Critical Groups of Rank 3 graphs

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The coauthors for various parts of this talk are: Andries Brouwer, David Chandler, Josh Ducey, Ian Hill, Venkata Raghu Tej Pantangi and Qing Xiang.
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Illustrative examples
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The cyclic decomposition of $S(A)$ is given by the Smith Normal Form of $A$: There exist unimodular $P, Q$ such that $D = PAQ$ has nonzero entries $d_1, \ldots, d_r$ only on the leading diagonal, and $d_i$ divides $d_{i+1}$.
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Other diagonal forms also describe $S(A)$. Generalizes from $\mathbb{Z}$ to principal ideal domains. For each prime $p$, can find $S(A)_p$ by working over a $p$-local ring. Then the $d_i$ are powers of $p$ called the $p$-elementary divisors.
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Definition and history

$A(\Gamma)$, an adjacency matrix of a (connected) graph $\Gamma = (V, E)$. 

$L(\Gamma) = D(\Gamma) - A(\Gamma)$, Laplacian matrix.

$K(\Gamma) = \text{Tor}(S(L(\Gamma)))$ is called the critical group of $\Gamma$.

$|K(\Gamma)| =$ number of spanning trees (Kirchhoff's Matrix-tree Theorem).

Origins and early work on $K(\Gamma)$ include: Sandpile model (Dhar 1990), Chip-firing game (Biggs), Cycle Matroids (Vince 1991), arithmetic graphs (Lorenzini, 1991).
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General problem

Compute the critical group for some graphs (families of graphs).
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Compute the critical group for some graphs (families of graphs).
Perhaps graphs with lots of automorphisms can be approached using group theory, representation theory.
Smith normal form

The critical group of a graph

**Rank 3 Graphs**

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Methods

Illustrative examples
Rank 3 group actions

Definition

The action of a group $G$ on a set $X$ is said to have rank 3 if it is transitive and a point stabilizer has exactly three orbits. Equivalently, $G$ has 3 orbits on $X \times X$.

$S_n$ acts on $\{1,\ldots,n\} := \{1,\ldots,n\}$ ($n \geq 4$). The induced action on unordered pairs has rank 3.

$\text{PGL}(n+1,q)$ acts on projective space $\text{PG}(n,q)$ ($n \geq 3$). Consider the induced action on lines.

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(G, X) rank 3 group of even order, orbits Δ, Φ, ψ on \( X \times X \). The graphs \((X, \Phi)\) and \((X, \Psi)\) are the associated rank 3 graphs.
Rank 3 graphs

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They are strongly regular graphs.
Smith normal form

The critical group of a graph

Rank 3 Graphs

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Illustrative examples
$K(\Gamma)$ for some families of rank 3 graphs

Paley graphs (Chandler-Xiang-S, (2015))
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Classical polar graphs (Pantangi-S, (2017))
Van Lint-Schrijver cyclotomic SRGs (Pantangi, 2018)
Some open cases to consider

There are many families of rank 3 graphs where $K(\Gamma)$ is not yet known, e.g. associated with primitive actions of the groups: $E_6(q)$, $O_{10}^+(q)$ (action on one orbit of t.i. 5-spaces); $U_5(q)$ (action on t.i. lines); $O_{2m}^\pm(p)$, $p = 2$ or $3$ and $O_{2m+1}(3)$ (action on nonisotropic points); wreathed cases.
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Some open cases to consider

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- Imprimitive rank 3 examples
- SRGs in general.
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Let $G \leq \text{Aut}(\Gamma)$. Then $\mathbb{Z}^\vee$ is a permutation module and $L(\Gamma)$ defines a $\mathbb{Z}G$-module homomorphism with cokernel $S(L(\Gamma))$, so $K(\Gamma)$ is a $\mathbb{Z}G$-module.
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- For each prime $\ell$, there is a canonical filtration of $\mathbb{F}_\ell^V$ by $\mathbb{F}_\ell G$-submodules, whose $i$-th subquotient has dimension equals the multiplicity of $\ell^i$ as an elementary divisor.
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- For each prime $\ell$, there is a canonical filtration of $\mathbb{F}_\ell^V$ by $\mathbb{F}_\ell G$-submodules, whose $i$-th subquotient has dimension equals the multiplicity of $\ell^i$ as an elementary divisor.
- Often there is natural characteristic prime $p$ that has to be treated differently.
$p$-modular group reps

$p$-adic character sums

special char. $p$

general module theory

cross-char. $\ell$

DFT

$\ell$-modular group reps

geometric vectors

SRG, eigenvalues
Smith normal form

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Illustrative examples
Paley graphs (Chandler-S-Xiang 2015)

Uses: DFT ($\mathbb{F}_q$-action) to get the $p'$-part, $\mathbb{F}_q^*$-action Jacobi sums and Transfer matrix method for $p$-part. The following gives the $p$-part of $K(\Gamma)$.

**Theorem**

Let $q = p^t$ be a prime power congruent to 1 modulo 4. Then the number of $p$-adic elementary divisors of $L(\text{Paley}(q))$ which are equal to $p^\lambda$, $0 \leq \lambda < t$, is

$$ f(t, \lambda) = \sum_{i=0}^{\min\{\lambda, t-\lambda\}} \frac{t}{t-i} \binom{t-i}{i} \binom{t-2i}{\lambda-i} (-p)^i \left( \frac{p+1}{2} \right)^{t-2i} $$

The number of $p$-adic elementary divisors of $L(\text{Paley}(q))$ which are equal to $p^t$ is $\left( \frac{p+1}{2} \right)^t - 2$. 
$K(\Gamma)$ examples for Paley graphs

$$K(\text{Paley}(5^3)) \cong (\mathbb{Z}/31\mathbb{Z})^{62} \oplus (\mathbb{Z}/5\mathbb{Z})^{36} \oplus (\mathbb{Z}/25\mathbb{Z})^{36} \oplus (\mathbb{Z}/125\mathbb{Z})^{25}.$$  

$$K(\text{Paley}(5^4)) \cong (\mathbb{Z}/156\mathbb{Z})^{312} \oplus (\mathbb{Z}/5\mathbb{Z})^{144} \oplus (\mathbb{Z}/25\mathbb{Z})^{176} \oplus (\mathbb{Z}/125\mathbb{Z})^{144} \oplus (\mathbb{Z}/625\mathbb{Z})^{79}.$$
Grassmann graph or Skew lines graph, (Ducey-S 2017)

$p'$-part of $K(\Gamma)$: Structure of $\mathbb{F}_\ell GL(n, q)$-permutation modules (G. James) depends on relation of $\ell$ to $n$. 

$D_1^F \ell V = \mathbb{F}_\ell \oplus D_2^F \ell$ 

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Examples:

<table>
<thead>
<tr>
<th>( \ell \mid \left\lfloor \frac{n-1}{2} \right\rfloor )</th>
<th>( \ell \mid \left\lfloor \frac{n-2}{1} \right\rfloor ), ( \ell \mid q+1 )</th>
</tr>
</thead>
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<tr>
<td>( \ell \nmid \left\lfloor \frac{n-2}{1} \right\rfloor )</td>
<td>( \mathbb{F}<em>\ell V = \mathbb{F}</em>\ell \oplus D_1 ) ( \mathbb{F}_\ell ) ( D_2 ) ( D_1 )</td>
</tr>
</tbody>
</table>
For the Grassmann graph, $|K(\Gamma)|$ is not divisible by $p$. 

Structure of $F_{q^t}GL(n, q)$-permutation module on points (Bardoe-S 2000) $p$-elementary divisors of $pt$-subspace inclusion matrices (Chandler-S-Xiang 2006). Subspace character sums (D. Wan) Much of the difficulty was handled in $n=4$ case (Brouwer-Ducey-S 2012).
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For skew lines graph, $K(\Gamma)$ has a large $p$-part.
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Much of the difficulty was handled in $n = 4$ case (Brouwer-Ducey-S 2012).
Example: $K(\Gamma)$ for Skew lines in $\text{PG}(3,9)$

\[ K(\Gamma) \cong (\mathbb{Z}/8\mathbb{Z})^{5824} \times (\mathbb{Z}/16\mathbb{Z})^{818} \]
\[ \times (\mathbb{Z}/7\mathbb{Z})^{6641} \times (\mathbb{Z}/13\mathbb{Z})^{6641} \times (\mathbb{Z}/41\mathbb{Z})^{818} \]
\[ \times (\mathbb{Z}/3\mathbb{Z})^{256} \times (\mathbb{Z}/9\mathbb{Z})^{6025} \times (\mathbb{Z}/81\mathbb{Z})^{202} \times (\mathbb{Z}/243\mathbb{Z})^{256} \]
\[ \times (\mathbb{Z}/729\mathbb{Z})^{361} \times (\mathbb{Z}/6561\mathbb{Z}) \]
(Pantangi-S 2017) Uses structure of cross characteristic permutation modules (S-Tiep, 2005). The $p$-part of $K(\Gamma)$ is trivial.
$K(\Gamma)$ for polar graph on $2m$-dimensional symplectic space

$$(f, g) := \left( \frac{q(q^m-1)(q^{m-1}+1)}{2(q-1)}, \frac{q(q^m+1)(q^{m-1}-1)}{2(q-1)} \right)$$

$$(a, b, c, d) := \left( \nu_\ell\left( \left[ \begin{array}{c} m-1 \\ 1 \end{array} \right]_q \right), \nu_\ell\left( \left[ \begin{array}{c} m \\ 1 \end{array} \right]_q \right), \nu_\ell(q^m + 1), \nu_\ell(q^{m-1} + 1) \right)$$

<table>
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<tr>
<th>Prime</th>
<th>conditions</th>
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</tr>
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<tbody>
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<td>$\ell = 2$</td>
<td>$m$ even, $q$ odd</td>
<td>$e_0 = g + 1$, $e_1 = f - g - 1$, $e_{d+1} = 1$, and $e_{d+b+1} = g - 1$.</td>
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<tr>
<td></td>
<td>$m$ odd, $q$ odd</td>
<td>$e_0 = g$, $e_a = 1$, $e_{a+c} = f - g - 1$, and $e_{a+c+1} = g$.</td>
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<td>$\ell \neq 2$</td>
<td>$b = d = 0$</td>
<td>$e_0 = g + \delta_{a,0}$, $e_a = \delta_{c,0}(f - 1) + 1 + \delta_{a,0}(g)$, and $e_{a+c} = f - 1 + \delta_{c,0}$.</td>
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<td>$a = c = 0$</td>
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Example: $q = 9, \ m = 3$

$\Gamma$ is an SRG$(66430, 7380, 818, 820)$. Eigenvalues $(7380, 80, -82)$ with multiplicities $(1, 33579, 32850)$.

$$K(\Gamma) \cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/4\mathbb{Z})^{728} \times (\mathbb{Z}/8\mathbb{Z})^{32851} \times (\mathbb{Z}/41\mathbb{Z})$$
$$\times (\mathbb{Z}/91\mathbb{Z})^{32580} \times (\mathbb{Z}/25\mathbb{Z})^{33578} \times (\mathbb{Z}/5\mathbb{Z}) \times (\mathbb{Z}/73\mathbb{Z})^{33579}$$
Thank you for your attention!
References


