Some Weyl Modules for Clasical Groups

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Outline

Joint work with Ogul Arslan.

- Let G be a semisimple algebraic group in characteristic $p > 0$.
- An important class of modules are the Weyl modules $V(\lambda)$.
- The characters of Weyl modules are given by Weyl's Character Formula.
- But their precise submodule structure is not fully understood.
- This talk is about a uniform description of the submodule structure of some infinite families of Weyl modules for classical groups.

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Outline

Groups and weights considered

(B) G of type B_ℓ, ($\ell \geq 2$) $\lambda = r(\omega_1)$, $0 \leq r \leq p-1$;

(D) G of type D_ℓ, ($\ell \geq 3$) $\lambda = r(\omega_1)$, $0 \leq r \leq p-1$;

(A) G of type A_ℓ, ($\ell \geq 3$) $\lambda = r(\omega_1 + \omega_\ell)$, $0 \leq r \leq p-1$;

(A′) G of type A_ℓ, ($\ell \geq 4$) $\lambda = (\omega_2 + \omega_{\ell-1})$; and

(A″) G of type A_4, $\lambda = (p-2)(\omega_2 + \omega_{\ell-1})$ or $(p-1)(\omega_2 + \omega_{\ell-1})$.

For each of the weights considered we obtain:

- The character of the simple module $L(\lambda)$
- The characters (and multiplicities) of the simple composition factors of $V(\lambda)$
- The submodule lattice of $V(\lambda)$

Theorem

Let G be of type B_ℓ, $\ell \geq 2$. Let ω_1 be the highest weight of the standard orthogonal module of dimension $2\ell + 1$. Assume $0 \leq r \leq p - 1$. Then the following hold.

(a) *$H^0(r\omega_1)$ is simple unless (i) $p = 2$ and $r = 1$ or (ii) $p > 2$ and there exists a positive odd integer m such that*

$$r + 2\ell - 1 \leq mp \leq 2r + 2\ell - 2.$$

(b) *If (i) holds then the quotient $H^0(\omega_1)/L(\omega_1)$ is the one-dimensional trivial module.*

(c) *If (ii) holds then m is unique and*

$$H^0(r\omega_1)/L(r\omega_1) \cong H^0(r_1\omega_1),$$

where $r_1 = mp - 2\ell + 1 - r$. Furthermore the module $H^0(r_1\omega_1)$ is simple.

Theorem

Let G be of type D_ℓ, $\ell \geq 3$. Let ω_1 be the highest weight of the standard orthogonal module of dimension 2ℓ. Assume $0 \leq r \leq p-1$. Then the following hold.

(a) *Suppose that there exists a positive even integer m such that*

$$r + 2\ell - 2 \leq mp \leq 2r + 2\ell - 3.$$

Then m is unique and

$$H^0(r\omega_1)/L(r\omega_1) \cong H^0(r_1\omega_1),$$

where $r_1 = mp - 2\ell + 2 - r$. Furthermore the module $H^0(r_1\omega_1)$ is simple.

(b) *Otherwise, $H^0(r\omega_1)$ is simple.*

Theorem

Let G be of type A_ℓ, $\ell \geq 3$. Assume $0 \leq r \leq p-1$. Then the following hold.

(a) *Suppose that here exists a positive integer m such that*

$$r + \ell \leq mp \leq 2r + \ell - 1.$$

Then m is unique and

$$H^0(r(\omega_1 + \omega_\ell))/L(r(\omega_1 + \omega_\ell)) \cong H^0(r_1(\omega_1 + \omega_\ell)),$$

where $r_1 = mp - \ell - r$. Furthermore the module $H^0(r_1(\omega_1 + \omega_\ell))$ is simple.

(b) *Otherwise, $H^0(r(\omega_1 + \omega_\ell))$ is simple.*

Theorem

Let G be of type A_ℓ, $\ell \geq 4$. If $p > 2$ then the following hold.

(a) *If $\ell \equiv 0 \pmod{p}$ then $H^0(\omega_2 + \omega_{\ell-1})/L(\omega_2 + \omega_{\ell-1}) \cong k$.*

(b) *If $\ell \equiv 1 \pmod{p}$ then*

$$H^0(\omega_2 + \omega_{\ell-1})/L(\omega_2 + \omega_{\ell-1}) \cong H^0(\omega_1 + \omega_\ell)$$

and this module is simple.

(c) *In all other cases $H^0(\omega_2 + \omega_{\ell-1})$ is simple.*

If $p = 2$ then the following hold.

(d) If $\ell \equiv 0 \pmod 4$ then

$$H^0(\omega_2 + \omega_{\ell-1})/L(\omega_2 + \omega_{\ell-1}) \cong k.$$

(e) If $\ell \equiv 1 \pmod 4$ then

$$H^0(\omega_2 + \omega_{\ell-1})/L(\omega_2 + \omega_{\ell-1}) \cong V(\omega_1 + \omega_\ell).$$

(f) If $\ell \equiv 2 \pmod 4$ then $H^0(\omega_2 + \omega_{\ell-1})$ is simple.

(g) If $\ell \equiv 3 \pmod 4$ then

$$H^0(\omega_2 + \omega_{\ell-1})/L(\omega_2 + \omega_{\ell-1}) \cong L(\omega_1 + \omega_\ell).$$

Theorem

Let G be of type A_4.

(a) *If $p = 2$, then $H^0((p-1)(\omega_2+\omega_3))/L((p-1)(\omega_2+\omega_3)) \cong k$.*

If $p > 2$, then the following hold.

(b) $H^0((p-1)(\omega_2+\omega_3))/L((p-1)(\omega_2+\omega_3)) \cong L((p-2)(\omega_2+\omega_3))$.

(c) $H^0((p-2)(\omega_2+\omega_3))/L((p-2)(\omega_2+\omega_3)) \cong H^0((p-2)(\omega_1+\omega_4))$*, which is simple.*

Outline

$V(\lambda)$ has a descending filtration, of submodules $V(\lambda)^i$, $i > 0$, such that

$$V(\lambda)^1 = \mathrm{rad}\, V(\lambda), \quad \text{so} \quad V(\lambda)/V(\lambda)^1 \cong L(\lambda).$$

and

$$\sum_{i>0} \mathrm{Ch}(V(\lambda)^i) = -\sum_{\alpha>0} \sum_{\{m: 0<mp<\langle\lambda+\rho,\alpha^\vee\rangle\}} v_p(mp)\chi(\lambda - mp\alpha)$$

The RHS of the Sum Formula can be computed by the following algorithm: For each positive root α,

(i) Compute $\langle \lambda + \rho, \alpha^\vee \rangle$

(ii) Compute $\lambda + \rho - mp\alpha$ for $0 < m < \langle \lambda + \rho, \alpha^\vee \rangle$

(iii) Find the Weyl group conjugate $w(\lambda + \rho - mp\alpha)$ in X_+ and note the sign of a Weyl group element w.

(iv) Compute $w(\lambda + \rho - mp\alpha) - \rho$.

(v) The contribution to the sum is $-\operatorname{sign}(w) v_p(mp) \chi(w(\lambda + \rho - mp\alpha) - \rho)$.

Keeping control

The main challenge lies in trying to do infinitely many Sum Formula computations at once. For fixed type the parameters of the problem are p, r and the rank ℓ.

Lemma

(a) *If R is of type B_ℓ or D_ℓ and $\lambda + \rho - mp\alpha$ has two coordinates with the same absolute value then the pair (α, m) contributes nothing to the final sum.*

(b) *If R is of type A_ℓ and $\lambda + \rho - mp\alpha$ has two equal coordinates, then the pair (α, m) contributes nothing to the final sum.*

$\square$

Eliminating multiplicities

- The Sum Formula *overestimates* the character of $\operatorname{rad} V(\lambda)$.
- Example. Type D_ℓ.
- $S^r(V^*)$ has a *good filtration* (Andersen-Jantzen). with subquotients of the form $H^0(s\omega_1)$, $s < r$.
- Then for $r_1 < r$,

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\begin{aligned}
\dim \operatorname{Hom}_G(V(r_1\omega_1), V(r\omega_1)) &= \dim \operatorname{Hom}_G(H^0(r\omega_1), H^0(r_1\omega_1)) \\
&\leq \dim \operatorname{Hom}_G(S^r(V^*), H^0(r_1\omega_1)) \\
&= \dim \operatorname{Hom}_G(V(r_1\omega_1), S^r(V^*)) \\
&\text{(by self-duality of } S^r(V^*)) \\
&= \text{multiplicity of } H^0(r_1\omega_1) \\
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Outline

Incidence of points and polar hyperplanes

- V vector space over $\mathbb{F}_q$ with nonsingular form $b(-,-)$.
- b may be alternating or symmetric or hermitian.
- $\widehat{P} = \{$all 1-dimensional subspaces of $V\}$
  $\supseteq P = \{$singular 1-dimensional subspaces$\}$,
- $\widehat{P}^* = \{$hyperplanes of $V\} \supseteq P^* = \{p^\perp \mid p \in P\}$, polar hyperplanes.
- $G(q) =$ group of linear transformations preserving $b(-,-)$.
- $A =$ incidence matrix of $(\widehat{P}^*, \widehat{P})$

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$$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

p-ranks

- We consider the p-ranks, where $q = p^t$.
- The p-rank of A is well known (Goethals-Delsarte, MacWilliams-Mann, Smith), and the p-rank of A_1 was found by Blokhuis and Moorhouse.
- Moorhouse (Linz, 2006): What is the p-rank of A_{11}?

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Permutation module structure

- (a) $G(q)$ acts on P with permutation rank 3
- (b) $k[P] \cong k.\mathbf{1} \oplus Y$,
- (c) $\mathrm{head}(Y) \cong \mathrm{soc}(Y)$.
- (a),(b),(c) $\implies \mathrm{head}(Y)$ is a simple $kG(q)$-module. Call it L.
- P and P^* are isomorphic $G(q)$-sets, so the incidence map induces

$$\phi \in \mathrm{End}_{kG(q)}(k[P]), \quad \phi(p) = \sum_{p' \in p^\perp} p'.$$

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- Outcome: $\mathrm{rank}_p A_{11} = 1 + \dim L$.

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- Outcome: $\mathrm{rank}_p A_{11} = 1 + \dim L$.

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- (a) $G(q)$ acts on P with permutation rank 3
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Identifying the simple module L

- $k[P] = \mathrm{ind}_{G(q)_x}^{G(q)}(k)$, $x \in P$, so Frobenius Reciprocity implies that $G(q)_x$ has a fixed point on L.
- The fixed point condition characterizes L:

$$L \cong L((q-1)\omega),$$

where $\omega = \omega_1$ in the orthogonal and symplectic cases, and $\omega_1 + \omega_\ell$ in the unitary case.
- By Steinberg's Tensor Product Theorem,

$$L((q-1)\omega) = L((p-1)\omega) \otimes L((p-1)\omega)^{(p)} \cdots \otimes L((p-1)\omega)^{(p^{t-1})}$$

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Outline

Oppositeness

- Let $(\Delta(q), S)$ be the spherical Tits building of a finite group of Lie type.
- Two types $I, J \subseteq S$ are *opposite* if $I^{w_0} = J$.
- Assume I and J are opposite types. We say the cosets gP_I and hP_J of the parabolic subgroups are opposite iff $P_I g^{-1} h P_J = P_I w_0 P_J$.
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- $\operatorname{Im} \eta$ is a simple module (follows from Carter and Lusztig (1976, PLMS))
- The incidences we looked at above can be described in terms of oppositeness.
- Oppositeness picks out a certain class of weights for further investigation. In the nontwisted case, the essential weights are those of the form $(p-1)\sum_{i \in J} \omega_i$, $J \subset S$.
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