Critical Groups of Rank 3 graphs

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Contents

Collaborators

The coauthors for various parts of this talk are: Andries Brouwer, David Chandler, Josh Ducey, Ian Hill, Venkata Raghu Tej Pantangi and Qing Xiang.

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For each prime p, can find $S(A)_p$ by working over a p-local ring. Then the d_i are powers of p called the p-elementary divisors.

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Origins and early work on $K(\Gamma)$ include: Sandpile model (Dhar 1990), Chip-firing game (Biggs), Cycle Matroids (Vince 1991), arithmetic graphs (Lorenzini, 1991).

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Perhaps graphs with lots of automorphisms can be approached using group theory, representation theory.

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  Primitive rank 3 permutation groups are known as consequence of CFSG: Bannai (1971), Cameron (1981), Kantor-Liebler (1982), Liebeck-Saxl (1986), Liebeck (1987).

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They are strongly regular graphs.

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Van Lint-Schrijver cyclotomic SRGs (Pantangi, 2018)

Some open cases to consider

- There are many families of rank 3 graphs where $K(\Gamma)$ is not yet known, e.g. associated with primitive actions of the groups: $E_6(q)$, $O_{10}^{+}(q)$ (action on one orbit of t.i. 5-spaces); $U_5(q)$ (action on t.i. lines); $O_{2m}^{\pm}(p)$, $p = 2$ or 3 and $O_{2m+1}(3)$ (action on nonisotropic points); wreathed cases.

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- Imprimitive rank 3 examples
- SRGs in general.

Let $G \leq \mathrm{Aut}(\Gamma)$. Then $\mathbb{Z}^V$ is a permutation module and $L(\Gamma)$ defines a $\mathbb{Z}G$-module homomorphism with cokernel $S(L(\Gamma))$, so $K(\Gamma)$ is a $\mathbb{Z}G$-module.

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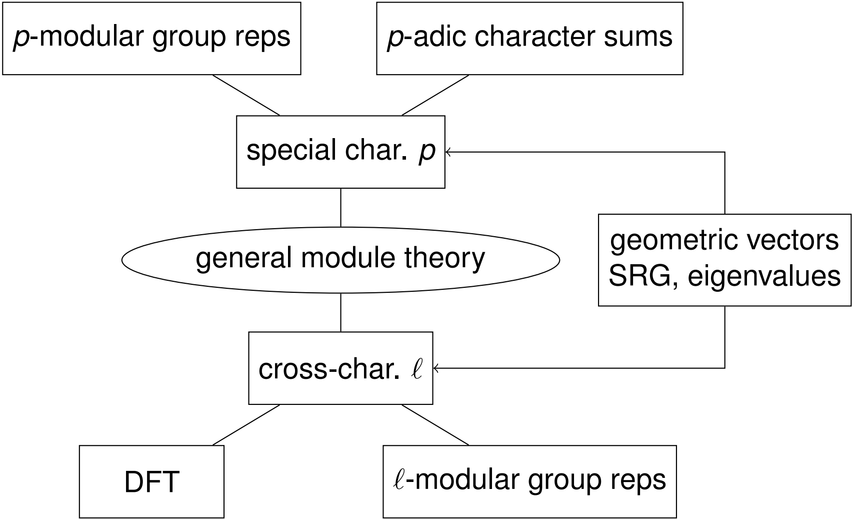
- Often there is natural characteristic prime p that has to be treated differently.


p-modular group reps
p-adic character sums
special char. p
general module theory
geometric vectors
SRG, eigenvalues
cross-char. ℓ
DFT
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Paley graphs (Chandler-S-Xiang 2015)

Uses: DFT ($\mathbb{F}_q$-action) to get the p'-part, $\mathbb{F}_q^*$-action Jacobi sums and Transfer matrix method for p-part. The following gives the p-part of $K(\Gamma)$.

Theorem

Let $q = p^t$ be a prime power congruent to 1 modulo 4. Then the number of p-adic elementary divisors of $L(\mathrm{Paley}(q))$ which are equal to p^λ, $0 \le \lambda < t$, is

$$f(t,\lambda) = \sum_{i=0}^{\min\{\lambda, t-\lambda\}} \frac{t}{t-i}\binom{t-i}{i}\binom{t-2i}{\lambda-i}(-p)^i \left(\frac{p+1}{2}\right)^{t-2i}.$$

The number of p-adic elementary divisors of $L(\mathrm{Paley}(q))$ which are equal to p^t is $\left(\frac{p+1}{2}\right)^t - 2$.

$K(\Gamma)$ examples for Paley graphs

$$K(\mathrm{Paley}(5^3)) \cong (\mathbb{Z}/31\mathbb{Z})^{62} \oplus (\mathbb{Z}/5\mathbb{Z})^{36} \oplus (\mathbb{Z}/25\mathbb{Z})^{36} \oplus (\mathbb{Z}/125\mathbb{Z})^{25}.$$

$$\begin{aligned} K(\mathrm{Paley}(5^4)) \cong (\mathbb{Z}/156\mathbb{Z})^{312} \oplus (\mathbb{Z}/5\mathbb{Z})^{144} \oplus (\mathbb{Z}/25\mathbb{Z})^{176} \\ \oplus (\mathbb{Z}/125\mathbb{Z})^{144} \oplus (\mathbb{Z}/625\mathbb{Z})^{79}. \end{aligned}$$

Grassmann graph or Skew lines graph, (Ducey-S 2017)

p'-part of $K(\Gamma)$: Structure of $\mathbb{F}_\ell GL(n,q)$-permutation modules (G. James) depends on relation of ℓ to n.

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Examples:

	$\ell \mid \begin{bmatrix} n \\ 1 \end{bmatrix}_q$, $\ell \mid \begin{bmatrix} n-2 \\ 1 \end{bmatrix}_q$, $\ell \mid q+1$	
	$\ell \nmid \begin{bmatrix} n \\ 2 \end{bmatrix}_q$	$\ell \mid \begin{bmatrix} n \\ 2 \end{bmatrix}_q$
$\ell \nmid \lfloor \frac{n-1}{2} \rfloor$	$\mathbb{F}_\ell{}^V = \mathbb{F}_\ell \oplus \begin{array}{ccccc} & & D_1 & & \\ & \diagup & & \diagdown & \\ D_2 & & & & \mathbb{F}_\ell \\ & \diagdown & & \diagup & \\ & & D_1 & & \end{array}$	$\mathbb{F}_\ell{}^V = \begin{array}{ccccc} & & D_1 & & \\ & \diagup & \mid & & \\ \mathbb{F}_\ell & & D_2 & & \mathbb{F}_\ell \\ & & \mid & \diagup & \\ & & D_1 & & \end{array}$
$\ell \mid \lfloor \frac{n-1}{2} \rfloor$	$\mathbb{F}_\ell{}^V = \mathbb{F}_\ell \oplus \begin{array}{c} D_1 \\ \mathbb{F}_\ell \\ D_2 \\ \mathbb{F}_\ell \\ D_1 \end{array}$	

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  Much of the difficulty was handled in $n = 4$ case (Brouwer-Ducey-S 2012).

Example: $K(\Gamma)$ for Skew lines in $\mathrm{PG}(3,9)$

$$\begin{aligned}K(\Gamma) \cong (\mathbb{Z}/8\mathbb{Z})^{5824} &\times (\mathbb{Z}/16\mathbb{Z})^{818} \\ &\times (\mathbb{Z}/7\mathbb{Z})^{6641} \times (\mathbb{Z}/13\mathbb{Z})^{6641} \times (\mathbb{Z}/41\mathbb{Z})^{818} \\ \times (\mathbb{Z}/3\mathbb{Z})^{256} &\times (\mathbb{Z}/9\mathbb{Z})^{6025} \times (\mathbb{Z}/81\mathbb{Z})^{202} \times (\mathbb{Z}/243\mathbb{Z})^{256} \\ &\times (\mathbb{Z}/729\mathbb{Z})^{361} \times (\mathbb{Z}/6561\mathbb{Z})\end{aligned}$$

Polar graphs

(Pantangi-S 2017) Uses structure of cross characteristic permutation modules (S-Tiep, 2005).
The p-part of $K(\Gamma)$ is trivial.

$K(\Gamma)$ for polar graph on $2m$-dimensional symplectic space

$(f,g) := \left(\frac{q(q^m-1)(q^{m-1}+1)}{2(q-1)}, \frac{q(q^m+1)(q^{m-1}-1)}{2(q-1)} \right)$		
$(a,b,c,d) := \left(v_\ell(\left[{m-1 \atop 1}\right]_q), v_\ell(\left[{m \atop 1}\right]_q), v_\ell(q^m+1), v_\ell(q^{m-1}+1) \right)$		
Prime	conditions	multiplicities
$\ell = 2$	m even, q odd	$e_0 = g+1$, $e_1 = f-g-1$, $e_{d+1} = 1$, and $e_{d+b+1} = g-1$.
	m odd, q odd	$e_0 = g$, $e_a = 1$, $e_{a+c} = f-g-1$, and $e_{a+c+1} = g$.
$\ell \neq 2$	$b = d = 0$	$e_0 = g + \delta_{a,0}$, $e_a = \delta_{c,0}(f-1) + 1 + \delta_{a,0}(g)$, and $e_{a+c} = f - 1 + \delta_{c,0}$.
	$a = c = 0$	$e_0 = f + \delta_{d,0}$, $e_d = \delta_{b,0}(g) + 1 + \delta_{d,0}(f)$, and $e_{b+d} = g - 1 + \delta_{b,0}$

Example: $q = 9$, $m = 3$

Γ is an $\mathrm{SRG}(66430, 7380, 818, 820)$. Eigenvalues $(7380, 80, -82)$ with multiplicities $(1, 33579, 32850)$.

$$K(\Gamma) \cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/4\mathbb{Z})^{728} \times (\mathbb{Z}/8\mathbb{Z})^{32851} \times (\mathbb{Z}/41\mathbb{Z}) \times (\mathbb{Z}/91\mathbb{Z})^{32580} \times (\mathbb{Z}/25\mathbb{Z})^{33578} \times (\mathbb{Z}/5\mathbb{Z}) \times (\mathbb{Z}/73\mathbb{Z})^{33579}$$

Thank you for your attention!

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