The p-rank of the $\mathrm{Sp}(4, q)$ generalized quadrangle

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AMS special session on Designs, Delaware, April 2nd, 2005

1. Introduction

- $k = \mathbf{F}_q$, $q = p^t$
- V, a 4-dimensional vector space over k with a nonsingular alternating bilinear form.
- $P = \mathbf{P}(V)$, the set of points of the projective space of V
- L, the set of totally isotropic 2-dimensional subspaces of V, considered as lines in P.

The sets P and L form the points and lines of the symplectic generalized quadrangle. Let A be the incidence matrix of (P, L), considered as a matrix over k. We would like to know the rank of A.

$p = 2$

(Sastry-Sin, 1997 [10]):

Theorem 1.1.

$$\operatorname{rank}(A) = 1 + \left(\frac{1+\sqrt{17}}{2}\right)^{2t} + \left(\frac{1-\sqrt{17}}{2}\right)^{2t}.$$

We can now solve the general case.

Theorem 1.2. *Let p be an odd prime.*
The rank of A is equal to $1+\alpha_1^t+\alpha_2^t$, where

$$\alpha_1, \alpha_2 = \frac{p(p+1)^2}{4} \pm \frac{p(p+1)(p-1)}{12}\sqrt{17}.$$

Note: The same formula also holds for $p = 2$.

1.1. Historical notes

When $q = p$, the rank was first found by De Caen and Moorhouse (unpublished), cf.[11]. Machine computations for the case $q = 9$ and the case $q = 27$ done by Eric Moorhouse and Dave Saunders respectively were helpful in the early stages of our investigations.

2. Representation-theoretic formulation

We consider the incidence map

$$\eta : k[L] \to k[P],$$

sending an isotropic 2-subspace to its characteristic function. This is a map of $k\mathrm{Sp}(V)$-modules, and the p-rank of the incidence matrix is the dimension of $\mathrm{Im}\eta$.

Theorem 1.2 is deduced from a stronger result describing the complete submodule lattice of the $k\mathrm{Sp}(V)$-module $\mathrm{Im}\eta$. To explain this deeper result we need some more notation.
Let

$$\mathcal{H} = \{\mathbf{s} = (s_0, s_1, \ldots, s_{t-1}) \mid 1 \leq s_j \leq 3, (0 \leq j \leq t-1)\}.$$

$\mathcal{H}$ has the natural product partial order.
Let $\mathcal{H}_2 \subset \mathcal{H}$ be the those tuples $\leq (2, 2, ...2)$.

Theorem 2.1. *(i)* $\operatorname{Im}\eta \cong k \oplus M$, *where* M *is a multiplicity-free module with* 2^t *composition factors* $L^+(\mathbf{s})$, $\mathbf{s} \in \mathcal{H}_2$.

(ii) $\dim L^+(\mathbf{s}) = \prod_{j=0}^{t-1} d_{ps_{j+1}-s_j}$, *where*

$$d_{p-1} = \frac{p(p+1)(p+2)}{6},$$
$$d_{p-2} = \frac{(p-1)p(p+1)}{6},$$
$$d_{2p-1} = \frac{2(p-1)p(p+1)}{3},$$
$$d_{2p-2} = \frac{p(p+1)(2p+1)}{6}.$$

(iii) The submodule lattice of M *is isomorphic to the lattice of ideals of* $\mathcal{H}_2$ *under the map taking a submodule to its set of composition factors.*

3. Theorem 2.1 $\Longrightarrow$ Theorem 1.2

We show that $\dim M = \mathrm{rank} A - 1$ satisfies a quadratic recursion in t.

- $\dim M(t)$ is a sum over $\mathcal{H}(t)_2$ of t-fold products of the d_λ.
- Let $r_{ab}(t)$ = contribution to $\dim M(t)$ from those $\mathbf{s} \in \mathcal{H}(t)_2$ with $s_0 = a$, $s_{t-1} = b$.

$$r_{21}(t) = r_{21}(t-1)d_{p-1} + r_{22}(t-1)\frac{d_{p-2}d_{2p-1}}{d_{2p-2}}.$$

$$r_{22}(t-1) = r_{21}(t-2)d_{2p-2} + r_{22}(t-2)d_{2p-2}.$$

These imply

$$r_{21}(t) = r_{21}(t-1)(d_{p-1}+d_{2p-2})+r_{21}(t-2)(d_{p-2}d_{2p-1}+d_{p-1}d_{2p-2}).$$

- r_{11}, r_{12} and r_{22} satisfy the same recursion. Theorem 1.2 now follows, using the known cases $t = 1$ and $t = 2$.

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4. $k[P]$ as a $k\mathrm{GL}(V)$-module

From (Bardoe-Sin, 2000 [1]) we recall the following facts.

- $k[P] = k \oplus Y$, where Y is a multiplicity-free, indecomposable module.
- The composition factors of Y are parametrized by $\mathcal{H}$.
- Given any $k\mathrm{GL}(V)$-submodule of Y, the set of its composition factors is an ideal in the partially ordered set $\mathcal{H}$ and that this correspondence is an order isomorphism from the submodule lattice of Y to the lattice of ideals in $\mathcal{H}$.
- Let $\mathbf{s} \in \mathcal{H}$ and let $\lambda_j = ps_{j+1} - s_j$. Let S^λ be the degree λ component in the truncated polynomial ring $k[x_1, x_2, x_3, x_4]/(x_i^p; 1 \le i \le 4)$. Then

$$L(\mathbf{s}) \cong S^{\lambda_0} \otimes (S^{\lambda_1})^{(p)} \otimes \cdots \otimes (S^{\lambda_{t-1}})^{(p^{t-1})}.$$

4.1. The submodule of all lines

- Let C = submodule of $k[P]$ generated by the characteristic functions of *all* 2-dimensional subspaces of V.
- $C = k \oplus Y_{\leq 2}$, where $Y_{\leq 2}$ is the submodule of of Y given by the set $\mathcal{H}_2$ of $\mathcal{H}$-tuples $\leq (2, 2, ...2)$.
- The possible λ_j are $p-2$, $p-1$, $2p-1$ and $2(p-1)$.
- Clearly, $\mathrm{Im}\eta \leq C$.

5. Action of $\mathbf{Sp}(V)$

We now consider the submodule structure of $k[P]$ and C under the action of $\mathrm{Sp}(V)$.

- The composition factors are known by work of Suprunenko-Zalesskii [12] and Lahtonen [7].
- How does a $\mathrm{GL}(V)$ composition factor $L(\mathbf{s})$ decompose upon restriction to $\mathrm{Sp}(V)$?
- The modules S^λ all remain simple except when $\lambda = 2(p-1)$, in which case we have

$$S^{2(p-1)} = S^+ \oplus S^-,$$

where S^+ and S^- are simple $k\mathrm{Sp}(V)$-modules of dimensions $\frac{p(p+1)(2p+1)}{6}$ and $\frac{p(p-1)(2p-1)}{6}$ respectively.

- The simple $k\mathrm{GL}(V)$-module

$$L(\mathbf{s}) \cong S^{\lambda_0} \otimes (S^{\lambda_1})^{(p)} \otimes \cdots \otimes (S^{\lambda_{t-1}})^{(p^{t-1})}. \qquad (1)$$

  decomposes as a direct sum of 2^r nonisomorphic simple modules, if r of the λ_j equal $2(p-1)$.

- Thus, the $k\mathrm{Sp}(V)$-composition factors of $k[P]$ are given by types, or $\mathcal{H}$-types, together with the additional choice of r signs.

Definition 5.1. Fix $\mathbf{s} \in \mathcal{H}$. $L^+(\mathbf{s}) :=$ the simple $k\mathrm{Sp}(V)$-submodule of $L(\mathbf{s})$ where all signs are chosen to be $+$, that is, we choose the S^+ summands of each $S^{2(p-1)}$ appearing in (1).

6. Proof of Theorem 2.1

- We construct a $k\mathrm{Sp}(V)$-submodule $E \leq C$ which contains $\mathrm{Im}\eta$ and which has the correct composition factors.
- Then we show that $E = \mathrm{Im}\eta$.

6.1. Upper bound: the module E

- Let $\overline{C}_j$ be the quotient of C corresponding to

$$\{\mathbf{s} \in \mathcal{H} \mid (1, \ldots, 1, 2, 2, 1, \ldots, 1) \leq \mathbf{s} \leq (2, \ldots, 2)\},$$

where the first of the two 2s occurs in the j-th position.

Theorem 6.1. *There exists a* $k\mathrm{GL}(V)$*-module* D_j *such that*

$$\overline{D}_j \otimes S^{2(p-1)^{(p^j)}} \cong \overline{C}_j.$$

Theorem 6.1 stems from the natural as a $\mathrm{GL}(V)$-algebra structure of $k[V]$.

Corollary 6.2. *As* $k\mathrm{Sp}(V)$*-modules, we have*

$$\overline{C}_j \cong (\overline{D}_j \otimes (S^+)^{(p^j)}) \oplus (\overline{D}_j \otimes (S^-)^{(p^j)}).$$

- Let C_j^+ be the preimage of the $+$ component of $\overline{C}_j$ in Corollary 6.2.
- Define $E := \cap_{j=0}^{t-1} C_j^+$.
- By construction, the $k\mathrm{Sp}(V)$-composition factors of E are precisely the modules $L^+(\mathbf{s})$, one for each $\mathbf{s} \in \mathcal{H}_2$.
- E contains the characteristic function of an isotropic 2-subspace, so $E \supseteq \mathrm{Im}\eta$.

6.2. Lower bound: submodule structure of E

- Each ideal I of $\mathcal{H}_2$, defines a submodule of C. whose intersection with E is a $k\mathrm{Sp}(V)$-submodule of C whose composition factors are precisely the modules $L^+(\mathbf{s})$ for $\mathbf{s} \in I$.

- Similarly, each coideal of $\mathcal{H}_2$ defines quotients of E and intersections of ideals with coideals correspond to subquotients of E.

- For any $\mathbf{s}$, $\mathbf{s}' \in \mathcal{H}_2$, with $\mathbf{s}'$ immediately below $\mathbf{s}$, there is a subquotient U of E giving a short exact sequence of $k\mathrm{Sp}(V)$-modules

$$0 \longrightarrow L^+(\mathbf{s}') \longrightarrow U \longrightarrow L^+(\mathbf{s}) \longrightarrow 0. \tag{2}$$

- Since E is multiplicity-free as a $k\mathrm{Sp}(V)$-module, this subquotient U is the unique one with these two composition factors.

Theorem 6.3. *The exact sequence (2) does not split.*

- This implies that any submodule which has $L^+(\mathbf{s})$ as a composition factor must also have $L^+(\mathbf{s}')$.
- Since $\mathrm{Im}\eta$ has $L^+(2,\dots,2)$ as a composition factor, it follows that all $L^+(\mathbf{s})$ with $\mathbf{s} \in \mathcal{H}_2$ are composition factors of $\mathrm{Im}\eta$, forcing $\mathrm{Im}\eta = E$.
- This completes the proof of Theorem 2.1 $\square$

- Theorem 6.3 is proved by explicit computations with a monomial basis. Certain *shift operators* defined in David Chandler's thesis [2] are crucial.

7. Related results and work in progress

- When $q = p$, the $k\mathrm{Sp}(V)$-submodule structure of $k[V]$ for arbitrary dimension was worked out in [11].
- Jeff Lataille [8],[9] worked out the $k\mathrm{Sp}(V)$-module structure over for $F[V]$ when $q = p \neq 2$ and arbitrary dimension, for F of characteristic $\neq p$. He also computed the integral invariants of the incidence maps from isotropic subspaces (and their perps) to points.
- In the case $q = p = 2$, the $k\mathrm{Sp}(V)$-submodule structure is not completely known, but there are several ways to describe it in terms of nice filtrations. The composition factors are known.
- CSX are currently trying to work out the submodule structure of $k[V]$ for arbitrary odd q and the integral invariants of the incidence map from isotropic subspaces (and their perps) to points. The invariants for arbitrary subspaces were computed in [3]

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