A NOTE ON POINT STABILIZERS IN SHARP PERMUTATION GROUPS OF TYPE $\{0, k\}$

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Abstract. We study sharp permutation groups of type $\{0, k\}$ and observe that, once the isomorphism type of a point stabilizer is fixed, there are only finitely many possibilities for such a permutation group. We then show that a sharp permutation group of type $\{0, k\}$ in which a point stabilizer is isomorphic to the alternating group on 5 letters must be a geometric group. There is, up to permutation isomorphism, one such permutation group.

1. Introduction

Let G be a finite group and $\hat{\theta}$ a (virtual) complex character of G. The *type* of $\hat{\theta}$ is the set $L = \{\hat{\theta}(g) \mid g \in G \setminus \{1\}\}$. As first observed by Blichfeldt [3],

$$|G| \;\Big|\; \prod_{\ell \in L} (\hat{\theta}(1) - \ell).$$

When equality holds $\hat{\theta}$ is said to be *a sharp character of type* L. If $G \leq \mathrm{Sym}(X)$ for some set X, then the permutation group (G, X) is said to be *sharp of type* L if the associated permutation character $\hat{\theta}$ is sharp of type L. (In the literature it is also common for G to be called a sharp permutation group of type $(L, |X|)$.)

Sharp permutation groups of type L have been studied for various choices of L. For example, a sharp permutation group of type $\{0, 1, \ldots, k-1\}$ is the same thing as a sharply k-transitive permutation group. Indeed, a sharp group G of type L is k-transitive on X if and only if $\{0, \ldots, k-1\} \subseteq L$ ([6, Corollary 5.5]).

In particular, the condition that $0 \in L$ is equivalent to the transitivity of the group so one line of inquiry has been to study sharp groups with $L = \{0, k\}$. The groups of this type which act primitively have been classified [5, Corollary 1.2]:

Theorem 1.1. *Let (G, X) be a primitive sharp permutation group of type $\{0, k\}$. Then either G is a 2-transitive Frobenius group, or $k = q$, an odd prime power, $|X| = q^3$, and $G \cong \mathrm{ASO}_3(q)$ acting on its natural module.*

Example 1.2. Let $G = \mathrm{ASO}_3(5)$ acting on its natural module V. Then $\mathrm{SO}_3(5) \cong \mathrm{PGL}_2(5)$ is non-solvable and acts irreducibly on V. Thus G acts primitively on V. Now for $1 \neq g \in \mathrm{SO}_3(5)$, g fixes the 5 points on its axis of rotation, but can fix no others without fixing all of V. Thus G is of type $\{0, 5\}$ on V. But $|G| = (5^3 - 0)(9^3 - 9)$, so G is a primitive sharp permutation group of type $(\{0, 5\}, 125)$ on V with $G_x \cong \mathrm{Sym}(5)$.

Another source of sharp permutation groups of type $L = \{0, k\}$ is the theory of geometric groups. We recall that a permutation group is called a *geometric group* if the

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pointwise stabilizer of any finite subset acts transitively on the points that it does not fix.

Geometric groups of type $\{0, k\}$ have been classified [16, Section 4.2, Theorem 1]:

Theorem 1.3. *Let (G, X) be a geometric group of type $\{0, k\}$. Then one of the following holds:*

(a) (G, X) *a sharply* 2*-transitive group of type* $\{0, 1\}$.
(b) $G \cong H \wr C_2$ *with* H *a regular permutation group of degree* k *and* $|X| = 2k$.
(c) $G \cong N : \mathrm{Sym}(3)$ *where* $N = \{(a, b, c) \in H^3 \mid abc = 1\}$, H *a abelian regular permutation group of order* k, *and* $|X| = 3k$.
(d) *"Twisted" versions of groups of type (c) when* k *is even.*
(e) G *is the stabilizer of* $m-2$ *independent vectors in* $\mathrm{GL}_m(q)$ *where* $|X| = q^m - q^{m-2}$ *and* $k = q^{m-1} - q^{m-2}$.
(f) G *is the semidirect product of the additive group of* $\mathrm{GF}(q)^m$ *by a group of type (e) above.*
(g) $G \cong C_{\frac{q-1}{2}} \times \mathrm{PSL}_2(q)$ *with* $q \equiv 3 \pmod 4$, $|X| = \frac{q^2-1}{2}$ *and* $k = \frac{q-1}{2}$.
(h) $G \cong C_{\frac{q-1}{2}} \times \mathrm{Sz}(q)$ *with* q *an odd power of* 2, $|X| = (q-1)(q^2+1)$ *and* $k = q - 1$.
(i) $G \cong \mathrm{PSL}_3(2)$, $|X| = 14$ *and* $k = 2$
(j) $G \cong \mathrm{PSL}_3(3)$, $|X| = 78$ *and* $k = 6$.

Example 1.4. Let Alt(5) act regularly on Y, and let $G = \mathrm{Alt}(5) \wr C_2$ with the corresponding natural imprimitive action on X, where X is a disjoint union of two copies of Y. Then for each $x \in X$, $G_x \cong \mathrm{Alt}(5)$ acts trivially on the block containing x and regularly on the other. In particular, $|\mathrm{fix}_X(G_x)| = \frac{|X|}{2}$ for every $x \in X$. It follows that G is a geometric group of type $\{0,\ 60\}$ on X and $|G| = (120 - 0)(120 - 60)$. So G is a sharp permutation group of type $(\{0,\ 60\},\ 120)$ on X with $G_x \cong \mathrm{Alt}(5)$.

If H is any non-trivial finite group, a construction following the lines of the previous example may be employed to produce a geometric group of type $\{0,\ |H|\}$ with point stabilizer isomorphic to H. So every finite group occurs as a point stabilizer of a geometric group of type $\{0,\ k\}$.

On the other hand, it is known [15, Theorem 1], that a point stabilizer in a non-geometric sharp permutation group of type $\{0, k\}$, must admit a nontrivial *partition*. By the work of Baer [1], [2] and Suzuki [17], a group with a nontrivial partition must be one of the following: a Frobenius group, a solvable group, or isomorphic to one of $\mathrm{PSL}(2, q)$, $\mathrm{PGL}(2, q)$ and $\mathrm{Sz}(q)$. Additional structural restrictions have been proved by Franchi in [11], [12], and [13].

With this classification in mind, one approach to sharp permutation groups of type $\{0, k\}$ is to fix the isomorphism type of point stabilizer and then attempt to classify the groups with the given point stabilizer. This approach is supported by the elementary observation (Proposition 2.4 below) that once the isomorphism type of a point stabilizer is fixed, there are only finitely many possibilities (with explicit upper bounds) for the values of $|X|$ and k. If, on closer inspection, each of the possibilities can be eliminated, then the given group cannot be a point stabilizer. This is the method used to prove our main result:

Theorem 1.5. *Let (G, X) be a sharp permutation group of type $\{0, k\}$ and G_x the stabilizer of $x \in X$. If $G_x \cong \mathrm{Alt}(5)$, then G is a geometric group. Indeed, $G \cong \mathrm{Alt}(5) \wr C_2$ acting as in Example 1.4.*

Note that by Theorem 1.1 such a group G cannot act primitively. On the other hand, if G is a geometric group of type $\{0, k\}$, then a simple check of Theorem 1.3 forces G to be as in conclusion (b), and so G is as in Example 1.4.

Consequently, we will focus our attention on non-geometric imprimitive sharp permutation groups of type $\{0, k\}$. Nevertheless, other primitive permutation groups still enter the picture and the list of primitive permutation groups of degree up to 50 (unpublished work of C. Sims, and now included in GAP) is very helpful.

2. Sharp permutation groups of type $\{0, k\}$

This section contains a number of general results on sharp permutation groups of type $\{0, k\}$.

Notation 2.1. For ease of reference, we list here some notation that is used throughout.

- (G, X) is a sharp permutation group of type $\{0, k\}$.
- $n := |X|$.
- G_x is the stabilizer in G of some (arbitrary but fixed) $x \in X$.
- $\mathrm{F} := \mathrm{fix}_X(G_x)$, the set of elements of X fixed by all elements of G_x.
- $f := |\mathrm{F}|$.
- $t := k - f$.
- $\hat{\theta}$ is the permutation character of G associated to the G-set X.
- θ is restriction of $\hat{\theta}$ to G_x.

Lemma 2.2. *Let (G, X) be a sharp permutation group of type $\{0, k\}$. Then the following hold.*

(a) *The permutation rank of (G, X) is $k + 1$.*
(b) *$n = |G_x| + k$.*
(c) *$f \mid (|G_x| + t)$.*
(d) *$k \mid n(n - 1)$.*
(e) *For each $g \in G$, $\mathrm{fix}_X(g)$ is a union of blocks of the system of imprimitivity F^G on X. In particular, $f \mid k$, $f \mid t$, and $f \mid |G_x|$.*

Proof. Part (a) is [4, Proposition 2.1]. Since (G, X) is sharp of type $\{0, k\}$, $|G| = n(n-k)$. So the orbit equation yields $|G_x| = n - k$, proving (b).

Observe that F is a (possibly trivial) block for the G-action, and so f divides $n = |G_x| + k = |G_x| + f + t$, whence f divides $|G_x| + t$, proving (c).

Let m be the number of elements of g such that $\theta(g) = m$. Then, by the Orbit Counting Lemma, $|G| \sum_g \theta(g) = n + mk$, so $k \mid |G| - n$. On multiplying (b) by n we also see that $k \mid |G| - n^2$. Therefore $k | n^2 - n$, proving (d).

To prove (e), we pick $g \in G_x$, $g \neq 1$, and set $K = \mathrm{fix}_X(g)$. For each $y \in K$, we let $F(y) = \mathrm{fix}_X(G_y)$. Then $|F(y)| = f$ for each y. We have $g \in G_y$, so $F(y) \subseteq K$ for each $y \in K$. We will show that for y, $z \in K$, either $F(y) = F(z)$ or $F(y) \cap F(z) = \emptyset$. Indeed if $w \in F(y) \cap F(z)$, then G_w contains and is therefore equal to both G_y and G_z, which means $F(y) = F(z)$. Since each $y \in K$ lies in F_y, the set K is partitioned by the distinct sets $F(y)$, and (e) follows. □

Lemma 2.3. *Let (G, X) be a sharp permutation group of type $\{0, k\}$, $n = |X|$, and $x \in X$.*

(a) *Let $g \in G_x$ be an element such that every G-conjugate of g lying in G_x is a G_x-conjugate of g. Then the number of G-conjugates of g is $\frac{n|G_x : C_{G_x}(g)|}{k}$.*

(b) *Let $T \leq G_x$ be any subgroup such that (i) $|\mathrm{fix}_X(T)| = k$ and (ii) every G-conjugate of T contained in G_x is a G_x-conjugate of T. Then the number of G-conjugates of T is $\frac{n|G_x:N_{G_x}(T)|}{k}$.*

(c) *If G_x contains an element g as in (a) or a subgroup T as in (b), then k divides n.*

Proof. There are n/f distinct conjugates of G_x and each contains $[G_x : C_{G_x}(g)]$ conjugates of g. We must determine the number of distinct conjugates of G_x which contain g. Now by Lemma 2.2(e) $\mathrm{fix}(g)$ is the disjoint union of k/f sets of the form $\mathrm{fix}(G_z)$ for various z, and these G_z are precisely the conjugates of G_x containing g. The formula is now immediate. The same argument proves (b). To prove (c), we note that if the conclusion of (a) or of (b) holds, then by Lemma 2.2(b) $k \mid n(n-k)$, so $k|n^2$, and then $k|n$ by Lemma 2.2(d). □

Since (G, X) is of type $\{0, k\}$, we have that every non-identify element g of G_x fixes k points in X, that is $\theta(g) = k$, where θ is the permutation character of G_x on X. In particular, if ρ_{G_x} denotes the regular character of G_x and 1_{G_x} the principal character of G_x, then we have

$$\theta = k \cdot 1_{G_x} + \rho_{G_x}. \tag{1}$$

Then the permutation character of G_x on $X \setminus \mathrm{F}$ is precisely

$$\theta - f \cdot 1_{G_x} = t \cdot 1_{G_x} + \rho_{G_x}, \tag{2}$$

by the previous equation. By Lemma 2.2(a), G_x has $k+1$ orbits on X, whence $t+1$ orbits on $X \setminus F$. Thus there are non-trivial, non-regular permutation characters ψ_i, $i = 0, \ldots t$ such that

$$\rho_{G_x} + t \cdot 1_{G_x} = \sum_{i=0}^{t} \psi_i, \tag{3}$$

and with $\deg(\psi_0) \geq \cdots \geq \deg(\psi_t)$.

Proposition 2.4. *Given any finite group H, there are only finitely many possible choices of k and n for which there can exist a sharp permutation group (G, X) of type $\{0, k\}$ and $|X| = n$, such that $G_x \cong H$.*

Proof. It suffices to prove that k and $|X|$ are bounded above by a function of $|G_x|$. We have $k = f + t$ and by Lemma 2.2(c) and (b) $f \mid (|G_x| + t)$ and $|X| = |G_x| + k$, so it suffices to bound t. By (3) we have $|G_x| + t \geq (t+1)d$, where $d = \psi_t(1) > 1$. Hence $t \leq (|G_x| - d)/(d-1) < |G_x|$. □

Proposition 2.5. *Let (G, X) be a sharp permutation group of type $\{0, k\}$. Then the following are equivalent:*

(i) *$f = k$.*

(ii) *G_x has a regular orbit on $X \setminus F$.*

(iii) *(G, X) is a transitive geometric group of rank 2 (that is of type L with $|L| = 2$).*

Proof. As G is sharp of type $\{0, k\}$ on X, by Lemma 2.2, $|G_x| = n - k$ and G has permutation rank $k+1$ on X. Hence, G_x has f orbits of length 1 on F and $k+1-f$ orbits of length greater than 1 on $X \setminus F$. If $f = k$, it follows at once that $X \setminus F$ is the unique regular orbit of G_x and (ii) holds. Assume that G_x has a regular orbit $\mathcal{O}$ on X. Then G_x has $k - f$ non-trivial orbits on the set $X \setminus (F \cup \mathcal{O})$ which has size $n - f - (n-k) = k - f$: a contradiction unless $k = f$ and G_x acts regularly on $X \setminus F$.

In particular, for each $y \in X \setminus \{x\}$, either $G_{x,y} = G_x$ (and $y \in F$) or $G_{x,y} = \{1\}$ (and $y \notin F$). In particular, for any finite sequence $x = x_1, \ldots, x_t$ distinct points in X, either $G_{x_1,x_2,\ldots,x_t} = G_x$ or $G_{x_1,x_2,\ldots,x_t} = \{1\}$. In either case, this stabilizer acts transitively on the points it does not fix and so G is a geometric group. Since $G_{x,y} = \{1\}$ for every $x \in X$ and $y \in X - \{x\}$, $\{x, y\}$ is an irredundant base for the geometric group G and thus G has rank 2 as a geometric group (see Section 4.0 of [16]). Thus (ii) implies (iii). Suppose finally that (G, X) is a transitive geometric group of rank 2. From Section 1 of [7] (bottom of page 221) it follows that the type of G is $\{0, f\}$ and so $k = f$, and (iii) implies (i). □

Corollary 2.6. *Let (G, X) be a non-geometric sharp permutation group of type $\{0, k\}$. Then f is a proper divisor of k. In particular,*

$$1 \leq f \leq \left\lfloor \frac{k}{2} \right\rfloor .$$

Proof. This is immediate from Proposition 2.5 and Lemma 2.2(e). □

3. Imprimitive sharp permutation groups of type $\{0, k\}$

By Lemma 2.2(e), when $|F| > 1$ the G-conjugates of F form a system of imprimitivity in X that we denote by $\mathcal{B}$. Let Z denote the kernel of the action of G on $\mathcal{B}$. If J is a non-empty subset of X, we let $G_{\{J\}}$ denote the *set-wise* stabilizer of J in G.

Lemma 3.1. *Let (G, X) be a non-geometric, imprimitive sharp permutation group of type $\{0, k\}$. Let $x \in X$ be arbitrary, Y be a block in a non-trivial system of imprimitivity for G with $x \in Y$, and let $G_{\{Y\}}$ denote the set-wise stabilizer in G of Y. If $f = 1$, then $G_x \ntrianglelefteq G_{\{Y\}}$.*

Proof. Suppose, on the contrary, that $G_x \trianglelefteq G_{\{Y\}}$. As $G_{\{Y\}}$ acts transitively on Y, it follows that G_x acts trivially on Y, whence $1 < |Y| \leq f$, contrary to hypothesis. □

Lemma 3.2. *Let (G, X) be a non-geometric, imprimitive sharp permutation group of type $\{0, k\}$. Fix $x \in X$ and suppose G_x is a non-abelian simple group. Let $N = N_G(G_x)$ and suppose that N/G_x is solvable. Let $\mathcal{B} = F^G$ and let Z denote the kernel of the G-action on $\mathcal{B}$. Then each of the following hold:*

- (a) *$G_x = N^{(\infty)}$. In particular, G_x is a characteristic subgroup of every subgroup L for which $G_x \leq L \leq N$.*
- (b) *$Z \cap G_x = \{1\}$ and $G_x \times Z \leq N$.*
- (c) *$|Z| \mid f$.*

Proof. Part (a) follows immediately from the solvability of N/G_x and the simplicity of G_x.

For (b), recall that $F = \mathrm{fix}_X(G_x)$, and so $N = N_G(G_x) = G_{\{F\}}$. Since Z acts trivially on $\mathcal{B}$, we have $Z \leq N$. Now suppose $Z \cap G_x \neq \{1\}$. Then $Z \trianglelefteq G$ and G_x simple force $G_x \leq Z \leq N$, and so G_x is a characteristic subgroup of Z by part (a), whence normal in G. Thus G_x acts trivially on X, a contradiction.

Part (c) follows from (b) together with the fact that $[N : G_x] = f$. □

Lemma 3.3. *Let (G, X) be a sharp permutation group of type $\{0, k\}$. Fix $x \in X$ and suppose G_x is a non-abelian simple group. Let $N = N_G(G_x)$, let $\mathcal{B} = F^G$ and suppose $f = |F| > 1$. Suppose, further, that N/G_x contains no subgroup isomorphic to G_x. Then G_x fixes no block in $\mathcal{B} - \{F\}$ set-wise.*

Proof. G acts transitively on the set $\mathcal{B}$, and the stabilizer of the block F^g is N^g, with G_x^g acting trivially on F^g. Therefore, if G_x stabilizes some $F^g \neq F$, we have a homomorphism $G_x \to N^g/(G_x^g)$ which must be trivial by our hypothesis, since $N^g/(G_x^g) \cong N/G_x$. Therefore G_x fixes F^g pointwise, and since $F^g \neq F$, this contradicts the definition of F. □

We end this section with a general lemma collecting together an assortment of facts about imprimitive group actions, which are easily derived from the definitions.

Lemma 3.4. *Suppose G acts imprimitively on X and let B be a block. Let H be a subgroup of G and $\mathcal{O}$ an H-orbit on X. Then*

(a) *$B \cap \mathcal{O}$ is a block for the action of H on $\mathcal{O}$. In particular if H acts primitively on $\mathcal{O}$, then either $\mathcal{O} \subseteq B$ or else $|B \cap \mathcal{O}| \leq 1$*

(b) *Suppose y and z are distinct elements of B, lying in H-orbits $\mathcal{O}_y$ and $\mathcal{O}_z$ respectively. If $|B \cap \mathcal{O}_y| = |B \cap \mathcal{O}_z| = 1$, then $|O_y| = |O_z|$. In particular, if $|B| = 2$, then either B is contained in a single H-orbit or else the two elements of B lie in distinct orbits of the same length. Thus, each orbit is either a union of blocks or else is paired with another orbit of the same length such that the two orbits intersect the same set of blocks.*

(c) *If $|\mathcal{O}| = c$ and some element of H acts as a c-cycle on $\mathcal{O}$, then either $|B \cap \mathcal{O}| \leq 1$ or else $B \supseteq \mathcal{O}$.*

4. Preliminaries for the proof of Theorem 1.5

We now begin to confine our attention to the case in which $G_x \cong \mathrm{Alt}(5)$, for which we fix some additional notation. For $p = 2,3,5$, we let S_p denote a fixed Sylow p-subgroup of G_x. Set $N_p = N_{G_x}(S_p)$ and let T be any fixed subgroup of order 2. Then every non-trivial non-regular transitive permutation character for G_x has the form $1_B^{G_x}$ where

$$B \in \{T, S_3, S_2, S_5, N_3, N_5, N_2\}.$$

For the next lemma we note that the permutation character for $(G_x, X \backslash F)$ is $\theta - f \cdot 1_{G_x}$.

Lemma 4.1. *Let (G, X) be a non-geometric sharp permutation group of type $\{0, k\}$. Suppose that $G_x \cong \mathrm{Alt}(5)$. Then $t \leq 8$. Moreover, one of the following hold:*

(a) *$t = 2$ and $(f, n) \in \{(1, 63), (2, 64)\}$, and $\theta - f \cdot 1_{G_x} = 1_T^{G_x} + 1_{S_3}^{G_x} + 1_{S_5}^{G_x}$.*

(b) *$t = 4$ and $(f, n) \in \{(1, 65), (2, 66)\}$, and $\theta - f \cdot 1_{G_x}$ is one of the following:*
   (i) $1_{S_3}^{G_x} + 2 \cdot 1_{S_5}^{G_x} + 2 \cdot 1_{N_3}^{G_x}$, *or*
   (ii) $2 \cdot 1_{S_3}^{G_x} + 1_{S_5}^{G_x} + 2 \cdot 1_{N_5}^{G_x}$, *or*
   (iii) $1_{S_2}^{G_x} + 1_{S_3}^{G_x} + 2 \cdot 1_{S_5}^{G_x} + 1_{N_2}^{G_x}$, *or*
   (iv) $1_T^{G_x} + 2 \cdot 1_{S_5}^{G_x} + 2 \cdot 1_{N_2}^{G_x}$.

(c) *$t = 5$ and $(f, n) \in \{(1, 66), (5, 70)\}$ and*

$$\theta - f \cdot 1_{G_x} = 1_{S_3}^{G_x} + 2 \cdot 1_{S_5}^{G_x} + 1_{N_2}^{G_x} + 1_{N_3}^{G_x} + 1_{N_5}^{G_x}.$$

(d) *$t = 6$ and $(f, n) = (6, 72)$ and $\theta - f \cdot 1_{G_x}$ is one of the following:*
   (i) $3 \cdot 1_{S_5}^{G_x} + 2 \cdot 1_{N_2}^{G_x} + 2 \cdot 1_{N_3}^{G_x}$, *or*
   (ii) $1_{S_3}^{G_x} + 2 \cdot 1_{S_5}^{G_x} + 2 \cdot 1_{N_2}^{G_x} + 2 \cdot 1_{N_5}^{G_x}$, *or*
   (iii) $1_{S_2}^{G_x} + 3 \cdot 1_{S_5}^{G_x} + 3 \cdot 1_{N_2}^{G_x}$.

(e) *$t = 8$ and $(f, n) \in \{(2, 70), (4, 72)\}$ and*

$$\theta - f \cdot 1_{G_x} = 3 \cdot 1_{S_5}^{G_x} + 4 \cdot 1_{N_2}^{G_x} + 2 \cdot 1_{N_5}^{G_x}.$$

Proof. Let ρ_{G_x} denote the regular character of G_x. From equation (3), we have

$$\rho_{G_x} + t \cdot 1_{G_x} = \sum_{i=0}^{t} \psi_i, \tag{4}$$

where the ψ_i are transitive, non-trivial, non-regular permutation characters for G_x satisfying

$$\deg(\psi_0) \geq \cdots \geq \deg(\psi_t).$$

Since $\deg(\psi_i) \geq 5$ for each i, it follows that $60 + t \geq 5(t+1)$, whence $t \leq 13$.

Next, Let $A \in \mathrm{Mat}_{7,5}(\mathbb{Z})$ denote the matrix whose rows correspond to the values of the 7 non-regular, non-trivial transitive permutation characters of G_x, listed in descending degree. For each $t \leq 13$ set $b_t = (60+t, t, t, t, t)$. Elementary calculations, which are easily performed on a computer algebra system such as sage or GAP, produce all non-negative integer vector solutions v to the matrix equation

$$vA = b_t.$$

for each choice of t. The entries of a solution v are the multiplicities of the various basic transitive permutation characters in equation (4) above. Thus we find that there are solutions only for $t \in \{2, 4, 5, 6, 7, 8\}$. For each such t, we then consider all f that satisfy the condition $f|60 + t$ of Lemma 2.2(c) and form the pair (f, n), where $n = 60 + f + t$. Then we eliminate all those pairs (f, n) that do not satisfy the conditions $f + t \mid n(n-1)$ of Lemma 2.2(d) and $f \mid t$ of Lemma 2.2(e). For $t = 7$, no pairs survive. For the other t, the surviving pairs (f, n) and the permutation characters obtained from solutions of the above matrix equation are those listed in (a)–(e). □

5. The proof of the Theorem 1.5

In light of the comments immediately following the statement of Theorem 1.5, it is enough to show that there are no non-geometric, imprimitive sharp permutation groups G of type $\{0, k\}$ with $G_x \cong \mathrm{Alt}(5)$. Thus we may assume that f is a proper divisor of n by Corollary 2.6 and in the case $f = 1$, that there is a system of imprimitivity for G different from F^G.

The remainder of our argument involves the consideration (and elimination) of the possibilities for t, f and n from the conclusion of Lemma 4.1. We pause to remind the reader of our notational conventions.

- $n = |X|$
- $k = |\mathrm{fix}_X(g)|$ for $g \in G_x - \{1\}$
- $F = \mathrm{fix}_X(G_x)$
- $|F| = f$,
- $t = k - f$
- $N := N_G(G_x)$
- $\mathcal{B} := F^G$
- Z the kernel of the G-action on $\mathcal{B}$
- $G^{\mathcal{B}}$ the group of permutations induced by G on $\mathcal{B}$

Lemma 5.1. *Let $H \cong \mathrm{Alt}(5)$ and let M be a 5-dimensional $\mathbb{F}_2[H]$-module containing a copy T of the trivial $\mathbb{F}_2[H]$-module and such that*

(a) *M/T is an irreducible 4-dimensional $\mathbb{F}_2[H]$-module, and*
(b) *H transitively permutes the non-zero elements of M/T.*

Then $|\mathrm{fix}_M(v)| \geq 8$ for every involution $v \in H$.

Proof. Note that there are two 4-dimensional irreducible $\mathbb{F}_2[H]$-modules. The first, which we denote by W, corresponds to the standard 2-dimensional module for $\mathrm{SL}_2(4)$ viewed as an $\mathbb{F}_2[\mathrm{SL}_2(4)]$-module. The second corresponds to the natural 4-dimensional $\mathbb{F}_2[\Omega_4^-(2)]$-module, which we denote by (V, Q) where Q is the underlying quadratic form. Note that $\Omega_4^-(2)$ has an orbit of length 5 on (V, Q) consisting of the five singular vectors with respect to Q. Since H acts transitively on the non-zero elements of M/T (by (b)), we must have

$$M/N \cong_{\mathbb{F}_2[H]} W.$$

Henceforth we identify H with $\mathrm{SL}_2(4)$. Observe that M may be identified with a quotient of a 3-dimensional $\mathbb{F}_4[\mathrm{SL}_2(4)]$-module $\tilde{M}$, where $\tilde{M}$ is an extension of W by a copy of the trivial $\mathbb{F}_4[\mathrm{SL}_2(4)]$-module.

Claim: $\dim_{\mathbb{F}_4}(C_{\tilde{M}}(v)) = 2$ for every involution $v \in \mathrm{SL}_2(4)$.

Proof of Claim: With respect to a suitable $\mathbb{F}_4$-basis of $\tilde{M}$, we may assume that the associated representation $\varphi : \mathrm{SL}_2(4) \to \mathrm{GL}_3(4)$ has the form

$$\varphi(g) = \begin{bmatrix} 1 & f(g) \\ \hat{0} & g \end{bmatrix}$$

where $f(g)$ is a row vector of length 2 over $\mathbb{F}_4$, and $\hat{0}$ is the zero column vector of length 2. Since $\mathrm{SL}_2(4)$ has a unique conjugacy class of involutions, we may assume

$$v = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

As φ is a homomorphism, we have $f(gh) = f(h) + f(g)h$ for every g, $h \in \mathrm{SL}_2(4)$. Since v is an involution and $f(1) = (0, 0)$, we have $(0, 0) = f(vv) = f(v) + f(v)v$. Thus $f(v)v = f(v)$, and so $f(v) = (x, 0)$ for some $x \in \mathbb{F}_4$. In particular,

$$\varphi(v) = \begin{bmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

which has 2-dimensional fixed points. The claim follows.

By the claim, $\dim_{\mathbb{F}_2}(C_{\tilde{M}}(v)) = 4$. But there exists a trivial $\mathbb{F}_2[\mathrm{SL}_2(4)]$-submodule T_0 of $\tilde{M}$ for which

$$M \cong_{\mathbb{F}_2[\mathrm{SL}_2(4)]} \tilde{M}/T_0.$$

Thus $\dim_{\mathbb{F}_2}(C_M(v)) \geq 3$ and v fixes at least 8 points of M. This completes the proof. □

We are now ready to analyze the possibilities for t, f, and n from Lemma 4.1(a)-(e). We do so in a sequence of lemmas.

Lemma 5.2. *Let (G, X) be a non-geometric sharp permutation group of type $\{0, k\}$ with $G_x \cong \mathrm{Alt}(5)$. Then $t \neq 2$.*

Proof. Suppose, by way of contradiction, that $t = 2$. By Lemma 4.1(a), we have $(f, n) \in \{(1, 63), (2, 64)\}$. Moreover G_x has exactly 3 orbits on $X \setminus F$, of cardinality 30, 20 and 12.

Case 1: $f = 1$, $n = 63$.

Since G is not primitive by Theorem 1.1, we choose a block Y for G with $x \in Y$ and let $\mathcal{Y} = Y^G$ denote the associated system of imprimitivity. Now $|Y| \mid 63$ and Y is G_x-stable and, as such, is a union of G_x orbits. By Lemma 4.1, G_x has orbits of size 30, 20 and 12 on $X \setminus F$. Thus it must be that $Y = \{x\} \cup \mathcal{O}$ where $|\mathcal{O}| = 20$, and $|\mathcal{Y}| = 3$.

Let $M_{\mathcal{Y}}$ denote the kernel of the G-action on $\mathcal{Y}$. As $G/M_{\mathcal{Y}}$ embeds in $\mathrm{Sym}(\mathcal{Y}) \cong \mathrm{Sym}(3)$, the simplicity of $G_x \cong \mathrm{Alt}(5)$ forces $G_x \leq M_{\mathcal{Y}}$ and every block of $\mathcal{Y}$ is G_x-stable. But this is incompatible with the possible G_x orbit sizes on X.

Case 2: $f = 2$, $n = 64$.

Here, we have $[N : G_x] = |F| = 2$ and we set $F = \{x, \hat{x}\}$. Note also that $N = G_{\{F\}}$, the set-wise stabilizer of F in G. From Lemma 4.1(a), G_x has three non-trivial orbits $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$ on $X \setminus F$ of size 30, 20 and 12 respectively.

Suppose first that G acts primitively on $\mathcal{B}$, with $|\mathcal{B}| = 32$. Then either G acts faithfully on $\mathcal{B}$, or else $N = G_x \times Z$ by Lemma 3.2. Thus either $|G^{\mathcal{B}}| = 3840$ or 1920. From the GAP primitive group Libraries [14], we see that there are precisely 7 primitive permutation groups of degree 32, none of which have order 3840 or 1920.

Henceforth we suppose that G acts imprimitively on $\mathcal{B}$ and let D be a proper block for G on X with $x \in F \subset D \subset X$. Since $|D| \mid |X|$ and D is G_x-stable it must be a union of G_x-orbits. Since $x \in D$, the cardinality information on the G_x-orbits from Lemma 4.1(a) forces $D = F \cup \mathcal{O}_1$ and $|D| = 32$. We set $\mathcal{B}_D = \{B \in \mathcal{B}; \mid B \subset D\}$, so $|\mathcal{B}_D| = 16$. Let $G_{\{D\}}$ denote the set-wise stabilizer of D in G. Then $G_{\{D\}}$ is a normal subgroup of G of index 2 and $G_x \leq N \leq G_{\{D\}}$ with $[G_{\{D\}} : G_x] = 32$.

Since $G_x \cong \mathrm{Alt}(5)$, G_x is core-free in $G_{\{D\}}$ and so $G_{\{D\}}$ acts faithfully on D. Since G_x has an orbit of size 30 on D, it follows that $G_{\{D\}}$ acts 2-transitively on $\mathcal{B}_D$. Since $\mathrm{Alt}(5) \cong G_x \lhd N$, we have either $N \cong \mathrm{Sym}(5)$ or $N \cong \mathrm{Alt}(5) \times C_2$.

By the remarks above, $G_{\{D\}}$ acts faithfully on D with permutation rank 3 and induces a 2-transitive group on both $\mathcal{B}_D$ and F.

Suppose first that $G_{\{D\}}$ acts faithfully on $\mathcal{B}_D$. Then in the language of [9], $G_{\{D\}}$ is *block faithful* with respect to $\mathcal{B}_D$, and so by [9, Corollary 3.5], $G_{\{D\}}$ is a quasiprimitive group of rank 3 on D. Whence $G_{\{D\}}$ is almost simple by [9, Corollary 1.4]. However, according to [10, Table B.2, Appendix B] this forces $\mathrm{Soc}(G_{\{D\}}) \cong \mathrm{Alt}(16)$, whence $64 \cdot 60 = |G| = 2|G_{\{D\}}| \geq 16!$, a contradiction.

Thus we may assume that $G_{\{D\}}$ does not act faithfully on $\mathcal{B}_D$. Let M_D denote the kernel of the $G_{\{D\}}$-action on $\mathcal{B}_D$. Then as $M_D \leq N = N_G(G_x)$ and $G_x \cong \mathrm{Alt}(5)$, we have

$$N = G_x \times M_D$$

and as above, the group $G_{\{D\}}^{\mathcal{B}_D} \cong G_{\{D\}}/M_D$ is a 2-transitive permutation group on $\mathcal{B}_D$. Recall that $G_{\{D\}}$ acts faithfully on D, and so $|G_{\{D\}}| = |D||G_x| = 1920$. Observe that as $[N : G_x] = |F| = 2$ and $N = G_x \times M_D$ with $M_D \neq \{1\}$, we must have $|M_D| = 2$ and so $G_x \cong N/M_D \leq G_{\{D\}}/M_D \cong G_{\{D\}}^{\mathcal{B}_D}$ with $|G_{\{D\}}^{\mathcal{B}_D}| = 960$. In particular, $G_{\{D\}}^{\mathcal{B}_D}$ is a 2-transitive permutation group on $\mathcal{B}_D$ with 'point' stabilizer isomorphic to $\mathrm{Alt}(5)$. Suppose $G_{\{D\}}^{\mathcal{B}_D}$ is almost simple with non-abelian simple socle $\hat{S}$ and let $\hat{N} \cong N/M_D \cong \mathrm{Alt}(5)$ denote the stabilizer of F in $G_{\{D\}}^{\mathcal{B}_D}$. As $G_{\{D\}}^{\mathcal{B}_D}$ is 2-transitive on $\mathcal{B}_D$, $\hat{N}$ is thus a maximal subgroup $G_{\{D\}}^{\mathcal{B}_D}$. As $\hat{S}$ is not a 2-group, $\hat{S} \cap \hat{N} \neq \{1\}$ and so the simplicity of $\hat{N}$ forces $\hat{N} \leq \hat{S}$. Since $\hat{N}$ does not act trivially on $\mathcal{B}$, it is not normal in $G_{\{D\}}^{\mathcal{B}_D}$. Thus $\hat{N} \neq \hat{S}$ and the maximality of $\hat{N}$ forces $\hat{S} = G_{\{D\}}^{\mathcal{B}_D}$, and so $G_{\{D\}}^{\mathcal{B}_D}$ is a non-abelian simple group of order 960, which is impossible by [8].

Thus $G_{\{D\}}^{\mathcal{B}_D}$ is a 2-transitive permutation group which cannot be almost simple and so by a theorem of Burnside [10, Theorem 4.1B]), $G_{\{D\}}/M_D$ is an affine 2-transitive permutation group of degree 16 with point stabilizer $N/M_D \cong \mathrm{Alt}(5)$. In particular,

$$G_{\{D\}}/M_D \cong R : \mathrm{Alt}(5),$$

with $R \cong \mathbb{F}_2^4$ acting regularly on $\mathcal{B}_D$ and Alt(5) acting transitively on $\mathcal{B}_D - \{F\}$. There are 2 non-isomorphic irreducible $\mathbb{F}_2[\mathrm{Alt}(5)]$-modules of dimension 4. Since Alt(5) acts transitively on the non-zero elements of our module, the action of N/M_D on R corresponds to the action of $\mathrm{SL}_2(4)$ on its natural module. In particular, it follows that the action of $G_{\{D\}}/M_D$ on $\mathcal{B}_D$ is permutation equivalent to the action of $\mathrm{ASL}_2(4)$ on the natural module of $\mathrm{SL}_2(4)$.

Let $A = G_{\{D\}}^{\mathcal{B}_D}$ denote the permutation group induced by $G_{\{D\}}$ on $\mathcal{B}_D$. In particular, $A \cong G_{\{D\}}/M_D \cong \mathrm{ASL}_2(4)$. Let $\pi : G_{\{D\}} \to A$ denote the associated permutation representation, $\hat{R}$ the unique regular normal subgroup of A of order 16, and set $T = \pi^{-1}(\hat{R})$. Then we have $T \trianglelefteq G_{\{D\}}$ with $|T| = 32$ and T acts transitively on $\mathcal{B}_D$ with kernel $M_D \leq Z(G_{\{D\}})$. Since each block of $\mathcal{B}_D$ has two elements, M_D fixes a point in a block of $\mathcal{B}_D$ if and only if it fixes every block of $\mathcal{B}_D$ point-wise. Since $G_{\{D\}}$ acts faithfully on D, it follows that M_D is fixed point free on each block in $\mathcal{B}_D$, and so T acts transitively, whence regularly, on D. Thus T is a regular normal subgroup of $G_{\{D\}}$ and $G_{\{D\}} = T : G_x$. Now T/M_D is elementary abelian, so T is either elementary abelian or extraspecial. Since G_x has an orbit of length 30 on D, it follows that G_x acts transitively on the elements of $T - M_D$, whence T is elementary abelian.

In particular, G_x acts trivially on M_D and irreducibly on T/M_D such that it is transitive on the non-identity elements of T/M_D. Thus by Lemma 5.1, every involution fixes 8 elements of T, and so $|\mathrm{fix}_D(v)| = 8$ for every involution v of G_x, contrary to (G, X) being of type $\{0, 4\}$. This completes the proof. □

Lemma 5.3. *Let (G, X) be a non-geometric sharp permutation group of type $\{0, k\}$ with $G_x \cong \mathrm{Alt}(5)$. Then $t \neq 4$.*

Proof. As in Lemma 5.2, we divide our argument into cases corresponding to Lemma 4.1(b).
Case 1: $f = 1$, $n = 65$.

Since G is not primitive by Theorem 1.1, we choose a block Y for G with $x \in Y$ and let $\mathcal{Y} = Y^G$ denote the associated system of imprimitivity. Now $|Y| \mid 65$ so $|Y| = 5$ or 13. As before, Y is a union at least two G_x-orbits (one of which is $\{x\}$). Since the minimal length of a non-trivial orbit is 5 and G_x has a single fixed point on X, we must have $|Y| = 13$.

Note that $G_x \leq G_{\{Y\}}$ and $[G_{\{Y\}} : G_x] = 13$. By Sylow's theorem, $G_{\{Y\}}$ has a normal Sylow 13-subgroup, call it P. As $G_x \cong \mathrm{Alt}(5)$, G_x must centralize P, whence $G_x \trianglelefteq G_{\{Y\}}$ and so G_x acts trivially on Y, forcing $Y \subseteq F$, a contradiction.
Case 2: $f = 2$, $n = 66$.

Let S be a Sylow 3-subgroup of G, a group of order 9. By Sylow counting, we see that $|N_G(S)/S|$ must be one of 440, 110, 44, 20, 11, 8, 5 and 2.

If this number is divisible by 11, then S is centralized by an element v of order 11. Then v acts on $\mathrm{fix}_X(s)$, where $s \in S$ may be chosen to have fixed points on X. Then since $|\mathrm{fix}_X(s)| = 6$, it follows that v fixes a point of X, a contradiction as $11 \nmid 60 = |G_x|$.

If $|N_G(S)/S|$ is divisible by 5, then, as above, $N_G(S)$ has an element u of order 5 that centralizes S. Then u acts on the set $\mathrm{fix}_X(s)$, where s is an element of S that has fixed points. Since $|\mathrm{fix}_X(s)| = 6$, it follows that u must fix a point of $\mathrm{fix}_X(s)$. Then the stabilizer of this point contains an element of order 15, which is absurd.

If $|N_G(S)/S| = 2$ then there are 220 conjugates of S in G. Let $T \in Syl_3(G_x)$ with $T \leq S$. Then T is cyclic of order 3 and satisfies the hypotheses of Lemma 2.3(b), which tells us that there are $33 \cdot 10/3 = 110$ G-conjugates of T. Thus $|N_G(T)| = 36$. Now $N_{G_x}(T) \cong \mathrm{Sym}(3)$ so $N_G(T)/C_G(T) \cong C_2$. Since $|S| = 9$, $T \leq S \leq C_G(T)$ and

$|C_G(T)| = 18$. Thus S is the unique Sylow 3-subgroup of $C_G(T)$, whence of $N_G(T)$. Since $T \subseteq \hat{S} \in Syl_3(G)$ if and only if $\hat{S} \leq C_G(T)$, it follows that there are no more G-conjugates of S than G-conjugates of T, a contradiction.

There remains the case $|N_G(S)/S| = 8$. If S is cyclic, then an element $s \in S$ of order 3 is conjugate to a subgroup of G_x so has 6 fixed points on X. So there are precisely two S-orbits of length 3 and the other S-orbits must have length 9, which is impossible since 9 does not divide $60 = 66 - 6$. So we may assume that S is elementary abelian.

Suppose first that S is centralized by an involution w. If, in addition, w has fixed points on X then S acts on $\mathrm{fix}_X(w)$, with no regular orbits, so some nontrivial element of S has a fixed point in $\mathrm{fix}_X(w)$, which implies that the stabilizer of that point has an element of order 6, a contradiction. So w has no fixed points on X. There also exists an element of S with no fixed points in X, for otherwise we would have $\sum_{s\in S} |\mathrm{fix}_X(s)| = 66+8(6) = 114$, which is not divisible by $|S|$. Therefore, there is cyclic subgroup H of order 6, acting semiregularly on X. Since $f = 2$ and H has 11 orbits, we see by Lemma 3.4(b) that at least one H-orbit is a union of blocks, which then contradicts Lemma 3.4(c).

Thus, we may assume that a Sylow 2-subgroup W of $N_G(S)$ embeds into $\mathrm{Aut}(S) \cong \mathrm{GL}(2, 3)$. Now W has order 8 and is not elementary abelian since $\mathrm{GL}(2, 3)$ has no such subgroup. Therefore W has an element y of order 4. Also, W is a Sylow 2-subgroup of G, so it has a conjugate in $N_G(G_x)$. Therefore y^2 has 6 fixed points on X, and it follows that the orbits of $\langle y\rangle$ on X consist of 3 orbits of length 2 and 15 of length 4. Then by Lemma 3.4(b), at least one of the orbits of length 4 contains a block, which is a contradiction to Lemma 3.4(c). This completes the proof of Case 2. □

Lemma 5.4. *Let (G, X) be a non-geometric sharp permutation group of type $\{0, k\}$ with $G_x \cong \mathrm{Alt}(5)$. Then $t \neq 5$.*

Proof. As in the previous lemmas, we divide our argument into cases corresponding to Lemma 4.1(c).

Case 1: $f = 1$, $n = 66$. By Theorem 1.1 we have that G acts imprimitively on X. By Lemma 4.1(c), G_x has orbits of length 1, 5, 6, 10, 12, 12, and 20 on X. We consider the possibilities for the size b of a non-trivial block, which must be a divisor of 66.

Suppose first that there is a non-trivial system of imprimitivity with more than 6 blocks. Then in fact there must be at least 11 blocks and so $b \leq 6$. Let $\mathcal{O}_r$ be the orbit of length r for $r = 5, 6, 10$. The nonempty intersections of $\mathcal{O}_{10}$ with blocks must be singleton sets, since G_x acts primitively on $\mathcal{O}_{10}$, by Lemma 3.4(a). Therefore the 11 elements of $\{x\} \cup \mathcal{O}_{10}$ belong to 11 distinct blocks. By Lemma 3.4(b), if either $\mathcal{O}_5$ or $\mathcal{O}_6$ intersects some block B' in a singleton, then B' would a twelfth distinct block, which would imply that $b < 6$, so $b = 2$ or 3. But in both of these cases, it is clear that G_x must fix all points in the block containing x, contradicting the assumption that $f = 1$. Therefore, by Lemma 3.4(a) both $\mathcal{O}_5$ and $\mathcal{O}_6$ are contained in blocks. This forces $b = 6$, so that $\mathcal{O}_6$ is itself a block and $\mathcal{O}_5 \cup \{x\}$ must also be block. Since $b = 6$, there are 11 blocks, but we have already seen that 10 blocks have nonempty intersection with $\mathcal{O}_{10}$, so we have a contradiction.

So there are either 2 blocks of size 33 or 3 blocks of size 22. Since G_x must act trivially on the set of blocks, each block is a union of G_x-orbits. This is impossible in the case of 3 blocks, from the orbit lengths. So we are left with the case of two blocks of size 33. From the orbit lengths, one of these blocks B_1, say must be the union of $\{x\}$ with an orbit of length 12 and an orbit of length 20. The setwise stabilizer $G_{\{B_1\}}$ of B_1 is a normal subgroup of index 2 in G. From the known actions of G_x on these orbits, we see that for an involution $g \in G_x$ we have $\mathrm{fix}_{B_1}(g) = \{x\}$. We can apply Lemma 2.3 to g to

conclude that $C_G(g)$ contains an element of order 3. This element must lie in $G_{\{B_1\}}$ and fix the unique point x fixed by g. Thus G_x has an element of order 6, which is absurd.
Case 2: $f = 5$, $n = 70$. Here we have $f = 5 = |F| = [N_G(G_x) : G_x]$ and $|\mathcal{B}| = 14$. It follows that N/G_x contains no subgroup isomorphic to G_x, while G_x (because of orbit size considerations) must fix a block in $\mathcal{B} - \{F\}$, contrary to Lemma 3.3. This completes the proof. □

Lemma 5.5. *Let* (G, X) *be a non-geometric sharp permutation group of type* $\{0, k\}$ *with* $G_x \cong \mathrm{Alt}(5)$*. Then* $t \neq 6$*.*

Proof. If $t = 6$, then by Lemma 4.1(d) we have $f = 6$ and $n = 72$. Then $|\mathcal{B}| = 12$ and the action of G_x on $\mathcal{B} - \{F\}$ must have an orbit of length 5 and one of length 6 (or else G_x will fix some block other than F setwise, contrary to Lemma 3.3). The first orbit gives a G_x-invariant set X_1 of size 30 and the second a X_2 of size 36. Since X_1 is the union of a G_x-orbit of length 5 on $\mathcal{B} - \{F\}$, it follows that the length of each G_x-orbit on X_1 is divisible by 5.

From the possible orbit structures from Lemma 4.1(d), it follows that if $t \in G_x$ has order 3, then its fixed points on $X \setminus F$ all lie in X_1, and form a block by Lemma 2.2(e). Each element of X_1 is fixed by some element of order 3, so X_1 is the disjoint union of blocks. Now in X_1 there are 5 blocks and there are 10 subgroups of order 3 in G_x, so there must be a block in X_1 that is fixed pointwise by two distinct subgroups of order 3, hence by the subgroup they generate, which is either G_x or isomorphic to $\mathrm{Alt}(4)$, and we can assume the latter, since the former gives an immediate contradiction to the definition of $X \setminus F$. Each element of this block must therefore belong to a G_x orbit of length 5 in X_1, which implies that X_1 is a union of 6 G_x-orbits of length 5, contrary to the possible orbit structures of Lemma 4.1(d). This completes the proof. □

Lemma 5.6. *Let* (G, X) *be a non-geometric sharp permutation group of type* $\{0, k\}$ *with* $G_x \cong \mathrm{Alt}(5)$*. Then* $t \neq 8$*.*

Proof. We assume by way of contradication that $t = 8$ and, as in the previous lemmas, we divide our argument into cases corresponding to Lemma 4.1(e).
Case 1: $f = 2$, $n = 70$.

Here we have $f = 2 = |F| = [N_G(G_x) : G_x]$ and $|\mathcal{B}| = 35$. We consider the Sylow 7-subgroups of G. Let S be a Sylow 7-subgroup. Then S is cyclic of order 7, acting semiregularly on X, since $7 \nmid 60 = |G_x|$. If S were to centralize any nontrivial element g that is conjugate to an element of G_x, then S would act on the 10 points of $\mathrm{fix}_X(g)$ and therefore S would have fixed points, a contradiction. Thus S centralizes no such elements. By Sylow counting, the number of Sylow 7-subgroups must be one of 1, 8, 15, 50 or 120. If S is normal in G, then G_x acts on S and would have to centralize it, a contradiction. If there are 8 Sylow 7-subgroups, then S is normalized, hence centralized by a Sylow 5-subgroup, which certainly contains conjugates of elements of G_x and we again have a contradiction. Likewise, if there are 15 Sylow 7-subgroups then $|N_G(S)/S| = 40$. A Sylow 2-subgroup of $N_G(S)$ is therefore conjugate to one in $N_G(G_x)$. Since $|N_G(G_x) : G_x| = 2$ and $4 \mid |C_G(S)|$, it follows S centralizes a conjugate of an involution in G_x. If there are 120 Sylow 7-subgroups, then $|N_G(S)| = 35$, so S is the center of its normalizer. By Burnside's Transfer Theorem, G has a normal 7-complement Q, of order 600. Then Q/G_x can be viewed as a G_x-invariant subset of X of size 10, containing F. But then G_x will have fixed points on the 8 points of Q/G_x outside F, a contradiction. Finally, suppose there are 50 Sylow 7-subgroups. Then $|N_G(S)| = 12 \cdot 7$, so S is centralized by an involution v. Now v cannot have fixed points on X or else S would act on them and fix some points. So S and v generate a cyclic subgroup $\langle w \rangle$ of order 14 acting semi-regularly on X. There are 5 $\langle w \rangle$-orbits, so by Lemma 3.4(b) one of the orbits must be a union of blocks of $\mathcal{B}$, contrary to Lemma 3.4(c).

Case 2: $f = 4$, $n = 72$.

By Lemma 4.1(e) we know that now that on $X \setminus F$, G_x has 3 orbits of length 12, 2 of length 6 and 4 of length 5. Each element of G_x has 8 fixed points on $X \setminus F$, made up of 2 blocks of $\mathcal{B}$, by Lemma 2.2(e). Let Y denote the union of the 2 orbits of length 6 and V the union of the 4 orbits of length 5. An involution $u \in G_x$ has 4 of fixed points on Y (2 in each orbit) and the other 4 in V, one in each orbit of length 5. Let s and s' be fixed points of u with $s \in V$ and $s' \in Y$. Then there is a 5-cycle h of G_x which fixes s'. Since $s^h \neq s$ and lies in the same orbit of length 5, we see that s^h is not fixed by u, so is in a different block from s. Hence s and s' are in different blocks. It follows that the 4 fixed points of u in V form a block and so also do the 4 fixed points of u on Y. Since all involutions of G_x are conjugate we see that every element of Y is fixed by some involution of G_x, and that the fixed points of any involution of G_x on Y form a block. Therefore, Y is a union of blocks. However, an element h' of order 5 has only one fixed point on each orbit of length 6, so only 2 fixed points in Y. But the fixed points of h' and Y are both unions of blocks, so their intersection should be a union of blocks, hence have size divisible by 4. This is a contradiction. □

By Lemma 4.1 together with Lemmas 5.2 - 5.6, since G is sharp of type $\{0, k\}$ with $G_x \cong \mathrm{Alt}(5)$, G must be a geometric group. Theorem 1.5 now follows from Theorem 1.3.

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