The $p$-rank of the $\text{Sp}(4, q)$ generalized quadrangle

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1. Introduction

- $k = \mathbf{F}_q$, $q = p^t$
- $V$, a 4-dimensional vector space over $k$ with a nonsingular alternating bilinear form.
- $P = \mathbf{P}(V)$, the set of points of the projective space of $V$
- $L$, the set of totally isotropic 2-dimensional subspaces of $V$, considered as lines in $P$.

The sets $P$ and $L$ form the points and lines of the symplectic generalized quadrangle. Let $A$ be the incidence matrix of $(P, L)$, considered as a matrix over $k$. We would like to know the rank of $A$. 
\( p = 2 \)

(Sastry-Sin, 1997 [10]):

**Theorem 1.1.**

\[
\text{rank}(A) = 1 + \left( \frac{1 + \sqrt{17}}{2} \right)^{2t} + \left( \frac{1 - \sqrt{17}}{2} \right)^{2t}.
\]
We can now solve the general case.

**Theorem 1.2.** Let $p$ be an odd prime.

The rank of $A$ is equal to $1 + \alpha_1^t + \alpha_2^t$, where

$$\alpha_1, \alpha_2 = \frac{p(p+1)^2}{4} \pm \frac{p(p+1)(p-1)}{12} \sqrt{17}.$$ 

Note: The same formula also holds for $p = 2$. 
1.1. Historical notes

When $q = p$, the rank was first found by De Caen and Moorhouse (unpublished), cf.[11]. Machine computations for the case $q = 9$ and the case $q = 27$ done by Eric Moorhouse and Dave Saunders respectively were helpful in the early stages of our investigations.
2. Representation-theoretic formulation

We consider the incidence map

\[ \eta : k[L] \rightarrow k[P], \]

sending an isotropic 2-subspace to its characteristic function. This is a map of \( k\text{Sp}(V) \)-modules, and the \( p \)-rank of the incidence matrix is the dimension of \( \text{Im}\eta \).
Theorem 1.2 is deduced from a stronger result describing the complete submodule lattice of the \( k\text{Sp}(V) \)-module \( \text{Im} \eta \). To explain this deeper result we need some more notation. Let

\[
\mathcal{H} = \{ \mathbf{s} = (s_0, s_1, \ldots, s_{t-1}) \mid 1 \leq s_j \leq 3, (0 \leq j \leq t - 1) \}.
\]

\( \mathcal{H} \) has the natural product partial order. Let \( \mathcal{H}_2 \subset \mathcal{H} \) be the those tuples \( \leq (2, 2, \ldots 2) \).
Theorem 2.1. (i) \( \text{Im} \eta \cong k \oplus M \), where \( M \) is a multiplicity-free module with \( 2^t \) composition factors \( L^+(s), s \in \mathcal{H}_2 \).

(ii) \( \dim L^+(s) = \prod_{j=0}^{t-1} d_{ps_j+1-s_j} \), where

\[
\begin{align*}
  d_{p-1} &= \frac{p(p+1)(p+2)}{6}, \\
  d_{p-2} &= \frac{(p-1)p(p+1)}{6}, \\
  d_{2p-1} &= \frac{2(p-1)p(p+1)}{3}, \\
  d_{2p-2} &= \frac{p(p+1)(2p+1)}{6}.
\end{align*}
\]

(iii) The submodule lattice of \( M \) is isomorphic to the lattice of ideals of \( \mathcal{H}_2 \) under the map taking a submodule to its set of composition factors.
3. Theorem 2.1 \implies Theorem 1.2

We show that \( \dim M = \text{rank}A - 1 \) satisfies a quadratic recursion in \( t \).

- \( \dim M(t) \) is a sum over \( \mathcal{H}(t)_2 \) of \( t \)-fold products of the \( d_\lambda \).
- Let \( r_{ab}(t) \) = contribution to \( \dim M(t) \) from those \( s \in \mathcal{H}(t)_2 \) with \( s_0 = a, s_{t-1} = b \).

\[
\begin{align*}
\quad r_{21}(t) &= r_{21}(t - 1)d_{p-1} + r_{22}(t - 1)\frac{d_{p-2}d_{2p-1}}{d_{2p-2}}. \\
\quad r_{22}(t - 1) &= r_{21}(t - 2)d_{2p-2} + r_{22}(t - 2)d_{2p-2}.
\end{align*}
\]

These imply

\[
r_{21}(t) = r_{21}(t-1)(d_{p-1}+d_{2p-2}) + r_{21}(t-2)(d_{p-2}d_{2p-1}+d_{p-1}d_{2p-2}).
\]

- \( r_{11}, r_{12} \) and \( r_{22} \) satisfy the same recursion. Theorem 1.2 now follows, using the known cases \( t = 1 \) and \( t = 2 \).
4. $k[P]$ as a $k\text{GL}(V)$-module

From (Bardoe-Sin, 2000 [1]) we recall the following facts.

- $k[P] = k \oplus Y$, where $Y$ is a multiplicity-free, indecomposable module.
- The composition factors of $Y$ are parametrized by $\mathcal{H}$.
- Given any $k\text{GL}(V)$-submodule of $Y$, the set of its composition factors is an ideal in the partially ordered set $\mathcal{H}$ and that this correspondence is an order isomorphism from the submodule lattice of $Y$ to the lattice of ideals in $\mathcal{H}$.
- Let $s \in \mathcal{H}$ and let $\lambda_j = ps_{j+1} - s_j$. Let $S^\lambda$ be the degree $\lambda$ component in the truncated polynomial ring $k[x_1, x_2, x_3, x_4]/(x_i^p; 1 \leq i \leq 4)$. Then

$$L(s) \cong S^{\lambda_0} \otimes (S^{\lambda_1}(p) \otimes \ldots \otimes (S^{\lambda_{t-1}}(p^{t-1})).$$
4.1. The submodule of all lines

- Let $C = \text{submodule of } k[P]$ generated by the characteristic functions of all 2-dimensional subspaces of $V$.
- $C = k \oplus Y_{\leq 2}$, where $Y_{\leq 2}$ is the submodule of $Y$ given by the set $\mathcal{H}_2$ of $\mathcal{H}$-tuples $\leq (2, 2, \ldots 2)$.
- The possible $\lambda_j$ are $p - 2$, $p - 1$, $2p - 1$ and $2(p - 1)$.
- Clearly, $\text{Im} \eta \leq C$. 
5. Action of $\text{Sp}(V)$

We now consider the submodule structure of $k[P]$ and $C$ under the action of $\text{Sp}(V)$.

- The composition factors are known by work of Suprunenko-Zalesskii [12] and Lahtonen [7].
- How does a $\text{GL}(V)$ composition factor $L(s)$ decompose upon restriction to $\text{Sp}(V)$?
- The modules $S^\lambda$ all remain simple except when $\lambda = 2(p - 1)$, in which case we have

$$S^{2(p-1)} = S^+ \oplus S^-,$$

where $S^+$ and $S^-$ are simple $k\text{Sp}(V)$-modules of dimensions $\frac{p(p+1)(2p+1)}{6}$ and $\frac{p(p-1)(2p-1)}{6}$ respectively.
• The simple \( k\text{GL}(V) \)-module

\[
L(s) \cong S^{\lambda_0} \otimes (S^{\lambda_1}(p)) \otimes \ldots \otimes (S^{\lambda_{t-1}}(p^{t-1})).
\] (1)

decomposes as a direct sum of \( 2^r \) nonisomorphic simple modules, if \( r \) of the \( \lambda_j \) equal \( 2(p - 1) \).

• Thus, the \( k\text{Sp}(V) \)-composition factors of \( k[P] \) are given by types, or \( \mathcal{H} \)-types, together with the additional choice of \( r \) signs.

**Definition 5.1.** Fix \( s \in \mathcal{H} \). \( L^+(s) := \) the simple \( k\text{Sp}(V) \)-submodule of \( L(s) \) where all signs are chosen to be +, that is, we choose the \( S^+ \) summands of each \( S^{2(p-1)} \) appearing in (1).
6. **Proof of Theorem 2.1**

- We construct a $k\text{Sp}(V)$-submodule $E \leq C$ which contains $\text{Im} \eta$ and which has the correct composition factors.
- Then we show that $E = \text{Im} \eta$. 
6.1. Upper bound: the module $E$

- Let $\overline{C}_j$ be the quotient of $C$ corresponding to
  \[
  \{ s \in \mathcal{H} \mid (1, \ldots, 1, 2, 2, 1, \ldots, 1) \leq s \leq (2, \ldots, 2) \},
  \]
  where the first of the two 2s occurs in the $j$-th position.

**Theorem 6.1.** There exists a $k\text{GL}(V)$-module $D_j$ such that

\[
\overline{D}_j \otimes S^{2(p-1)p^j} \cong \overline{C}_j.
\]

Theorem 6.1 stems from the natural as a $\text{GL}(V)$-algebra structure of $k[V]$. 
Corollary 6.2. As $k\text{Sp}(V)$-modules, we have

$$
\overline{C}_j \cong (D_j \otimes (S^+)^{(p^j)}) \oplus (D_j \otimes (S^-)^{(p^j)}).
$$

- Let $C_j^+$ be the preimage of the $+$ component of $\overline{C}_j$ in Corollary 6.2.
- Define $E := \cap_{j=0}^{t-1} C_j^+$.
- By construction, the $k\text{Sp}(V)$-composition factors of $E$ are precisely the modules $L^+(s)$, one for each $s \in \mathcal{H}_2$.
- $E$ contains the characteristic function of an isotropic 2-subspace, so $E \supseteq \text{Im} \eta$. 
6.2. Lower bound: submodule structure of $E$

- Each ideal $I$ of $\mathcal{H}_2$, defines a submodule of $C$. whose intersection with $E$ is a $k\text{Sp}(V)$-submodule of $C$ whose composition factors are precisely the modules $L^+(s)$ for $s \in I$.

- Similarly, each coideal of $\mathcal{H}_2$ defines quotients of $E$ and intersections of ideals with coideals correspond to subquotients of $E$.

- For any $s, s' \in \mathcal{H}_2$, with $s'$ immediately below $s$, there is a subquotient $U$ of $E$ giving a short exact sequence of $k\text{Sp}(V)$-modules

$$0 \to L^+(s') \to U \to L^+(s) \to 0. \quad (2)$$

- Since $E$ is multiplicity-free as a $k\text{Sp}(V)$-module, this subquotient $U$ is the unique one with these two composition factors.
Theorem 6.3. The exact sequence (2) does not split.

- This implies that any submodule which has $L^+(s)$ as a composition factor must also have $L^+(s')$.
- Since $\text{Im} \eta$ has $L^+(2, \ldots, 2)$ as a composition factor, it follows that all $L^+(s)$ with $s \in \mathcal{H}_2$ are composition factors of $\text{Im} \eta$, forcing $\text{Im} \eta = E$.
- This completes the proof of Theorem 2.1

- Theorem 6.3 is proved by explicit computations with a monomial basis. Certain shift operators defined in David Chandler’s thesis [2] are crucial.
7. Related results and work in progress

- When $q = p$, the $k\text{Sp}(V)$-submodule structure of $k[V]$ for arbitrary dimension was worked out in [11].

- Jeff Lataille [8],[9] worked out the $k\text{Sp}(V)$-module structure over for $F[V]$ when $q = p \neq 2$ and arbitrary dimension, for $F$ of characteristic $\neq p$. He also computed the integral invariants of the incidence maps from isotropic subspaces (and their perps) to points.

- In the case $q = p = 2$, the $k\text{Sp}(V)$-submodule structure is not completely known, but there are several ways to describe it in terms of nice filtrations. The composition factors are known.

- CSX are currently trying to work out the submodule structure of $k[V]$ for arbitrary odd $q$ and the integral invariants of the incidence map from isotropic subspaces (and their perps) to points. The invariants for arbitrary subspaces were computed in [3].
References


[12] I. D. Suprunenko, A. E. Zalesskii, Reduced symmetric powers of natural realizations of the groups $\text{SL}_m(P)$ and $\text{Sp}_m(P)$ and their restrictions to subgroups, Siberian Mathematical Journal (4) 31, (1990), 33–46.