Assignment 2 Solutions

2.2.16 We are given finite dimensional vector spaces V, W with $\dim(V) = \dim(V) = n$, and a linear transformation $T : V \to W$. By Theorem 2.1 we have that $\mathrm{R}(T)$ and $\mathrm{N}(T)$ are sub spaces of V and W, respectively.

If $R(T) = \{0_W\}$ (i.e $\dim(\mathrm{R}(T)) = 0$), then $[T]_\beta^\gamma$ is the zero matrix for any ordered basis β of V and any ordered basis γ of W. Let $\dim(\mathrm{R}(T)) = k > 0$ and $\gamma_1 = (w_1, \ldots, w_k)$ be an ordered basis for $\mathrm{R}(T)$. For each w_i, pick a vector $v_i \in V$ satisfying $T(v_i) = w_i$. By Dimension theorem we have $\dim(\mathrm{N}(T)) = n - k$. We may now find vectors $v_{k+1}, \ldots, v_n$ in V such that $\{v_{k+1}, \ldots v_n\}$ is a basis for $\mathrm{N}(T)$.

Let $c_1, \ldots c_n$ be scalars such that $\sum_{i=0}^{n} c_i v_i = 0_V$. Since $v_i \in \mathrm{N}(T)$ for $i > k$ and $T(v_i) = w_i$ for $i \leq k$, we have

$$0_W = T(0_V) = T(\sum_{i=0}^{n} c_i v_i) = \sum_{i=0}^{n} c_i T(v_i) = \sum_{i=0}^{k} c_i w_i.$$

Linear independence of $\{w_1, \ldots, w_k\}$ implies that $c_i = 0$ for all $i \leq k$. So we now have $\sum_{i>k} c_i v_i = 0_v$. As $\{v_{k+1}, \ldots v_n\}$ is a linearly independent set, we must have $c_i = 0$ for $i > k$ as well. We have proved $\{v_1, \ldots, v_n\}$ is a linearly independent set of size $n = \dim(V)$. Thus $\beta := (v_1, \ldots, v_n)$ is an ordered basis for V (see Cor.2 of Thm 1.11). Let γ be an ordered basis of W extending the ordered basis γ_1 of $\mathrm{R}(T)$. Now $[T]_\beta^\gamma$ is a diagonal matrix.

2.3.9 Let $U : \mathrm{F}^2 \to \mathrm{F}^2$ and $T : \mathrm{F}^2 \to \mathrm{F}^2$ be non-zero linear transformations satisfying $\mathrm{R}(T) \subseteq \mathrm{N}(U)$ and $\mathrm{R}(U) \not\subseteq \mathrm{N}(T)$. Then we have $UT = T_0$ and $TU \neq T_0$.

Define T to be the transformation satisfying $T(x, y) = (0, x + y)$ for all $(x, y) \in \mathrm{F}^2$. Define U to be the transformation satisfying $U(x, y) = (x, 0)$ for all $(x, y) \in \mathrm{F}^2$. Observe that $\mathrm{R}(T)$ and $\mathrm{N}(U)$ are both equal to the subspace of F^2 generated by $(0, 1)$. Also observe that $\mathrm{R}(U)$ contains $(1, 0)$, but $\mathrm{N}(T)$ does not contain $(1, 0)$. So we have $\mathrm{R}(T) \subseteq \mathrm{N}(U)$ and $\mathrm{R}(U) \not\subseteq \mathrm{N}(T)$ and thus $UT = T_0$ and $TU \neq T_0$.

Let β be the standard basis for F^2. If $A = [U]_\beta^\beta$ and $B = [T]_\beta^\beta$, then $AB = [T_0]_\beta^\beta = 0$ and $BA \neq [T_0]_\beta^\beta = 0$.