On the dimensions of LDPC codes defined by equations over finite fields

Peter Sin

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Overview

- LDPC (low density parity check) codes have attracted much attention recently, due to their good performance in theory and practice.
- A main division is into random and structured types.
- One structured family, constructed using certain bipartite graphs was studied by: J.-L. Kim, U. Peled, I. Perepelitsa, V. Pless, and S. Friedland (2004)
- They conjectured the dimensions of the codes.
- We’ll describe the conjecture and its proof (with Q. Xiang).
- The proof involves the geometry of generalized quadrangles and the representation theory of $\text{Sp}(4, q)$. 
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The codes $LU(3, q)$

- $q$, any prime power
- $P^*, L^*$ be two sets in bijection with $\mathbb{F}_q^3$
- $(a, b, c) \in P^*$ is incident with $[x, y, z] \in L^*$ if and only if
  \[ y = ax + b \quad \text{and} \quad z = ay + c. \]  (1)
- The binary incidence matrix $M_2(P^*, L^*)$ and its transpose can be taken as parity check matrices of two codes.
- These codes are designated $LU(3, q)$. We have:
  \[ \dim LU(3, q) = q^3 - \text{rank} M_2(P^*, L^*). \]
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Conjecture: If $q$ is odd, the dimension of $\text{LU}(3, q)$ is 
$$(q^3 - 2q^2 + 3q - 2)/2.$$ 
This number was known to be a lower bound when $q$ is an odd prime.
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The symplectic generalized quadrangle

- $q$, any prime power
- $(V, \langle \cdot, \cdot \rangle)$, a 4-dimensional $F_q$-vector space with a nonsingular alternating bilinear form
- $e_0, e_1, e_2, e_3$, a symplectic basis such that $(e_0, e_3) = (e_1, e_2) = 1$
- $x_0, x_1, x_2, x_3$, coordinates for basis
- $P = P(V)$, the set of points of the projective space of $V$
- $L$, the set of totally isotropic 2-dimensional subspaces of $V$, considered as lines in $P$
- $(P, L)$ is called the *symplectic generalized quadrangle*. 
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Quadrangle property

Given any line and any point not on the line, there is a unique line which passes though the given point and meets the given line.
\( p_0 = \langle e_0 \rangle \) and \( \ell_0 = \langle e_0, e_1 \rangle \).

\( p^\perp \), the set of points on lines through the point \( p \)

\( P_1 = P \setminus p_0^\perp \)

\( L_1 \), the set of lines in \( L \) which do not meet \( \ell_0 \)

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- We have new incidence systems $(P_1, L_1), (P, L_1), (P_1, L)$. 
We will see below that \((P_1, L_1)\) is equivalent to the system \((P^*, L^*)\).

So we want to prove:

**Theorem**
Assume \(q\) is odd. The rank of \(M_2(P_1, L_1)\) equals \((q^3 + 2q^2 - 3q + 2)/2\).

A known result is:

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(Bagchi-Brouwer-Wilbrink) Assume \(q\) is a power of an odd prime. Then the rank of \(M_2(P, L)\) is \((q^3 + 2q^2 + q + 2)/2\).

Note that the difference in ranks is \(2q\).
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Note that the difference in ranks is \(2q\).
Next, see $(P_1, L_1) \cong (P^*, L^*)$, for $q$ any prime power.
Coordinates of $P_1$

- $x_0, x_1, x_2, x_3$ be homogeneous coordinates of $P$
- $p_0 = \langle e_0 \rangle$
- $P_1 = \{(x_0 : x_1 : x_2 : x_3) \mid x_3 \neq 0\}$
  $= \{(a : b : c : 1) \mid a, b, c \in F_q\} \cong F_q^3$. (2)
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Coordinates of lines in $P(V)$

- $e_i \wedge e_j, \ 0 \leq i < j \leq 3$, basis of the exterior square $\wedge^2(V)$
- $p_{01}, p_{02}, p_{03}, p_{12}, p_{13}, p_{23}$, homogeneous coordinates for $P(\wedge^2(V))$
- If $W$ is a 2-dimensional subspace of $V$ then $\wedge^2(W) \in P(\wedge^2(V))$.
- If $W = \langle (a_0 : a_1 : a_2 : a_3), (b_0 : b_1 : b_2 : b_3) \rangle$ then $\wedge^2(W)$ has coordinates $p_{ij} = a_i b_j - a_j b_i$, its Grassmann-Plücker coordinates.
- The totality of points of $P(\wedge^2(V))$ obtained from all $W$ forms the set with equation $p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 0$, called the Klein Quadric.
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Coordinates of $L$ and $L_1$

- $L$ corresponds to the subset of points of the Klein quadric which satisfy the additional linear equation $p_{03} = -p_{12}$.
- $\ell_0 = \langle (1 : 0 : 0 : 0), (0 : 1 : 0 : 0) \rangle$
- $L_1$ is the subset of $L$ given by $p_{23} \neq 0$.
- The quadratic relation yields

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L_1 \cong \{(z^2 + xy : x : z : -z : y : 1) \mid x, y, z \in \mathbf{F}_q\} 
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Incidence equations

- When is \((a : b : c : 1) \in P_1\) on \((z^2 + xy : x : z : -z : y : 1) \in L_1\)?

- If the line is spanned by points with homogeneous coordinates \((a_0 : a_1 : a_2 : a_3)\) and \((b_0 : b_1 : b_2 : b_3)\). The given point and line are incident if and only if all \(3 \times 3\) minors of the matrix

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Hence \((P_1, L_1)\) and \((P^*, L^*)\) are equivalent.
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Hence \((P_1, L_1)\) and \((P^*, L^*)\) are equivalent.
Relative dimensions and a bound

$q$ is any prime power.

- $\mathbb{F}_2[P]$, the vector space of all $\mathbb{F}_2$-valued functions on $P$
- Abuse notation slightly, identify points and lines with their characteristic functions in $\mathbb{F}_2[P]$.
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Lemma

$Z \cup X_0 \cup Y$ is linearly independent over $F_2$.

Corollary

$$\dim_{F_2} LU(3, q) \geq q^3 - \dim_{F_2} C(P, L) + 2q.$$  (6)
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Proof of Theorem 1

Assume that $q$ is odd. By Corollary 4 the proof of Theorem 1 will be completed if we can show that $Z \cup X_0 \cup Y$ spans $C(P, L)$ as a vector space over $F_2$. 
Lemma

Let $\ell \in L$. Then the sum of all lines which meet $\ell$ (excluding $\ell$ itself) is the constant function 1.

Proof.
The function given by the sum takes the value $q \equiv 1$ at any point of $\ell$ and value 1 at any point off $\ell$, by the quadrangle property.
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Similarly:

Lemma
Let $\ell \neq \ell_0$ be a line which meets $\ell_0$ at a point $p$. Let $\Phi_\ell$ be the sum of all lines in $L_1$ which meet $\ell$. Then

$$\Phi_\ell(p') = \begin{cases} 
0, & \text{if } p' = p; \\
q, & \text{if } p' \in \ell \setminus \{p\}; \\
0, & \text{if } p' \in p^\perp \setminus \ell; \\
1, & \text{if } p' \in P \setminus p^\perp. 
\end{cases} \quad (7)$$

Corollary
Let $p \in \ell_0$ and let $\ell$, $\ell'$ be two lines through $p$, neither equal to $\ell_0$. Then $\ell - \ell' \in C(P, L_1)$. 
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Lemma
\[ \ker \pi_{p_1} \cap C(P, L) \text{ has dimension } q + 1, \text{ with basis } X. \]

Proof:

- Let \( G_{p_0} \) be the stabilizer in \( \text{Sp}(V) \) of \( p_0 \).

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\ker \pi_{p_1} = F_2[p_0^\perp] = F_2[\{p_0\}] \oplus F_2[p_0^\perp \setminus \{p_0\}] \quad (8)
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as an \( F_2 G_{p_0} \)-module. Clearly \( F_2[\{p_0\}] \) is a one-dimensional trivial \( F_2 G_{p_0} \)-module.
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We consider the following subgroups of $G_{p_0}$.

$$Q = \left\{ \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in F_q \right\}, \quad Z(Q) = \left\{ \begin{pmatrix} 1 & 0 & 0 & c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid c \in F_q \right\}$$

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$Q \triangleleft G_{p_0}$, $Q/Z(Q)$ is elementary abelian of order $q^2$ and $Z(Q)$ acts trivially on $p_0^\perp$.

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Since $Q$ has odd order, it acts semisimply on $\mathbf{F}_2[p_0^\perp]$ and we can compute the decomposition.
Applying Clifford’s Theorem, we have a $\mathbb{F}_2 G_{p_0}$-module decomposition
\[ \mathbb{F}_2[p_{0}^\perp] = T \oplus W, \tag{10} \]
where $T$ is the $q + 2$-dimensional space of $Q$-fixed points and $W$ is simple of dimension $q^2 - 1$.

The intersection
\[ \ker \pi_{P_1} \cap C(P, L) = \mathbb{F}_2[p_{0}^\perp] \cap C(P, L), \tag{11} \]
is an $\mathbb{F}_2 G_{p_0}$-submodule of $\mathbb{F}_2[p_{0}^\perp]$.

The $q + 1$ lines through $p_0$ lie in the intersection, accounting for $q + 1$ dimensions of $T$.

We must argue that the intersection is no bigger than their span. If it were, then by (10), $\mathbb{F}_2[p_{0}^\perp] \cap C(P, L)$ must contain either $W$ or all the $Q$-fixed points on $\mathbb{F}_2[p_{0}^\perp]$.

Both possibilities lead immediately to contradictions.
Applying Clifford’s Theorem, we have a $\mathbb{F}_2 G_{\rho_0}$-module decomposition

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$\ker \pi_{P_1} \cap C(P, L_1)$ has dimension $q - 1$, and basis the set of functions $\ell - \ell'$, where $\ell \neq \ell_0$ is an arbitrary but fixed line through $p_0$ and $\ell'$ varies over the $q - 1$ lines through $p_0$ different from $\ell_0$ and $\ell$. 
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- By Lemma 9, the span of $X_0$ and $Z$ is equal to the span of $X_0$ and $L_1$, since $\ker \pi_{P_1} \cap C(P, L_1)$ is contained in the span of $X_0$.
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Further research

- Consider the binary code $LU(3, q)$ when $q = 2^t$, $t \geq 1$.
- Corollary 4 provides a lower bound for the dimension.
- Note, however, that $\dim_{\mathbb{F}_2} C(P, L)$ is quite different:

**Theorem** (Sastry-Sin) Assume $q = 2^t$. Then the rank of $M_2(P, L)$ is

$$1 + \left(\frac{1 + \sqrt{17}}{2}\right)^{2t} + \left(\frac{1 - \sqrt{17}}{2}\right)^{2t}.$$ (13)

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