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A Note on Brauer's Induction Theorem

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INTRODUCTION

Many variants of Brauer's induction theorem have been proved, including the one in [2], described below, which relates it to the theory of vertices of modular representations. In this note we shall prove a global version of this result, which at the same time generalizes [1, Theorem 3.7].

Throughout, G will denote a finite group and $\omega \in \mathbb{C}$ a primitive $|G|$th root of unity.

Let p be a prime and $\mathfrak{D}_p$ a nonempty collection of p-subgroups of G which is closed under taking subgroups and conjugates. Let A be a discrete valuation ring containing $\mathbb{Z}[\omega]$ and having residue field of characteristic p. We recall that for any subgroup H of G, an (A-free, finitely generated) AH-module is said to be $\mathfrak{D}_p$-projective if it is isomorphic to a direct summand of a direct sum of modules induced from subgroups of H belonging to $\mathfrak{D}_p$. The $\mathbb{Z}$-span of the characters afforded by these modules forms an ideal $\mathfrak{J}(H, \mathfrak{D}_p)$ in the character ring Char(H). Induction and restriction preserve $\mathfrak{D}_p$-projectivity. The result in [2] is that $\mathfrak{J}(G, \mathfrak{D}_p)$ is the span of the characters induced from (linear characters of) Brauer elementary subgroups with p-Sylow subgroups in $\mathfrak{D}_p$. This makes it clear that $\mathfrak{J}(G, \mathfrak{D}_p)$ does not in fact depend on the choice of A.

More generally, let Π be any set primes and suppose that a collection $\mathfrak{D}_p$ as above is given for each prime $p \in \Pi$. Let $R \supseteq \mathbb{Z}[\omega]$ be a principal ideal domain in which each $p \in \Pi$ lies in a prime ideal $\mathfrak{m}_p$. We consider (R-free, finitely generated) RG-modules M such that for every $p \in \Pi$, the localization $M \otimes_R R_{\mathfrak{m}_p}$ is $\mathfrak{D}_p$-projective. Their characters span an ideal $\mathfrak{J}(G, \mathfrak{D})$ of Char(G) which is clearly contained in $\bigcap_p \mathfrak{J}(G, \mathfrak{D}_p)$.

Finally, we denote by $\mathfrak{E}(G, \mathfrak{D})$ the ideal of Char(G) spanned by the characters induced from (linear characters of) Brauer elementary subgroups

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of G whose p-Sylow subgroups lie in $\mathfrak{D}_p$ for all $p \in \Pi$. Evidently, $\mathfrak{E}(G, \mathfrak{D})$ is contained in $\mathfrak{J}(G, \mathfrak{D})$.

We shall prove the reverse inclusions.

THEOREM. *The ideals* $\bigcap_p \mathfrak{J}(G, \mathfrak{D}_p)$, $\mathfrak{J}(G, \mathfrak{D})$ *and* $\mathfrak{E}(G, \mathfrak{D})$ *are the same.*

Before giving the proof we remark that if $\mathfrak{D}_p$ contains a p-Sylow subgroup of G then $\mathfrak{J}(G, \mathfrak{D}_p) = \mathrm{Char}(G)$, so that if this is the case for all primes in Π except for at most one, then the theorem reduces to the local version quoted above.

PROOF OF THE THEOREM

We need one auxiliary result.

LEMMA. *Let x be a p-element of G. Then the function which takes the value $|G|_p$ on elements of G whose p-parts are conjugate to x and the value zero on all other elements lies in the $\mathbb{Z}[\omega]$-span of characters induced from Brauer elementary subgroups with p-Sylow subgroups conjugate to $\langle x \rangle$.*

Proof. Using Brauer's induction theorem, we can write the function as

$$f = f \cdot 1 = f \sum_E c_E \,\mathrm{Ind}_{E,G}(\alpha_E)$$

$$= \sum_E c_E \,\mathrm{Ind}_{E,G}(\mathrm{Res}_{G,E}(f)\, \alpha_E), \qquad c_E \in \mathbb{Z},\ \alpha_E \in \mathrm{Char}(E),$$

where E runs through the Brauer elementary subgroups of G. Since the intersection of the p-section of x in G with elementary subgroup E is a union of p-sections of E, we are reduced, by means of this equation, to proving the lemma when G is elementary. In this case G is the direct product of its p-Sylow subgroup and a p-complement K. Set

$$\mu = \sum_{\lambda \in \mathrm{Irr}\langle x \rangle} \lambda(x^{-1})\, \lambda \in \mathrm{Char}(\langle x \rangle) \otimes_{\mathbb{Z}} \mathbb{Z}[\omega]$$

and extend μ to $\langle x \rangle K$ by extending each λ to be 1 on K. Then a multiple of $\mathrm{Ind}_{\langle x \rangle K, G}(\mu)$ is the required funtion.

We are now ready to prove the theorem. In view of the remark following the statement of the theorem, we may assume that there are two primes in Π, p and q say, for which $\mathfrak{D}_p$ and $\mathfrak{D}_q$ do not contain Sylow subgroups of G. By adding together the functions of the lemma for various elements we see that the function δ'_p which is $|G|_p$ on those elements with p-parts in an element of $\mathfrak{D}_p$ and zero on all other elements has the form

$$\delta_p = \sum_E a_E \operatorname{Ind}_{E,G}(\beta_E), \qquad a_E \in \mathbb{Z}[\omega],\ \beta_E \in \operatorname{Char}(E),$$

where E ranges over Brauer elementary subgroups whose p-Sylow subgroups belong to $\mathfrak{D}_p$. Our hypothesis on p implies that these are all proper subgroups of G, so that we can assume by induction that the theorem holds for these subgroups. Let $\theta \in \bigcap_p \mathfrak{J}(G, \mathfrak{D}_p)$. Then θ vanishes on elements whose p-parts do not belong to an element of $\mathfrak{D}_p$ (and also on those whose q-parts do not belong to an element of $\mathfrak{D}_q$). Therefore,

$$|G|_p\,\theta = \theta\delta_p = \sum_E a_E \operatorname{Ind}_{E,G}(\operatorname{Res}_{G,E}(\theta)\,\beta_E).$$

Since $\operatorname{Res}_{G,E}(\theta)\,\beta_E \in \bigcap_p \mathfrak{J}(E, \mathfrak{D}_p) = \mathfrak{E}(E, \mathfrak{D})$, we have $|G|_p\,\theta \in \mathfrak{E}(G, \mathfrak{D}) \otimes_{\mathbb{Z}} \mathbb{Z}[\omega]$. Arguing with q in place of p we obtain $|G|_q\,\theta \in \mathfrak{E}(G, \mathfrak{D}) \otimes_{\mathbb{Z}} \mathbb{Z}[\omega]$ as well. Thus, $\theta \in \mathfrak{E}(G, \mathfrak{D}) \otimes_{\mathbb{Z}} \mathbb{Z}[\omega] \cap \operatorname{Char}(G) = \mathfrak{E}(G, \mathfrak{D})$. This completes the proof.

We single out the most important special case.

COROLLARY. *Let χ be an irreducible character of G and for each prime p let D_p be a defect group of the p-block to which χ belongs. Then χ can be written as a $\mathbb{Z}$-linear combination of characters induced from those Brauer elementary subgroups which have their p-Sylow subgroups conjugate in G to a subgroup of D_p for every p.*

(Of course, a suitable refinement of the corollary with "defect group" replaced by "vertex" is also true.)

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