Weyl Modules, simple modules and invariants of incidence maps

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Outline

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Let G be a semisimple algebraic group in characteristic $p > 0$. For each of the types and weights considered below we find:

- The character of the simple module $L(\lambda)$
- The characters (and multiplicities) of the simple composition factors of $V(\lambda)$
- The submodule lattice of $V(\lambda)$

Groups and weights considered

(B) G of type B_ℓ, ($\ell \geq 2$) $\lambda = r(\omega_1)$, $0 \leq r \leq p-1$;

(D) G of type D_ℓ, ($\ell \geq 3$) $\lambda = r(\omega_1)$, $0 \leq r \leq p-1$;

(A) G of type A_ℓ, ($\ell \geq 3$) $\lambda = r(\omega_1 + \omega_\ell)$, $0 \leq r \leq p-1$;

Note: For type A and type C, the Weyl modules $V(r\omega_1)$ are simple.

Theorem B

Let G be of type B_ℓ, $\ell \geq 2$. Let ω_1 be the highest weight of the standard orthogonal module of dimension $2\ell+1$. Assume $0 \leq r \leq p-1$. Then the following hold.

(a) *$H^0(r\omega_1)$ is simple unless (i) $p=2$ and $r=1$ or (ii) $p>2$ and there exists a positive odd integer m such that*

$$r+2\ell-1 \leq mp \leq 2r+2\ell-2.$$

(b) *If (i) holds then the quotient $H^0(\omega_1)/L(\omega_1)$ is the one-dimensional trivial module.*

(c) *If (ii) holds then m is unique and*

$$H^0(r\omega_1)/L(r\omega_1) \cong H^0(r_1\omega_1),$$

where $r_1 = mp-2\ell+1-r$. Furthermore the module $H^0(r_1\omega_1)$ is simple.

Theorem D

Let G be of type D_ℓ, $\ell \geq 3$. Let ω_1 be the highest weight of the standard orthogonal module of dimension 2ℓ. Assume $0 \leq r \leq p-1$. Then the following hold.

(a) *Suppose that there exists a positive even integer m such that*

$$r + 2\ell - 2 \leq mp \leq 2r + 2\ell - 3.$$

Then m is unique and

$$H^0(r\omega_1)/L(r\omega_1) \cong H^0(r_1\omega_1),$$

where $r_1 = mp - 2\ell + 2 - r$. Furthermore the module $H^0(r_1\omega_1)$ is simple.

(b) *Otherwise, $H^0(r\omega_1)$ is simple.*

Theorem A

Let G be of type A_ℓ, $\ell \geq 3$. Assume $0 \leq r \leq p-1$. Then the following hold.

(a) *Suppose that here exists a positive integer m such that*

$$r + \ell \leq mp \leq 2r + \ell - 1.$$

Then m is unique and

$$H^0(r(\omega_1 + \omega_\ell))/L(r(\omega_1 + \omega_\ell)) \cong H^0(r_1(\omega_1 + \omega_\ell)),$$

where $r_1 = mp - \ell - r$. Furthermore the module $H^0(r_1(\omega_1 + \omega_\ell))$ is simple.

(b) *Otherwise, $H^0(r(\omega_1 + \omega_\ell))$ is simple.*

Outline

Sum Formula

The *Jantzen filtration* $V(\lambda)^i$, $i > 0$, of $V(\lambda)$ satisfies

$$V(\lambda)^1 = \operatorname{rad} V(\lambda), \quad \text{so} \quad V(\lambda)/V(\lambda)^1 \cong L(\lambda).$$

and

$$\sum_{i>0} \operatorname{Ch}(V(\lambda)^i) = -\sum_{\alpha>0} \sum_{\{m: 0<mp<\langle\lambda+\rho,\alpha^\vee\rangle\}} v_p(mp)\chi(\lambda - mp\alpha)$$

Keeping control

- The main challenge lies in trying to do infinitely many Sum Formula computations at once. For fixed type the parameters of the problem are p, r and the rank ℓ.
- Use coordinate descriptions of root systems.
- If R is of type B_ℓ or D_ℓ and $\lambda + \rho - mp\alpha$ has two coordinates with the same absolute value then the pair (α, m) contributes nothing to the final sum.
- If R is of type A_ℓ and $\lambda + \rho - mp\alpha$ has two equal coordinates, then the pair (α, m) contributes nothing to the final sum.

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Eliminating multiplicities

- The Sum Formula *overestimates* the character of $\mathrm{rad}\, V(\lambda)$ and multiplicities of composition factors may be greater than the actual composition multiplicity.
- Example. Type D_ℓ.
- For $r \leq p-1$, $S^r(V^*)$ has a *good filtration* . with subquotients of the form $H^0(s\omega_1)$, $s < r$.
- Then for $r_1 < r$,

$$\begin{aligned}
\dim \mathrm{Hom}_G(V(r_1\omega_1), V(r\omega_1)) &= \dim \mathrm{Hom}_G(H^0(r\omega_1), H^0(r_1\omega_1)) \\
&\leq \dim \mathrm{Hom}_G(S^r(V^*), H^0(r_1\omega_1)) \\
&= \dim \mathrm{Hom}_G(V(r_1\omega_1), S^r(V^*)) \\
&\text{(by self-duality of } S^r(V^*)) \\
&= \text{multiplicity of } H^0(r_1\omega_1) \\
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Outline

Incidence of points and polar hyperplanes

- V vector space over $\mathbb{F}_q$ with nonsingular form $b(-,-)$.
- b may be alternating or symmetric or hermitian.
- $\widehat{P} = \{$all 1-dimensional subspaces of $V\}$
  $\supseteq P = \{$singular 1-dimensional subspaces$\}$,
- $\widehat{P}^* = \{$hyperplanes of $V\} \supseteq P^* = \{p^\perp \mid p \in P\}$, polar hyperplanes.
- $G(q) =$ group of linear transformations preserving $b(-,-)$.
- $A =$ incidence matrix of $(\widehat{P}^*, \widehat{P})$

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p-ranks

- Problem is to find the p-ranks, where $q = p^t$.
- The p-rank of A is well known (Goethals-Delsarte, MacWilliams-Mann, Smith), and the p-rank of A_1 was found by Blokhuis and Moorhouse.
- Moorhouse (Linz, 2006): What is the p-rank of A_{11}?

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Permutation module structure

- P and P^* are isomorphic $G(q)$-sets, $k[P] :=$ permutation module.
- $k[P] \cong k.\mathbf{1} \oplus Y$,
- $\mathrm{head}(Y) \cong \mathrm{soc}(Y)$, a simple module L.
- Incidence map induces

$$\phi \in \mathrm{End}_{kG(q)}(k[P]), \quad \phi(p) = \sum_{p' \in p^\perp} p'.$$

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$$\mathrm{Im}\,\phi = k.\mathbf{1} \oplus L.$$

- Outcome: $\mathrm{rank}_p A_{11} = 1 + \dim L$.

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Identifying the simple module L

- $$L \cong L((q-1)\omega),$$

  where $\omega = \omega_1$ in the orthogonal and symplectic cases, and $\omega_1 + \omega_\ell$ in the unitary case.
- By Steinberg's Tensor Product Theorem,

  $$L((q-1)\omega) = L((p-1)\omega) \otimes L((p-1)\omega)^{(p)} \cdots \otimes L((p-1)\omega)^{(p^{t-1})}$$
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- By Steinberg's Tensor Product Theorem,
  $$L((q-1)\omega) = L((p-1)\omega) \otimes L((p-1)\omega)^{(p)} \cdots \otimes L((p-1)\omega)^{(p^{t-1})}$$
- Conclusion: $\operatorname{rank}_p A_{11} = 1 + (\dim L((p-1)\omega))^t$.

Oppositeness

- Let $(\Delta(q), S)$ be the spherical Tits building of a finite group of Lie type.
- Two types $I, J \subseteq S$ are *opposite* if $I^{w_0} = J$.
- Assume I and J are opposite types. We say the cosets gP_I and hP_J of the parabolic subgroups are opposite iff $P_I g^{-1} h P_J = P_I w_0 P_J$.
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Outline

A. E. Brouwer, J. Ducey, P. Sin, To appear, Proc. AMS

Skew lines

- We consider the relation of *skewness* between lines in $PG(3, q)$, $q = p^t$.
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Notation

- V, a 4-dimensional vector space over $\mathbf{F}_q$
- A incidence matrix of skewness between lines in $\mathbb{P}(V)$
- A is square of size $(q^2+q+1)(q^2+1)$.
- For any matrix M, let $e_i(M) =$ number of invariant factors in the Smith Normal Form of M which are exactly divisible by p^i.
- Problem: Compute $e_i(A)$

Outline

p-filtrations

- $R = \mathbb{Z}_p[\zeta]$, $\zeta^{q^4-1} = 1$, $F = R/pR$.
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Left and right SNF Bases

- For a given homomorphism $\eta\colon R^m \to R^n$, we will be interested in pairs of bases $(\mathcal{B}, \mathcal{C})$ with respect to which the matrix of η is in Smith normal form.
- We define a *left* SNF basis for η to be any basis of R^m that belongs to such a pair. Similarly, a *right* SNF basis for η is any basis of R^n belonging to such a pair.
- A left SNF basis can be constructed by lifting a basis of F^m compatible with the descending p-filtration $\{M_i(\eta)\}$.
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Outline

The equation of a strongly regular graph

- $A^2 = q^4 I + (q^4 - q^3 - q^2 + q)A + (q^4 - q^3)(J - A - I)$
- Eigenvalues of A are q, $-q^2$, and q^4 with respective multiplicities $q^4 + q^2$, $q^3 + q^2 + q$, and 1.
- The elementary divisors of A are all powers of p.

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Theorem 1

Let $e_i = e_i(A)$*.*

1. $e_i = e_{3t-i}$ *for* $0 \leq i < t$*.*
2. $e_i = 0$ *for* $t < i < 2t$*,* $3t < i < 4t$*, and* $i > 4t$*.*
3. $\sum_{i=0}^{t} e_i = q^4 + q^2$*.*
4. $\sum_{i=2t}^{3t} e_i = q^3 + q^2 + q$*.*
5. $e_{4t} = 1$*.*

Thus we get all the elementary divisor multiplicities once we know t of the numbers $e_0, \ldots, e_t$ (or the numbers $e_{2t}, \ldots, e_{3t}$).

More notation

- $[3]^t = \{(s_0, \ldots, s_{t-1}) \,|\, s_i \in \{1,2,3\} \text{ for all } i\}$
- $\mathcal{H}(i) = \left\{(s_0, \ldots, s_{t-1}) \in [3]^t \,\middle|\, \#\{j | s_j = 2\} = i\right\}$
- For $\vec{s} = (s_0, \ldots, s_{t-1}) \in [3]^t$

  $$\lambda_i = ps_{i+1} - s_i,$$

  (subscripts mod t) and

  $$\vec{\lambda} = (\lambda_0, \ldots, \lambda_{t-1})$$

- For an integer k, set d_k to be the coefficient of x^k in the expansion of $(1 + x + \cdots + x^{p-1})^4$. Set $d(\vec{s}) = \prod_{i=0}^{t-1} d_{\lambda_i}$.

- Theorem 2

  Let $e_i = e_i(A)$ denote the multiplicity of p^i as an elementary divisor of A. Then, for $0 \leq i \leq t$,

$$e_{2t+i} = \sum_{\vec{s} \in \mathcal{H}(i)} d(\vec{s}).$$

Example, $q = 9$

- $(1 + x + x^2)^4 =$
  $1 + 4x + 10x^2 + 16x^3 + 19x^4 + 16x^5 + 10x^6 + 4x^7 + x^8$
- $\mathcal{H}(0) = \{(11), (13), (31), (33)\}$,
  $\mathcal{H}(1) = \{(21), (23), (12), (32)\}$, $\mathcal{H}(2) = \{(22)\}$.
- $e_4 = d(11) + d(13) + d(31) + d(33) = 202$
- $e_5 = d(21) + d(23) + d(12) + d(32) = 256$
- $e_6 = d(22) = 361$

Table: The elementary divisors of the incidence matrix of lines vs. lines in $\mathrm{PG}(3,9)$, where two lines are incident when skew.

Elem. Div.	1	3	3^2	3^4	3^5	3^6	3^8
Multiplicity	361	256	6025	202	256	361	1

Outline

- Let $\mathcal{L}_r = \{r\text{-diml subspaces of } V\}$
- B denote the incidence matrix with rows indexed by $\mathcal{L}_1$ and columns indexed by $\mathcal{L}_2$, where incidence again means zero intersection.
- B^t denotes the transpose of B, and is just the incidence matrix of lines vs. points.
- $$B^tB = (q^3+q^2)I + (q^3+q^2-q-1)A + (q^3+q^2-q)(J-A-I). \tag{1}$$
- $(\mathbf{1})B^tB = q^4(q^2+q+1)(q+1)\mathbf{1}$,
- $B^tB = -[A+(q^2-q)I] + q^2I + (q^3+q^2-q)J$
- $e_i(B^tB) = e_i(A+(q^2-q)I)$ for $0 \le i \le t$.
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Outline

Proof of Theorem 2

- Suppose we can obtain the SNF of B^t and B by:

$$PB^tE^{-1} = D_{2,1}$$

and

$$EBQ^{-1} = D_{1,2}$$

where E is the same matrix in both equations

- Then we can find the SNF of the product:

$$PB^tBQ^{-1} = D_{r,1}D_{1,s},$$

- E exists iff there is basis of $R^{\mathcal{L}_1}$ which is simultaneously a *right* SNF basis for B^t and a *left* SNF basis for B
- In general such as basis does not exist, but we get it here from $R^{\mathcal{L}_1}$ being multiplicity-free.
- Finally, we know the elementary divisors of B^t and B from [D. Chandler, P. Sin, Q. Xiang, Trans. AMS 358 (2006) 3537-3559].

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Recapitulation

- In both problems we considered p-filtrations with gaps.
- elementary divisors with multiplicity zero $\equiv$ empty layers in the p-filtration $\implies$ excessive multiplicities in the Jantzen Sum formula.
- Is there a way to predict these gaps?
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Thank you for your attention!