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# A Green Ring Version of the Brauer Induction Theorem

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## 1. INTRODUCTION

In [1], Alperin showed that the Green correspondence may be used to give a short proof of Brauer's theorem on induced characters, thus establishing a connection between the elementary subgroups of that theorem and the vertices in the modular theory. In this paper we use similar reductions to prove an analogue (Theorem A) of Brauer's result, valid in the integral Green ring, which may be seen as an attempt to make this connection more precise. Section 2 will be devoted to a proof of this result and a discussion of some of its implications. In Section 3 we prove a kind of converse to Theorem A, namely that the set $\mathscr{E}(P)$ of subgroups, which is defined below, is the *defect base* for the *G-functor* $a(\mathcal{O}G)$, in the terminology of [7]. This generalises the corresponding result of Green [6], which shows that the set of elementary subgroups is minimal for Brauer's theorem.

## 2. THE INDUCTION THEOREM

Let $G$ be a finite group and $p$ a prime. Choose a finite extension $R_0$ of $\mathbf{Z}_p$, the ring of $p$-adic integers, whose field of quotients is a splitting field for all subgroups of $G$, and let $R$ be the completion of the maximal unramified extension of $R_0$, and $S$ the field of quotients of $R$. Let $(\pi)$ be the maximal ideal of $R$. Then the residue field $\bar{R} := R/(\pi)$ is algebraically closed of characteristic $p$. Let $\mathcal{O} \in \{R, \bar{R}\}$. We shall be concerned with $\mathcal{O}G$-modules which are finitely generated and free over $\mathcal{O}$. Our choice of $\mathcal{O}$ means that every indecomposable $\mathcal{O}G$-module is *absolutely indecomposable.* We shall use the following notation:



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|---|---|
| $a(\mathcal{O}G)$ | the integral Green ring of $\mathcal{O}G$; |
| $a(\mathcal{O}G, H)$ | the $H$-projective ideal in $a(\mathcal{O}G)$, for $H \leqslant G$; |
| $G_0(\mathcal{O}G)$ | the Grothendieck ring of $\mathcal{O}G$; |
| $\mathrm{Char}_s(G)$ | the (ordinary) character ring of $G$. |

Variations of the above may be used but their meanings should be clear. We use the symbol "$\otimes$" for multiplication in the Green ring. No confusion results since this product extends the usual tensor product of modules over $\mathcal{O}$. As this tensor product preserves exactness it induces a product on $G_0(\mathcal{O}G)$. We also abuse notation by identifying a module with the class of $a(\mathcal{O}G)$ which it represents.

If $H \leqslant G$ and $N$ is a $\mathcal{O}H$-module, the mapping $N \mapsto N^G$ extends to a linear map $a(\mathcal{O}H) \to a(\mathcal{O}G)$. Moreover, induction preserves exactness so we have an induced induction map $G_0(\mathcal{O}H) \to G_0(\mathcal{O}G)$. Thus for any collection of subgroups of $G$, we may define in an obvious way a linear map, which we shall also call induction, from the disjoint sums of their Green rings to $a(\mathcal{O}G)$. The same applies to Grothendieck rings and in both cases the image of the induction map is an ideal.

Let $V$ be a fixed $p$-subgroup of $G$ and define $\mathscr{E}(V)$ to be the set

$$\{H \leqslant G \mid H = W \rtimes E,\ W \leqslant_G V,\ E \text{ is an elementary } p'\text{-subgroup}\}.$$

Clearly, if $H \in \mathscr{E}(V)$ then the image of $a(\mathcal{O}H)$ under the induction map lies in $a(\mathcal{O}G, V)$. Our main result can now be stated.

THEOREM A. *The induction map*

$$\sum_{H \in \mathscr{E}(V)} a(\mathcal{O}H) \xrightarrow{\text{ind}} a(\mathcal{O}G, V)$$

*is surjective.*

*Remark.* Theorem A can be obtained via results of A. Dress on relative Grothendieck rings [5]. We shall present a direct proof.

Before starting on the proof of Theorem A we state a reformulation in terms of modules, and some familiar facts which can be read off from the theorem.

THEOREM A′. *Let $M$ be a $V$-projective $\mathcal{O}G$-module. Then there exist modules $N_1$ and $N_2$ which are sums of modules induced from subgroups in $\mathscr{E}(V)$ such that*

$$M \oplus N_1 \cong N_2.$$

In fact, it will be evident from the proof of Theorem A that given $M$, one can find a finite extension of $\mathbf{Z}_p$ for which this holds.

COROLLARY 1. *Let $\mathcal{O}' \subseteq \mathcal{O}$ be an extension (of fields or local principal ideal domains) and let $M$ be a $V$-projective $\mathcal{O}'G$-module. Then* $\operatorname{rank}_{\mathcal{O}'} M$ *is divisible by $|P : V|$, where $V \leqslant P \in \mathrm{Syl}_p(G)$.*

*Proof.* All summands of $M \otimes_{\mathcal{O}'} \mathcal{O}$ are $V$-projective and

$$\operatorname{rank}_{\mathcal{O}'}(M \otimes_{\mathcal{O}'} \mathcal{O}) = \operatorname{rank}_{\mathcal{O}'} M.$$

COROLLARY 2. *Let $\chi$ be the character afforded by a $V$-projective $R'G$-module ($R' \subseteq R$ an extension of local p.i.d.s). Then $\chi(x) = 0$ unless the $p$-part of $x$ is conjugate to an element of $V$.*

*Proof.* We may assume that $R' = R$ without loss of generality, and apply Theorem $A'$.

COROLLARY 3. *The Brauer character of a $V$-projective $\bar{R}G$-module is the restriction to the $p$-regular elements of the difference of two (ordinary) characters afforded by $V$-projective $RG$-modules.*

*Proof.* For $H \in \mathscr{E}(V)$ it is clear that the Brauer characters are restrictions of ordinary characters, automatically $V$-projective.

Corollary 3 generalises the Brauer lifting and was first proved by M. Broué in [2].

COROLLARY 4. *Let $\chi$ be the character afforded by a $V$-projective $RG$-module. Then $\chi \in \sum_H \operatorname{ind}_{H,G}(\mathrm{Char}_s(H))$, where the sum runs over elementary subgroups $H$ with $\mathcal{O}_p(H) \leqslant V$.*

*Proof.* By Theorem A we are reduced to the case $G \in \mathscr{E}(V)$. In this case every elementary subgroup satisfies $\mathcal{O}_p(H) \leqslant V$ so Brauer's theorem gives the result.

We remark that the hypotheses of Corollary 4 hold when $\chi$ lies in a $p$-block with defect group $V$.

Of course, these corollaries may be proved separately by much quicker methods. In fact we shall need to prove Corollary 4 in the case $V = 1$ in Lemma 1. This result is the first step in the proof of Theorem A by induction on $|V|$ and $|G|$.

LEMMA 1.

$$\sum_{H \in \mathscr{E}(1)} a(\mathcal{O}H) \xrightarrow{\text{ind}} a(\mathcal{O}G, V)$$

*is surjective.*

*Proof.* It suffices to prove the corresponding statement for the characters of projective $RG$-modules, since all projective indecomposable $\bar{R}G$-modules lift uniquely to these. We must show that for any projective character $\Phi$, we have

$$\Phi = \sum_{H \in \mathscr{E}(1)} \lambda_H{}^G, \tag{1}$$

where $\lambda_H \in \mathrm{Char}_S(H)$. By Brauer's theorem,

$$1_G = \sum_H \rho_H{}^G, \qquad \rho_H \in \mathrm{Char}_S(H),$$

where the summation is over all elementary subgroups. Thus

$$\Phi = \Phi \cdot 1_G = \sum_H (\Phi_H \rho_H)^G.$$

This is not yet in the form of (1), since we still have to eliminate those subgroups $H$ for which $\mathcal{O}_p(H) \neq 1$. However, $\Phi_H \rho_H$ is a projective character of $H$, and $H = \mathcal{O}_p(H) \times Q$ for some $p'$-subgroup $Q$. Thus all projective characters are induced from $Q$, which permits the desired elimination and completes the proof of the lemma.

Next we state two well-known results which we shall use to prove Theorem A.

LEMMA 2 (see [4], p. 448). *Let $\mathscr{E}$ denote the set of elementary subgroups of $G$. Then*

$$\sum_{H \in \mathscr{E}} G_0(\bar{R}H) \xrightarrow{\text{ind}} G_0(\bar{R}G)$$

*is surjective.*

LEMMA 3 ([3], Lemma 1). *Suppose $H \lhd G$ and let $x \in a(\mathcal{O}G, H)$. If $y$ and $z$ are elements of $a(\mathcal{O}(G/H))$, regarded as elements of $a(\mathcal{O}G)$ by inflation, which both represent the same element of $G_0(\mathcal{O}G)$, then*

$$x \cdot y = x \cdot z.$$

We may now prove Theorem A, arguing by induction on $|V|$ and $|G|$. The case $|V| = 1$ is Lemma 1. Suppose then that $M$ is an indecomposable $\mathcal{O}G$-module with nontrivial vertex $V$, and let $f(M)$ be its Green correspondent with respect to $N_G(V)$. Then

$$f(M)^G \cong M \oplus (\textstyle\bigoplus_i N_i),$$

where $N_i$ has vertices properly contained in $V$. By induction, we may therefore assume that $V \lhd G$. Now regard the trivial module $I_G$ as a $\mathcal{O}(G/V)$-module. By Lemma 2 in the case $\mathcal{O} = \bar{R}$, and by Brauer's theorem if $\mathcal{O} = R$, both applied to $G/V$, there exists an element $j \in a(\mathcal{O}(G/V))$ which represents the identity element of $G_0(\mathcal{O}(G/V))$ and which is virtually induced from elementary subgroups of $G/V$. Inflating to $a(\mathcal{O}G)$ we have an element $j$ virtually induced from the inverse images in $G$ of elementary subgroups of $G/V$ which represents the identity element of $G_0(\mathcal{O}(G))$. By Lemma 3 we have

$$M = M \cdot I_G = M \cdot j,$$

so we conclude that $M$ is virtually induced from the subgroups described above. Therefore, by induction, we may assume that $G$ is itself one of these subgroups, that is, that $G/V$ is elementary. Let $C$ be a $p$-complement in $G$. Then $VC \lhd G$ and $G/VC$ is a $p$-group. Theorem A is now a consequence of Green's indecomposability theorem.

*Remark.* It is only in applying Green's indecomposability theorem that we have used the full strength of the hypothesis on $\mathcal{O}$. Thus, it is possible to obtain analogous induction theorems over arbitrary fields $K$ by replacing the set $\mathscr{E}(V)$ by a suitable set $\mathscr{E}_K(V)$ of subgroups which are "$K$-elementary modulo $V$," though one may no longer insist that the Sylow $p$-subgroups of elements of $\mathscr{E}_K(V)$ be contained in a conjugate of $V$.

## 3. MINIMALITY OF $\mathscr{E}(V)$

*The classical case.* Green [6] has observed that a set $\mathscr{D}$ of subgroups of $G$ such that

(a) $\mathscr{D}$ is closed under taking subgroups and conjugates;

(b) $\sum_{H \in \mathscr{D}} \mathrm{Char}_S(H) \to^{\mathrm{ind}} \mathrm{Char}_S(G)$ is surjective

necessarily contains all elementary subgroups of $G$. We investigate the analogous question for $a(\mathcal{O}G, V)$ and the set $\mathscr{E}(V)$.

DEFINITION. *Let* $\mathscr{X}$ be a set of subgroups of $G$. Define $j_G(\mathscr{X})$ to be the collection of all subgroups of all conjugates of subgroups in $\chi$. We say that $\mathscr{X}$ is *$j_G$-closed* if $\mathscr{X} = j_G(\mathscr{X})$.

Suppose that for each $p$-subgroup $V$ of $G$, $\mathscr{D}(G, V)$ is a set of subgroups of $G$, minimal subject to

(a) $\mathscr{D}(G, V)$ is $j_G$-closed;

(b) $\sum_{H \in \mathscr{D}(G,V)} a(\mathcal{O}H, H \cap V) \to^{\mathrm{ind}} a(\mathcal{O}G, V)$ is surjective;

(c) $W \leqslant V$ implies $\mathscr{D}(G, W) \subseteq \mathscr{D}(G, V)$.

We notice immediately that we have $\mathscr{D}(G, V) \subseteq \mathscr{E}(V)$, because $\mathscr{D}(G, V) \cap \mathscr{E}(V)$ clearly satisfies (a) and (c) and if $H \in \mathscr{D}(G, V)$, then by Theorem A, $a(\mathcal{O}H, H \cap V)$ is the image of the induction map from those subgroups of $H$ lying in $\mathscr{E}(H \cap V) \subseteq \mathscr{E}(V)$. In particular the Sylow $p$-subgroup of $H$ is contained in a conjugate of $V$, for each $H \in \mathscr{D}(G, V)$.

LEMMA 4. $\mathscr{D}(G, 1) = \mathscr{E}(1)$

*Sketch of proof.* First we observe that $\mathscr{E}(1)$ satisfies (a)–(c). We now follow Green's argument [6]; it is clear that $\mathscr{D}(G, 1)$ must contain all cyclic $p'$-subgroups. One then shows that if a $p'$-elementary subgroup $\langle x \rangle \times Q$ is not in $\mathscr{D}(G, 1)$, where $Q$ is a nontrivial $q$-subgroup, then for every projective character $\Phi$, we have $\Phi(x)/q$ is an algebraic integer, which is contrary to the fact that the span of the projective characters over the algebraic integers contains a function which has $p$-power value on every $p$-regular element.

LEMMA 5. *Suppose $V \lhd G$ and $\mathscr{X}$ is a $j_G$-closed set of subgroups of $G$ satisfying*:

(a) *If $H \in \mathscr{X}$ then the Sylow $p$-subgroups of $H$ are contained in $V$*;

(b) *Every indecomposable $\mathcal{O}G$-module $M$ with vertex $V$ and $V$ contained in its kernel may be written as*

$$M = \sum_{H} y_H{}^G,$$

*where $y_H \in a(\mathcal{O}H)$ and $H \in \mathscr{X}$.*

*Then $\mathscr{X} \supseteq \mathscr{E}(V)$. In particular $\mathscr{D}(G, V) = \mathscr{E}(V)$.*

*Proof.* The indecomposable $\mathcal{O}G$-modules with vertex $V$ and $V$ in their kernels are precisely the projective indecomposable $\mathcal{O}(G/V)$-modules. By Lemma 4, for each subgroup $X \in \mathscr{E}(V/V)$, there is a subgroup $H \in \mathscr{X}$ such that $HV/V = X$. We show that in fact $H$ may be chosen to contain $V$. Let $M$ be a projective indecomposable $\mathcal{O}(G/V)$-module, regarded as a $\mathcal{O}G$-module, and suppose we have

$$M \oplus \sum_{H} \sum_{i} a_i N_{i,H}{}^G \cong \sum_{K} \sum_{j} b_j L_{j,K}{}^G, \tag{2}$$

where the $a_i$ and $b_j$ are natural numbers, the groups $H$, $K$ run through $\mathscr{X}$, and $N_{i,H}$, $L_{j,K}$ are indecomposable $\mathcal{O}H$- and $\mathcal{O}K$-modules, respectively. Now $N_{i,H}{}^G$ has a summand which has vertex $V$ and $V$ in its kernel if and only if

$N_{i,H}$ has both these properties, because $V \lhd G$. Therefore, $V \leqslant H$ in this case. Thus, we must have

$$M \oplus \sum_H{}' \sum_i{}' a_i N_{i,H}{}^G \cong \sum_K{}' \sum_j{}' b_j L_{j,K}{}^G,$$

where the primes indicate that we omit from (2) those subgroups which do not contain $V$ and those modules not having $V$ as vertex and $V$ in their kernels. This shows that $\mathscr{X}$ must contain all the inverse images of subgroups in $\mathscr{E}(V/V)$. Let $W \rtimes E \in \mathscr{E}(V)$. Then $VE/V \in \mathscr{E}(V/V)$, so $VE \in \mathscr{X}$, and then $WE \in \mathscr{X}$ by $j_G$-closure, which proves the main statement. The other statement is immediate since $\mathscr{E}(V)$ satisfies (a)–(c).

THEOREM B. $\mathscr{D}(G, V) = \mathscr{E}(V)$.

*Proof.* We argue by induction on $|V|$, using Lemma 4 to start. By condition (c) and the inductive hypothesis, it will suffice to show that $\mathscr{D}(G, V)$ contains all subgroups of the form $V \rtimes E$, where $E$ is an elementary $p'$-subgroup of $G$. Now all these subgroups lie in $N_G(V)$, so by Lemma 5, we are reduced to showing that for any $\mathscr{O}G$-module $M$ with vertex $V$, the Green correspondent $f(M)$ with respect to $N_G(V)$ can be written as an integral linear combination of modules induced from subgroups of $N_G(V)$ which belong to $\mathscr{D}(G, V)$. By (b), we have

$$M = \sum_{H \in \mathscr{D}(G,V)} y_H{}^G,$$

for some $y_H \in a(\mathscr{O}H)$. We split this sum as

$$M = \sum_{H \supseteq_G V} z_H{}^G + \sum_K x_K{}^G, \tag{3}$$

where $z_H$ is the part of $y_H$ involving indecomposable modules with vertex $V$. Every indecomposable module occurring in the second term has a vertex properly contained in $V$. By replacing the subgroups in the first term by suitable conjugates, we may assume that they are contained in $N_G(V)$. We consider the element

$$N = \sum_H z_H{}^{N_G(V)}$$

of $a(\mathscr{O}N_G(V), V)$ obtained from the subgroups and modules in the first term of (3). Each indecomposable module occurring with nonzero coefficient in $N$ has vertex $V$, since $V \lhd N_G(V)$. Therefore by (3) and the Green correspondence, we must have $N = f(M)$, which completes the proof of Theorem B.

*Remark.* Our definition of $\mathscr{D}(G, V)$ differs from Green's definition of a *defect base* in the addition of condition (c). If $V$ is a Sylow $p$-pubgroup then Theorem B can be proved using only (a) and (b): Let $W \leqslant V$. Composing the map of (b) with restriction to $N_G(W)$ and applying Mackey decomposition, we see that the unit of $a(\mathcal{O}N_G(W))$ is in the image of the induction map from subgroups in $\mathscr{D}(G, V)$. By Theorem A, $a(\mathcal{O}N_G(W), W)$ is the image of the induction map from subgroups in $\mathscr{E}(W)$. Thus, multiplying the images of these two maps and applying the Mackey tensor identity, we see that $a(\mathcal{O}N_G(W), W)$ is in the image of the induction map from $\mathscr{E}(W) \cap \mathscr{D}(G, V)$. Then by Lemma 5, every subgroup of $N_G(W)$ which belongs to $\mathscr{E}(W)$ also belongs to $\mathscr{D}(G, V)$. Since every subgroup in $\mathscr{E}(V)$ is conjugate to one of this form for a suitable choice of $W \leqslant V$, this shows that $\mathscr{E}(V) \subseteq \mathscr{D}(G, V)$.

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