Assignment 3 and 4 Solutions

3.4.9) The map $r : M_{2\times 2}(\mathbb{R}) \to \mathbb{R}^4$ satisfying $r\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = [a, b, c, d]^t$ is an isomorphism. Under this map, the set S maps to the set

$r(S) := \{[0, -1, -1, 1]^t,\ [1, 2, 2, 3]^t,\ [2, 1, 1, 9]^t,\ [1, -2, -2, 4]^t,\ [-1, 2, 2, -1]^t\}$. As r is an isomorphism, $r(S)$ generates the subspace $r(W)$ of $\mathbb{R}^4$ and moreover if $A \subset r(S)$ is a basis set of $r(W)$, then $r^{-1}(A) := \{r^{-1}(a) |\ a \in A\}$ is a subset of S generating W.

The matrix whose columns are elements of S is

$$X = \begin{pmatrix} 0 & 1 & 2 & 1 & -1 \\ -1 & 2 & 1 & -2 & 2 \\ -1 & 2 & 1 & -2 & 2 \\ 1 & 3 & 9 & 4 & -1 \end{pmatrix}.$$

The reduced row-echelon form of X is

$$\begin{pmatrix} 1 & 0 & 3 & 0 & 4 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

By Theorem 3.16 *c*) & *d*), the set $A = \{[0, -1, -1, 1]^t,\ [1, 2, 2, 3]^t, [1, -2, -2, 4]^t\}$ is a basis for $\text{Span}(r(S)) = r(W)$. Thus the set consisting of the first, second and fourth elements of S is a basis for W.

4.2.23) We prove this by induction on the size of the matrix. It is clear that the determinant of all 1×1 upper triangular matrices is equal to the product of its diagonal entries. Now we assume that the statement is true for all $n \times n$ matrices for some positive integer n. Consider the following generic $n + 1 \times n + 1$ upper triangular matrix.

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n+1} \\ & \ddots & & \\ & & \ddots & \\ & 0 & & \ddots \\ & & & a_{n+1,n+1} \end{pmatrix}.$$

Using cofactor expansion along the first column, we have $det(A) = a_{11}det(\tilde{A}_{11})$. Now $\tilde{A}_{11}$ is an $n \times n$ upper triangular matrix whose diagonal entries are $a_{22}, \ldots a_{n+1,n+1}$. Thus by induction hypothesis $det(\tilde{A}_{11}) = \prod_{j=2}^{n+1} a_{jj}$, and thus $det(A) = \prod_{j=1}^{n+1} a_{jj}$.

We can now conclude that the statement is true by principles of mathematical induction.

5.2.3 a) With respect to the standard ordered basis $\beta = (1, x, x^2, x^3)$ of $P_3(\mathbb{R})$, we have

$$A := [T]_\beta = \begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The characteristic polynomial of A is $det(A - tI) = t^4$. So 0 is the only eigenvalue of A. We observe that $rank(A) = 3$ and thus $4 - rank(A - 0I) = 1$ is not equal to the multiplicity of the eigenvalue 0 of A. So by Theorem 5.9, T is not diagonalizable.

b) With respect to the standard ordered basis $\beta = (1, x, x^2)$ of $P_2(\mathbb{R})$, we have

$$A := [T]_\beta = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The characteristic polynomial of T is $det(tI - A) = (t + 1)(t - 1)^2$. Thus T has two eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$, with multiplicities $m_{\lambda_1} = 2$ and $m_{\lambda_2} = -1$ respectively.

Consider the eigenspace $E_1 = \{f(x) \in P_2(\mathbb{R}) |\ T(f(x)) = f(x)\}$. If $ax^2 + bx + c$ is an element of E_1, then we must have $(ax^2 + bx + c) = cx^2 + bx + a$. Thus $a = c$, and therefore $E_1 = \{ax^2 + bx + c |\ a, b \in \mathbb{R}\}$. The set $\beta_1 = \{x^2 + 1, x\}$ generates E_1 and is linearly independent. We can now conclude that $dim(E_1) = 2 = m_{\lambda_1}$

Consider the eigenspace $E_{-1} = \{f(x) \in P_2(\mathbb{R}) |\ T(f(x)) = -f(x)\}$. If $ax^2 + bx + c$ is an element of E_{-1}, then we must have $-(ax^2 + bx + c) = cx^2 + bx + a$. Thus $a = -c$ and $b = 0$, and therefore $E_{-1} = \{ax^2 - a |\ a \in \mathbb{R}\}$. The set $\beta_2 = \{x^2 - 1\}$ generates E_{-1} and thus $dim(E_{-1}) = 1 = m_{\lambda_2}$.

Now by Theorem 5.9 we can conclude that T is diagonalizable and $\beta = (x^2 + 1, x, x^2 - 1)$ is an ordered basis of $P_2(\mathbb{R})$ such that $[T]_\beta$ is a diagonal matrix.

6.1.11 We have

$$||x + y||^2 + ||x - y||^2 = \langle x + y,\ x + y\rangle + \langle x - y,\ x - y\rangle$$

(; by the definition of innerproducts we have)

$$= \langle x,\ x + y\rangle + \langle y,\ x + y\rangle$$
$$+ \langle x,\ x - y\rangle - \langle y,\ x + y\rangle\,;$$

(now application of Theorem 6.1 yields)

$$= \langle x,\ x\rangle + \langle x,\ x\rangle + \langle y,\ x\rangle + \langle y,\ y\rangle$$
$$+ \langle x,\ x\rangle - \langle x,\ y\rangle - \langle y,\ x\rangle + \langle y,\ y\rangle$$
$$= 2(||x||^2 + ||y||^2).$$

In $\mathbb{R}^2$, this law can be interpreted as “The sum of the squares of the lengths of diagonals of a parallelogram is equal to the sum of the squares of the lengths of its sides.”