s < r \).

Then for \( r_1 < r \),

\[
\dim \text{Hom}_G(V(r_1 \omega_1), V(r_\omega_1)) = \dim \text{Hom}_G(H^0(r_\omega_1), H^0(r_1 \omega_1)) \\
\leq \dim \text{Hom}_G(S^r(V^*), H^0(r_1 \omega_1)) \\
= \dim \text{Hom}_G(V(r_1 \omega_1), S^r(V^*)) \\
(\text{by self-duality of } S^r(V^*)) \\
= \text{multiplicity of } H^0(r_1 \omega_1) \text{ in a good filtration of } S^r(V^*) \\
\leq 1.
\]
Eliminating multiplicities

► The Sum Formula overestimates the character of \( \text{rad} \ V(\lambda) \).
► Example. Type \( D_\ell \).
► \( S^r(V^*) \) has a good filtration (Andersen-Jantzen). with subquotients of the form \( H^0(s\omega_1), s < r \).
► Then for \( r_1 < r \),

\[
\dim \text{Hom}_G(V(r_1\omega_1), V(r\omega_1)) = \dim \text{Hom}_G(H^0(r\omega_1), H^0(r_1\omega_1)) \\
\leq \dim \text{Hom}_G(S^r(V^*), H^0(r_1\omega_1)) \\
= \dim \text{Hom}_G(V(r_1\omega_1), S^r(V^*)) \\
(\text{by self-duality of} \ S^r(V^*)) \\
= \text{multiplicity of} \ H^0(r_1\omega_1) \ \\
\text{in a good filtration of} \ S^r(V^*) \\
\leq 1.
\]
Eliminating multiplicities

- The Sum Formula overestimates the character of \( \text{rad } V(\lambda) \).
- Example. Type \( D_\ell \).
- \( S^r(V^*) \) has a good filtration (Andersen-Jantzen). with subquotients of the form \( H^0(s_\omega_1), s < r \).
- Then for \( r_1 < r \),

\[
\dim \text{Hom}_G(V(r_1 \omega_1), V(r \omega_1)) = \dim \text{Hom}_G(H^0(r \omega_1), H^0(r_1 \omega_1)) \leq \dim \text{Hom}_G(S^r(V^*), H^0(r_1 \omega_1)) = \dim \text{Hom}_G(V(r_1 \omega_1), S^r(V^*))
\]
(by self-duality of \( S^r(V^*) \))

\[
= \text{multiplicity of } H^0(r_1 \omega_1)
\]
in a good filtration of \( S^r(V^*) \)
\[
\leq 1.
\]
Eliminating multiplicities

- The Sum Formula overestimates the character of $\text{rad } V(\lambda)$.
- Example. Type $D_\ell$.
- $S^r(V^*)$ has a good filtration (Andersen-Jantzen). with subquotients of the form $H^0(s\omega_1)$, $s < r$.
- Then for $r_1 < r$,

$$\dim \text{Hom}_G(V(r_1\omega_1), V(r\omega_1)) = \dim \text{Hom}_G(H^0(r\omega_1), H^0(r_1\omega_1))$$
$$\leq \dim \text{Hom}_G(S^r(V^*), H^0(r_1\omega_1))$$
$$= \dim \text{Hom}_G(V(r_1\omega_1), S^r(V^*))$$
(by self-duality of $S^r(V^*)$)
$$= \text{multiplicity of } H^0(r_1\omega_1)$$
in a good filtration of $S^r(V^*)$
$$\leq 1.$$
Eliminating multiplicities

- The Sum Formula *overestimates* the character of $\text{rad } V(\lambda)$.
- Example. Type $D_\ell$.
- $S^r(V^*)$ has a *good filtration* (Andersen-Jantzen). with subquotients of the form $H^0(s\omega_1)$, $s < r$.
- Then for $r_1 < r$,

\[
\dim \text{Hom}_G(V(r_1\omega_1), V(r\omega_1)) = \dim \text{Hom}_G(H^0(r\omega_1), H^0(r_1\omega_1)) \\
\leq \dim \text{Hom}_G(S^r(V^*), H^0(r_1\omega_1)) \\
= \dim \text{Hom}_G(V(r_1\omega_1), S^r(V^*)) \\
(\text{by self-duality of } S^r(V^*)) \\
= \text{multiplicity of } H^0(r_1\omega_1) \\
in a *good filtration* of $S^r(V^*)$ \\
\leq 1.
\]
Incidence of points and polar hyperplanes

- $V$ vector space over $\mathbb{F}_q$ with nonsingular form $b(-, -)$.
- $b$ may be alternating or symmetric or hermitian.
- $\hat{P} = \{\text{all 1-dimensional subspaces of } V\}$
- $\supseteq P = \{\text{singular 1-dimensional subspaces}\}$,
- $\hat{P}^* = \{\text{hyperplanes of } V\} \supseteq P^* = \{p^\perp \mid p \in P\}$, polar hyperplanes.
- $G(q)$ = group of linear transformations preserving $b(-, -)$.
- $A =$ incidence matrix of $(\hat{P}^*, \hat{P})$

$$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$
Incidence of points and polar hyperplanes

- $V$ vector space over $\mathbb{F}_q$ with nonsingular form $b(-,-)$.
- $b$ may be alternating or symmetric or hermitian.
- $\hat{P} = \{\text{all 1-dimensional subspaces of } V\}$
  $\supseteq P = \{\text{singular 1-dimensional subspaces}\}$,
- $\hat{P}^* = \{\text{hyperplanes of } V\} \supseteq P^* = \{p^\perp | p \in P\}$, polar hyperplanes.
- $G(q) = \text{group of linear transformations preserving } b(-,-)$.
- $A = \text{incidence matrix of } (\hat{P}^*, \hat{P})$

$$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$
Incidence of points and polar hyperplanes

- $V$ vector space over $\mathbb{F}_q$ with nonsingular form $b(−, −)$.
- $b$ may be alternating or symmetric or hermitian.
- $\hat{P} = \{all \ 1\text{-}dimensional \ subspaces \ of \ V\}$
- $\hat{P}^* = \{hyperplanes \ of \ V\} \supseteq P^* = \{p^\perp \mid p \in P\}$, polar hyperplanes.
- $G(q) = \text{group of linear transformations preserving } b(−, −)$.
- $A = \text{incidence matrix of } (\hat{P}^*, \hat{P})$

$$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$
Incidence of points and polar hyperplanes

- $V$ vector space over $\mathbb{F}_q$ with nonsingular form $b(−, −)$.
- $b$ may be alternating or symmetric or hermitian.
- $\hat{P} = \{\text{all 1-dimensional subspaces of } V\}$
  $\supseteq P = \{\text{singular 1-dimensional subspaces}\}$,
- $\hat{P}^* = \{\text{hyperplanes of } V\} \supseteq P^* = \{p^\perp \mid p \in P\}$, polar hyperplanes.
- $G(q) = \text{group of linear transformations preserving } b(−, −)$.
- $A = \text{incidence matrix of } (\hat{P}^*, \hat{P})$

$$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$
Incidence of points and polar hyperplanes

- $V$ vector space over $\mathbb{F}_q$ with nonsingular form $b(\cdot, \cdot)$.
- $b$ may be alternating or symmetric or hermitian.
- $\hat{P} = \{\text{all 1-dimensional subspaces of } V\}$
  $\supseteq P = \{\text{singular 1-dimensional subspaces}\}$,
- $\hat{P}^* = \{\text{hyperplanes of } V\}$
- $G(q) = \text{group of linear transformations preserving } b(\cdot, \cdot)$.
- $A = \text{incidence matrix of } (\hat{P}^*, \hat{P})$

\[
A = \begin{bmatrix}
A_1 \\
A_2
\end{bmatrix}
\]
Incidence of points and polar hyperplanes

- $V$ vector space over $\mathbb{F}_q$ with nonsingular form $b(\cdot, \cdot)$.
- $b$ may be alternating or symmetric or hermitian.
- $\hat{P} = \{\text{all 1-dimensional subspaces of } V\}$
  $\supseteq P = \{\text{singular 1-dimensional subspaces}\}$,
- $\hat{P}^* = \{\text{hyperplanes of } V\} \supseteq P^* = \{p^\perp \mid p \in P\}$, polar hyperplanes.
- $G(q) = \text{group of linear transformations preserving } b(\cdot, \cdot)$.
- $A = \text{incidence matrix of } (\hat{P}^*, \hat{P})$
  
  \[ A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \]
Incidence of points and polar hyperplanes

- $V$ vector space over $\mathbb{F}_q$ with nonsingular form $b(-,-)$.
- $b$ may be alternating or symmetric or hermitian.
- $\hat{P} = \{\text{all 1-dimensional subspaces of } V\}$
  $\supseteq P = \{\text{singular 1-dimensional subspaces}\}$,
- $\hat{P}^* = \{\text{hyperplanes of } V\} \supseteq P^* = \{p^\perp | p \in P\}$, polar hyperplanes.
- $G(q) = \text{group of linear transformations preserving } b(-,-)$.
- $A = \text{incidence matrix of } (\hat{P}^*, \hat{P})$

$$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$
Incidence of points and polar hyperplanes

- $V$ vector space over $\mathbb{F}_q$ with nonsingular form $b(-, -)$.
- $b$ may be alternating or symmetric or hermitian.
- $\hat{P} = \{\text{all 1-dimensional subspaces of } V\}$
  $\supseteq P = \{\text{singular 1-dimensional subspaces}\}$,
- $\hat{P}^* = \{\text{hyperplanes of } V\} \supseteq P^* = \{p^\perp | p \in P\}$, polar hyperplanes.
- $G(q) = \text{group of linear transformations preserving } b(-, -)$.
- $A = \text{incidence matrix of } (\hat{P}^*, \hat{P})$

\[
A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}
\]
Incidence of points and polar hyperplanes

- $V$ vector space over $\mathbb{F}_q$ with nonsingular form $b(−, −)$.
- $b$ may be alternating or symmetric or hermitian.
- $\hat{P} = \{\text{all 1-dimensional subspaces of } V\} \supseteq P = \{\text{singular 1-dimensional subspaces}\}$,
- $\hat{P}^* = \{\text{hyperplanes of } V\} \supseteq P^* = \{p^\perp | p \in P\}$, polar hyperplanes.
- $G(q)$ = group of linear transformations preserving $b(−, −)$.
- $A = \text{incidence matrix of } (\hat{P}^*, \hat{P})$

$$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$
We consider the $p$-ranks, where $q = p^t$.

The $p$-rank of $A$ is well known (Goethals-Delsarte, MacWilliams-Mann, Smith), and the $p$-rank of $A_1$ was found by Blokhuis and Moorhouse.

Moorhouse (Linz, 2006): What is the $p$-rank of $A_{11}$?

$$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$
\( p \)-ranks

- We consider the \( p \)-ranks, where \( q = p^t \).
- The \( p \)-rank of \( A \) is well known (Goethals-Delsarte, MacWilliams-Mann, Smith), and the \( p \)-rank of \( A_1 \) was found by Blokhuis and Moorhouse.
- Moorhouse (Linz, 2006): What is the \( p \)-rank of \( A_{11} \) ?

\[
A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}
\]
We consider the $p$-ranks, where $q = p^t$.

The $p$-rank of $A$ is well known (Goethals-Delsarte, MacWilliams-Mann, Smith), and the $p$-rank of $A_1$ was found by Blokhuis and Moorhouse.

Moorhouse (Linz, 2006): What is the $p$-rank of $A_{11}$?

$$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$
Permutation module structure

- (a) $G(q)$ acts on $P$ with permutation rank 3
- (b) $k[P] \cong k \cdot 1 \oplus Y$,
- (c) $\text{head}(Y) \cong \text{soc}(Y)$.
- (a),(b),(c) $\implies \text{head}(Y)$ is a simple $kG(q)$-module. Call it $L$.
- $P$ and $P^*$ are isomorphic $G(q)$-sets, so the incidence map induces

$$\phi \in \text{End}_{kG(q)}(k[P]), \quad \phi(p) = \sum_{p' \in p^\perp} p'.$$

$$\text{Im} \phi = k \cdot 1 \oplus L.$$

- Outcome: $\text{rank}_p A_{11} = 1 + \dim L$. 

Permutation module structure

- (a) $G(q)$ acts on $P$ with permutation rank 3

\[ \Leftrightarrow \dim \text{End}_{kG(q)}(k[P]) = 3 \]

- (b) $k[P] \cong k.1 \oplus Y,$

- (c) $\text{head}(Y) \cong \text{soc}(Y).$

- (a),(b),(c) $\implies \text{head}(Y)$ is a simple $kG(q)$-module. Call it $L.$

- $P$ and $P^*$ are isomorphic $G(q)$-sets, so the incidence map induces

\[ \phi \in \text{End}_{kG(q)}(k[P]), \quad \phi(p) = \sum_{p' \in p^\perp} p'. \]

\[ \text{Im} \phi = k.1 \oplus L. \]

- Outcome: $\text{rank}_p A_{11} = 1 + \dim L.$
Permutation module structure

- (a) $G(q)$ acts on $P$ with permutation rank 3
  $\iff \dim \operatorname{End}_{kG(q)}(k[P]) = 3$
- (b) $k[P] \cong k.1 \oplus Y$,
- (c) $\operatorname{head}(Y) \cong \operatorname{soc}(Y)$.
- (a),(b),(c) $\implies \operatorname{head}(Y)$ is a simple $kG(q)$-module. Call it $L$.
- $P$ and $P^*$ are isomorphic $G(q)$-sets, so the incidence map induces

$$\phi \in \operatorname{End}_{kG(q)}(k[P]), \quad \phi(p) = \sum_{p' \in p^\perp} p'.$$

$$\text{Im } \phi = k.1 \oplus L.$$

- Outcome: $\operatorname{rank}_p A_{11} = 1 + \dim L$. 
Permutation module structure

(a) $G(q)$ acts on $P$ with permutation rank 3

$\iff \dim \operatorname{End}_{kG(q)}(k[P]) = 3$

(b) $k[P] \cong k.1 \oplus Y$,

(c) $\operatorname{head}(Y) \cong \operatorname{soc}(Y)$.

(a),(b),(c) $\iff \operatorname{head}(Y)$ is a simple $kG(q)$-module. Call it $L$.

$P$ and $P^*$ are isomorphic $G(q)$-sets, so the incidence map induces

$$\phi \in \operatorname{End}_{kG(q)}(k[P]), \quad \phi(p) = \sum_{p' \in p^\perp} p'.$$

$$\operatorname{Im} \phi = k.1 \oplus L.$$

Outcome: $\operatorname{rank}_p A_{11} = 1 + \dim L.$
Permutation module structure

- (a) $G(q)$ acts on $P$ with permutation rank 3
  $\iff \dim \text{End}_{kG(q)}(k[P]) = 3$

- (b) $k[P] \cong k.1 \oplus Y$,

- (c) $\text{head}(Y) \cong \text{soc}(Y)$.

- (a),(b),(c) $\implies \text{head}(Y)$ is a simple $kG(q)$-module. Call it $L$.

- $P$ and $P^*$ are isomorphic $G(q)$-sets, so the incidence map induces

  $$\phi \in \text{End}_{kG(q)}(k[P]), \quad \phi(p) = \sum_{p' \in p^\perp} p'.$$

  $$\text{Im} \phi = k.1 \oplus L.$$

- Outcome: $\text{rank}_p A_{11} = 1 + \dim L$. 
Permutation module structure

- (a) $G(q)$ acts on $P$ with permutation rank 3 \iff $\dim \text{End}_{kG(q)}(k[P]) = 3$
- (b) $k[P] \cong k \cdot 1 \oplus Y$,
- (c) $\text{head}(Y) \cong \text{soc}(Y)$.
- (a),(b),(c) \implies $\text{head}(Y)$ is a simple $kG(q)$-module. Call it $L$.
- $P$ and $P^*$ are isomorphic $G(q)$-sets, so the incidence map induces

$$\phi \in \text{End}_{kG(q)}(k[P]), \quad \phi(p) = \sum_{p' \in p^\perp} p'.$$

$$\text{Im} \phi = k \cdot 1 \oplus L.$$

- Outcome: $\text{rank}_p A_{11} = 1 + \dim L$. 
Permutation module structure

- (a) \(G(q)\) acts on \(P\) with permutation rank 3
  \(\iff\) \(\dim \operatorname{End}_{kG(q)}(k[P]) = 3\)
- (b) \(k[P] \cong k.1 \oplus Y\),
- (c) \(\text{head}(Y) \cong \operatorname{soc}(Y)\).
- (a), (b), (c) \(\implies\) \(\text{head}(Y)\) is a simple \(kG(q)\)-module. Call it \(L\).
- \(P\) and \(P^*\) are isomorphic \(G(q)\)-sets, so the incidence map induces
  \[
  \phi \in \operatorname{End}_{kG(q)}(k[P]), \quad \phi(p) = \sum_{p' \in p^\perp} p'.
  \]
- \(\text{Im} \phi = k.1 \oplus L\).
- Outcome: \(\operatorname{rank}_p A_{11} = 1 + \dim L\).
Permutation module structure

- (a) $G(q)$ acts on $P$ with permutation rank 3 $\iff \dim \text{End}_{kG(q)}(k[P]) = 3$

- (b) $k[P] \cong k.1 \oplus Y$,

- (c) $\text{head}(Y) \cong \text{soc}(Y)$.

- (a),(b),(c) $\implies \text{head}(Y)$ is a simple $kG(q)$-module. Call it $L$.

- $P$ and $P^*$ are isomorphic $G(q)$-sets, so the incidence map induces

$$\phi \in \text{End}_{kG(q)}(k[P]), \quad \phi(p) = \sum_{p' \in p^\perp} p'.$$

- $\text{Im} \phi = k.1 \oplus L$.

- Outcome: $\text{rank}_p A_{11} = 1 + \dim L$. 

\[ \]
Identifying the simple module $L$

- $k[P] = \text{ind}^{G(q)}_{G(q)_x}(k)$, $x \in P$, so Frobenius Reciprocity implies that $G(q)_x$ has a fixed point on $L$.
- The fixed point condition characterizes $L$:

$$L \cong L((q - 1)\omega),$$

where $\omega = \omega_1$ in the orthogonal and symplectic cases, and $\omega_1 + \omega_\ell$ in the unitary case.
- By Steinberg’s Tensor Product Theorem,

$$L((q-1)\omega) = L((p-1)\omega) \otimes L((p-1)\omega)^{(p)} \cdots \otimes L((p-1)\omega)^{(p^{t-1})}$$

- Conclusion: $\text{rank}_p A_{11} = 1 + (\dim L((p - 1)\omega))^t$. 
Identifying the simple module $L$

- $k[P] = \text{ind}^{G(q)}_{G(q)x}(k)$, $x \in P$, so Frobenius Reciprocity implies that $G(q)_x$ has a fixed point on $L$.
- The fixed point condition characterizes $L$:

$$L \cong L((q - 1)\omega),$$

where $\omega = \omega_1$ in the orthogonal and symplectic cases, and $\omega_1 + \omega_\ell$ in the unitary case.

- By Steinberg’s Tensor Product Theorem,

$$L((q-1)\omega) = L((p-1)\omega) \otimes L((p-1)\omega)^{(p)} \cdots \otimes L((p-1)\omega)^{(p^{t-1})}$$

- Conclusion: $\text{rank}_p A_{11} = 1 + (\text{dim} L((p - 1)\omega))^t$. 
Identifying the simple module $L$

- $k[P] = \text{ind}_{G(q)}^{G(q)_x} (k)$, $x \in P$, so Frobenius Reciprocity implies that $G(q)_x$ has a fixed point on $L$.
- The fixed point condition characterizes $L$:

$$L \cong L((q - 1)\omega),$$

where $\omega = \omega_1$ in the orthogonal and symplectic cases, and $\omega_1 + \omega_\ell$ in the unitary case.

- By Steinberg’s Tensor Product Theorem,

$$L((q-1)\omega) = L((p-1)\omega) \otimes L((p-1)\omega)^{(p)} \cdots \otimes L((p-1)\omega)^{(p^t-1)}$$

- Conclusion: $\text{rank}_p A_{11} = 1 + (\text{dim} L((p - 1)\omega))^t$. 
Identifying the simple module $L$

- $k[P] = \text{ind}_{G(q)}^G(k)$, $x \in P$, so Frobenius Reciprocity implies that $G(q)_x$ has a fixed point on $L$.
- The fixed point condition characterizes $L$:
  \[
  L \cong L((q - 1)\omega),
  \]
  where $\omega = \omega_1$ in the orthogonal and symplectic cases, and $\omega_1 + \omega_\ell$ in the unitary case.
- By Steinberg’s Tensor Product Theorem,
  \[
  L((q-1)\omega) = L((p-1)\omega) \otimes L((p-1)\omega)^{(p)} \cdots \otimes L((p-1)\omega)^{(p^{t-1})}
  \]
- Conclusion: $\text{rank}_p A_{11} = 1 + (\dim L((p - 1)\omega))^t.$
Outline

Introduction

Statement of results

Jantzen Sum Formula

Applications

Further Research

Conclusion
Oppositeness

Let \((\Delta(q), S)\) be the spherical Tits building of a finite group of Lie type.

Two types \(I, J \subseteq S\) are opposite if \(I^{w_0} = J\).

Assume \(I\) and \(J\) are opposite types. We say the cosets \(gP_I\) and \(hP_J\) of the parabolic subgroups are opposite iff

\[ P_Ig^{-1}hP_J = P_Iw_0P_J. \]

Oppositeness map:

\[ \eta : \text{ind}^{G(q)}_{P_I}(k) \rightarrow \text{ind}^{G(q)}_{P_J}(k), \quad gP_I \mapsto \sum_{hP_J \subseteq gP_Iw_0P_J} hP_J \]

\(\text{Im } \eta\) is a simple module (follows from Carter and Lusztig (1976, PLMS))

The incidences we looked at above can be described in terms of oppositeness.

Oppositeness picks out a certain class of weights for further investigation. In the nontwisted case, the essential weights are those of the form \((p-1) \sum_{i \in J} \omega_i, J \subseteq S\).

Work is in progress.
Oppositeness

- Let \((\Delta(q), S)\) be the spherical Tits building of a finite group of Lie type.
- Two types \(I, J \subseteq S\) are opposite if \(I^{w_0} = J\).
- Assume \(I\) and \(J\) are opposite types. We say the cosets \(gP_I\) and \(hP_J\) of the parabolic subgroups are opposite iff \(P_I g^{-1} hP_J = P_I w_0 P_J\).
- Oppositeness map:
  \[\eta: \text{ind}_{P_I}^{G(q)}(k) \rightarrow \text{ind}_{P_J}^{G(q)}(k), \quad gP_I \mapsto \sum_{hP_J \subseteq gP_I w_0 P_J} hP_J\]
- \(\text{Im } \eta\) is a simple module (follows from Carter and Lusztig (1976, PLMS))
- The incidences we looked at above can be described in terms of oppositeness.
- Oppositeness picks out a certain class of weights for further investigation. In the nontwisted case, the essential weights are those of the form \((p - 1) \sum_{i \in J} \omega_i, J \subseteq S\).
- Work is in progress.
Oppositeness

Let \( (\Delta(q), S) \) be the spherical Tits building of a finite group of Lie type.

Two types \( I, J \subseteq S \) are opposite if \( I^{w_0} = J \).

Assume \( I \) and \( J \) are opposite types. We say the cosets \( gP_I \) and \( hP_J \) of the parabolic subgroups are opposite iff \( P_I g^{-1} hP_J = P_I w_0 P_J \).

Oppositeness map:

\[
\eta : \text{ind}^{G(q)}_{P_I}(k) \to \text{ind}^{G(q)}_{P_J}(k), \quad gP_I \mapsto \sum_{hP_J \subseteq gP_I w_0 P_J} hP_J
\]

\( \text{Im} \eta \) is a simple module (follows from Carter and Lusztig (1976, PLMS))

The incidences we looked at above can be described in terms of oppositeness.

Oppositeness picks out a certain class of weights for further investigation. In the nontwisted case, the essential weights are those of the form \( (p - 1) \sum_{i \in J} \omega_i \), \( J \subseteq S \).

Work is in progress.
Oppositeness

- Let \((\Delta(q), S)\) be the spherical Tits building of a finite group of Lie type.
- Two types \(I, J \subseteq S\) are opposite if \(I^{w_0} = J\).
- Assume \(I\) and \(J\) are opposite types. We say the cosets \(gP_I\) and \(hP_J\) of the parabolic subgroups are opposite iff 
  \(P_Ig^{-1}hP_J = P_Iw_0P_J\).
- Oppositeness map:
  \[
  \eta : \text{ind}_{P_I}^{G(q)}(k) \rightarrow \text{ind}_{P_J}^{G(q)}(k), \quad gP_I \mapsto \sum_{hP_J \subseteq gP_Iw_0P_J} hP_J
  \]
  \(\text{Im } \eta\) is a simple module (follows from Carter and Lusztig (1976, PLMS))
- The incidences we looked at above can be described in terms of oppositeness.
- Oppositeness picks out a certain class of weights for further investigation. In the nontwisted case, the essential weights are those of the form \((p - 1) \sum_{i \in J} \omega_i, J \subseteq S\).
- Work is in progress.
Let $(\Delta(q), S)$ be the spherical Tits building of a finite group of Lie type.

Two types $I, J \subseteq S$ are opposite if $I^{w_0} = J$.

Assume $I$ and $J$ are opposite types. We say the cosets $gP_I$ and $hP_J$ of the parabolic subgroups are opposite iff $P_Ig^{-1}hP_J = P_Iw_0P_J$.

Oppositeness map:

$$\eta : \text{ind}_{P_I}^{G(q)}(k) \rightarrow \text{ind}_{P_J}^{G(q)}(k), \quad gP_I \mapsto \sum_{hP_J \subseteq gP_Iw_0P_J} hP_J$$

$\text{Im} \eta$ is a simple module (follows from Carter and Lusztig (1976, PLMS))

The incidences we looked at above can be described in terms of oppositeness.

Oppositeness picks out a certain class of weights for further investigation. In the nontwisted case, the essential weights are those of the form $(p - 1) \sum_{i \in J} \omega_i, J \subset S$.

Work is in progress.
Let $(\Delta(q), S)$ be the spherical Tits building of a finite group of Lie type.

Two types $I, J \subseteq S$ are opposite if $I^{w_0} = J$.

Assume $I$ and $J$ are opposite types. We say the cosets $gP_I$ and $hP_J$ of the parabolic subgroups are opposite iff $P_I g^{-1} hP_J = P_I w_0 P_J$.

Oppositeness map:

$$\eta : \text{ind}_{P_I}^{G(q)}(k) \rightarrow \text{ind}_{P_J}^{G(q)}(k), \quad gP_I \mapsto \sum_{hP_J \subseteq gP_I w_0 P_J} hP_J$$

$\text{Im } \eta$ is a simple module (follows from Carter and Lusztig (1976, PLMS))

The incidences we looked at above can be described in terms of oppositeness.

Oppositeness picks out a certain class of weights for further investigation. In the nontwisted case, the essential weights are those of the form $(p - 1) \sum_{i \in J} \omega_i$, $J \subseteq S$.

Work is in progress.
Let \((\Delta(q), S)\) be the spherical Tits building of a finite group of Lie type.

Two types \(I, J \subseteq S\) are opposite if \(I^{w_0} = J\).

Assume \(I\) and \(J\) are opposite types. We say the cosets \(gP_I\) and \(hP_J\) of the parabolic subgroups are opposite iff \(P_ig^{-1}hP_J = P_Iw_0P_J\).

Oppositeness map:

\[
\eta : \text{ind}^{G(q)}_{P_I}(k) \rightarrow \text{ind}^{G(q)}_{P_J}(k), \quad gP_I \mapsto \sum_{hP_J \subseteq gP_Iw_0P_J} hP_J
\]

\(\text{Im} \eta\) is a simple module (follows from Carter and Lusztig (1976, PLMS))

The incidences we looked at above can be described in terms of oppositeness.

Oppositeness picks out a certain class of weights for further investigation. In the nontwisted case, the essential weights are those of the form \((p - 1) \sum_{i \in J} \omega_i, J \subseteq S\).
Oppositeness

- Let $\Delta(q), S$ be the spherical Tits building of a finite group of Lie type.
- Two types $I, J \subseteq S$ are opposite if $I^{w_0} = J$.
- Assume $I$ and $J$ are opposite types. We say the cosets $gP_I$ and $hP_J$ of the parabolic subgroups are opposite iff $P_I g^{-1} hP_J = P_I w_0 P_J$.
- Oppositeness map:
  \[ \eta : \text{ind}^{G(q)}_{P_I}(k) \to \text{ind}^{G(q)}_{P_J}(k), \quad gP_I \mapsto \sum_{hP_J \subseteq gP_I w_0 P_J} hP_J \]
- $\text{Im} \eta$ is a simple module (follows from Carter and Lusztig (1976, PLMS))
- The incidences we looked at above can be described in terms of oppositeness.
- Oppositeness picks out a certain class of weights for further investigation. In the nontwisted case, the essential weights are those of the form $(p - 1) \sum_{i \in J} \omega_i, J \subset S$.
- Work is in progress.
Outline

Introduction

Statement of results

Jantzen Sum Formula

Applications

Further Research

Conclusion
Thank you for your attention!