SOME UNISERIAL REPRESENTATIONS OF CERTAIN SPECIAL LINEAR GROUPS

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Abstract. In an earlier paper a construction was given for an infinite-dimensional uniserial module over $\mathbf{Q}$ for $\mathrm{SL}(2,\mathbf{Z})$ whose composition factors are all isomorphic to the standard (two-dimensional) module. In this note we consider generalizations of this construction to other composition factors and to other rings of algebraic integers.

1. Introduction

The group $\Gamma = \mathrm{SL}(2,\mathbf{Z})$ is generated by the matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The element S has order 4 and ST has order 6. Moreover, Γ is isomorphic to the free product of the two cyclic groups generated by S and $R = ST$, with amalgamation of the elements S^2 and R^3.

Let V be the standard 2-dimensional module for Γ over $\mathbf{C}$ (and later for other subgroups of $\mathrm{GL}(2,\mathbf{C})$). In [1, Theorem 5.1] a module E for Γ of countably infinite dimension was constructed over $\mathbf{C}$ with the following properties.

(1) T acts indecomposably on E.
(2) E has a filtration $0 = E_0 \subset E_1 \subset E_2 \subset \cdots$ such that each quotient E_{i+1}/E_i is isomorphic to V.
(3) The elements of Γ are represented by integer matrices.

In [1, Theorem 8.1], it was shown that properties (1) and (2) characterize the module E up to isomorphism. By (3), we have $E = \mathbf{C} \otimes_{\mathbf{Q}} E'$ for some $\mathbf{Q}\Gamma$ module E' and the same proof shows that E' also satisfies (1) and (2) and is uniquely characterized by them.

More generally, we can study Γ-modules on which which T acts indecomposably (as a single Jordan block). We shall call these T-*indecomposable* modules for short. It is clear that every composition

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factor of such a module would have the same property so the first problem is to identify some simple modules with this property. The symmetric powers $S^m(V)$ provide an infinite class of such modules, as can be seen by reduction mod $p >> m$. We therefore consider when one symmetric power $S^m(V)$ can extend another $S^n(V)$ and when the resulting extension is T-indecomposable. It is immediately clear from the action of $-I$ that a necessary condition for the existence of a nonsplit extension is that m and n have the same parity. The condition is also sufficient, and in §3 we compute the dimension of $\mathrm{Ext}^1_{\mathbf{C}\Gamma}(S^m(V), S^n(V))$. In §1 we consider T-indecomposable modules constructed from symmetric powers. In such a module the degrees of the symmetric powers must all have the same parity. The main result of §1 is a construction, for any sequence of positive integers of the same parity, of a T-indecomposable module whose composition factors are the symmetric powers with degrees equal to the terms of the sequence. In the final section we discuss generalizations of [1, Theorem 5.1] to rings $\mathcal{O}$ of integers in number fields , other than $\mathbf{Q}$, and consider T-indecomposable modules for $\mathrm{SL}(2, \mathcal{O})$ whose composition factors are all isomorphic to V. We show that such modules can exist only for imaginary quadratic fields and study some examples of existence and nonexistence.

2. Uniserial modules constructed from symmetric powers

Theorem 2.1. *Let* $\mathbf{a} = a_1, a_2, \ldots$ *be any finite or infinite sequence of positive integers, all of the same parity. Then there exists a* T*-indecomposable* $\mathbf{Q}\Gamma$*-module* $M(\mathbf{a})$ *with increasing filtration* $0 = F_0 \subset F_1 \subset F_2 \subset \cdots$ *such that such that for* $k \geq 1$ *we have* $F_k/F_{k-1} \cong S^{a_k}(V)$.

Proof. The elements S, T and R are represented on $S^m(V)$ with respect to the basis of monomials (ordered in a standard way) by the matrices

$$\rho_m(S) = \begin{bmatrix} 0 & 0 & \ldots & 0 & (-1)^m \\ 0 & 0 & \ldots & (-1)^{m-1} & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & -1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{bmatrix},$$

$$\rho_m(T) = \begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 & 1 \\ 0 & 1 & 2 & 3 & \dots & (m-1) & m \\ 0 & 0 & 1 & 3 & \dots & \binom{m-1}{2} & \binom{m}{2} \\ 0 & 0 & 0 & 1 & \dots & \binom{m-1}{3} & \binom{m}{3} \\ \vdots & \vdots & \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & m \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

and

$$\rho_m(R) = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 & (-1)^m \\ 0 & 0 & 0 & 0 & \dots & (-1)^{m-1} & (-1)^{m-1}m \\ \vdots & \vdots & \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 0 & 0 & -1 & \dots & -\binom{m-1}{3} & -\binom{m}{3} \\ 0 & 0 & 1 & 3 & \dots & \binom{m-1}{2} & \binom{m}{2} \\ 0 & -1 & -2 & -3 & \dots & -(m-1) & -m \\ 1 & 1 & 1 & 1 & \dots & 1 & 1 \end{bmatrix}$$

respectively.

For positive integers m and n, we define $(m+1) \times (n+1)$ matrices

$$B(m,n) = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & -1 \end{bmatrix}, \qquad C(m,n) = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ (-1)^{m-1} & 0 & \dots & 0 & 0 \end{bmatrix}$$

and

$$E(m,n) = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}.$$

Let $\mathbf{a} = (a_1, a_2, \dots)$ be a sequence of positive integers, all of the same parity. Let

$$R(\mathbf{a}) = \begin{bmatrix} \rho_{a_1}(R) & 0 & 0 & \dots \\ 0 & \rho_{a_2}(R) & 0 & \dots \\ 0 & 0 & \rho_{a_3}(R) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

be the infinite block-diagonal matrix with blocks $\rho_{a_i}(R)$

Let $B_{ij} = B(a_i, a_j)$, $C_{ij} = C(a_i, a_j)$. We use these to define the infinite block upper-triangular matrix

$$S(\mathbf{a}) = \begin{bmatrix} \rho_{a_1}(S) & B_{12} & C_{13} & B_{14} & 2C_{15} & B_{16} & 3C_{17} & \dots \\ 0 & \rho_{a_2}(S) & B_{23} & C_{24} & B_{25} & 2C_{26} & B_{27} & \dots \\ 0 & 0 & \rho_{a_3}(S) & B_{34} & C_{35} & B_{36} & 2C_{37} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The following matrix equations are easily seen to hold when m, n, j and k are positive integers that are either all odd or all even.

(1) $\rho_m(S)^4 = I, \quad \rho_m(R)^6 = I.$
(2) $\rho_m(S)B(m,n) + B(m,n)\rho_n(S) = 0.$
(3) $B(m,n)C(n,k) + C(m,j)B(j,k) = 0.$
(4) $\rho_m(S)C(m,n) + C(m,n)\rho_n(S) = -E(m,n).$
(5) $B(m,j)B(j,n) = E(m,n).$
(6) $C(m,n)C(n,k) = 0.$

These equations imply that $S(\mathbf{a})^4 = I$, $R(\mathbf{a})^6 = I$ and $S(\mathbf{a})^2 = R(\mathbf{a})^3$, so there is a representation of Γ sending S to $S(\mathbf{a})$ and R to $R(\mathbf{a})$. It is clear from the block forms of $R(\mathbf{a})$ and $S(\mathbf{a})$ that the underlying module has a filtration described in the statement of the theorem.

Finally, it remains to show that the $\mathbf{Q}\Gamma$-module $M(\mathbf{a})$ that we have just constructed is T-indecomposable.

Since $T = -SR$, and each diagonal block $\rho_{a_i}(T)$ of $-S(\mathbf{a})R(\mathbf{a})$ is an upper unitriangular and acts indecomposably, T-indecomposability will follow if we show that the bottom left entry of each super-diagonal block $-S(\mathbf{a})R(\mathbf{a})$ is nonzero. If $a_i = m$ and $a_{i+1} = n$, then the $(i, i+1)$

block of $-S(\mathbf{a})R(\mathbf{a})$ is

$$-B(m,n)\rho_n(R) = -\begin{bmatrix} 1 & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & \dots & 0 & (-1)^n \\ 0 & 0 & \dots & (-1)^{n-1} & (-1)^{n-1}n \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 0 & \dots & -\binom{n-1}{3} & -\binom{n}{3} \\ 0 & 0 & \dots & \binom{n-1}{2} & \binom{n}{2} \\ 0 & -1 & \dots & -(n-1) & -n \\ 1 & 1 & \dots & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \dots & \dots \\ 0 & \dots & \dots \\ 0 & \dots & \dots \\ \vdots & \vdots & \ddots \\ 0 & \dots & \dots \\ 0 & \dots & \dots \\ 1 & \dots & \end{bmatrix}.$$

This completes the proof of the theorem. □

Remark 2.2. We note that if $S^m(V)$ is reduced modulo a prime $p \leq m$, then T will not act indecomposably. This is in contrast with the module E constructed in [1, Theorem 5.1] which is T-indecomposable modulo every prime.

3. Extensions between symmetric powers

In this section we compute the groups $\mathrm{Ext}^1_{\mathbf{C}\Gamma}(S^m(V), S^n(V))$. The uniqueness result [1, Theorem 8.1] depends on the fact that $\mathrm{Ext}^1_{\mathbf{C}\Gamma}(V,V) \cong \mathbf{C}$. In the case of symmetric powers there may be inequivalent non-split extensions and, as we shall see, inequivalent T-indecomposable extensions.

Let $M_{k,\ell}(\mathbf{C})$ denote the space of $k \times \ell$ matrices with entries from $\mathbf{C}$. If ρ and σ are complex matrix representations of a group G of degrees k and ℓ respectively we let

$$I(\rho,\sigma) = \{M \in M_{k,\ell}(\mathbf{C})\} \mid \forall g \in G, \rho(g)M = M\sigma(g)\}$$

and set $i(\rho,\sigma) = \dim_{\mathbf{C}} I(\rho,\sigma)$.

The first two lemmas are proved by considering the decomposition of $S^m(V)$ into simple one-dimensional submodules for $\langle R\rangle$. We omit the details.

Lemma 3.1. *The trace of R on $S^m(V)$ is as follows.*

(1) *If m is odd then*

$$\operatorname{tr}_{\rho_m}(R) = \begin{cases} 0 & \text{if } m \equiv 5 \pmod 6, \\ 1 & \text{if } m \equiv 1 \pmod 6, \\ -1 & \text{if } m \equiv 3 \pmod 6. \end{cases}$$

(2) *If m is even then*

$$\operatorname{tr}_{\rho_m}(R) = \begin{cases} 0 & \text{if } m \equiv 2 \pmod 3, \\ 1 & \text{if } m \equiv 0 \pmod 3, \\ -1 & \text{if } m \equiv 1 \pmod 3. \end{cases}$$

Lemma 3.2. *Let m and n be nonnegative integers of the same parity. Then*

$$i(\rho_m|_{\langle R\rangle}, \rho_m|_{\langle R\rangle}) = \frac{1}{3}[(m+1)(n+1) + 2\operatorname{tr}_{\rho_m}(R)\operatorname{tr}_{\rho_n}(R)].$$

Let m and n be nonnegative integers of the same parity. Let $S(m,n)$ be the subspace of $(m+1)\times(n+1)$ complex matrices M such that $\rho_m(S)M + M\rho_n(S) = 0$ and let $s(m,n)$ be its dimension.

Lemma 3.3. *We have*

$$s(m,n) = \begin{cases} \frac{(m+1)(n+1)}{2} & \text{if } m \text{ and } n \text{ are odd}, \\ \frac{(m+1)(n+1)-1}{2} + \epsilon(m,n) & \text{if } m \text{ and } n \text{ are even}, \end{cases}$$

where $\epsilon(m,n) = 1$ if $m+n \equiv 2 \pmod 4$ and zero otherwise.

Theorem 3.4. *Let m and n be nonnegative integers. If they differ in parity then $\operatorname{Ext}^1_{\mathbf{C}\Gamma}(S^m(V), S^n(V)) = 0$. If they have the same parity then*

$$\dim_{\mathbf{C}} \operatorname{Ext}^1_{\mathbf{C}\Gamma}(S^m(V), S^n(V)) = s(m,n) + \delta_{m,n} - i(\rho_m|_{\langle R\rangle}, \rho_n|_{\langle R\rangle}).$$

(Here $\delta_{m,n}$ is the Kronecker delta.)

Proof. Since $-I \in \Gamma$ acts as $(-1)^m$ on $S^m(V)$, it is clear that

$$\operatorname{Ext}^1_{\mathbf{C}\Gamma}(S^m(V), S^n(V)) = 0$$

if m and n differ in parity.

Therefore we assume for the rest of this proof that m and n have the same parity.

Let $\hom_{m,n}(\Gamma)$ denote the set of matrix representations of the form

$$\phi(g) = \begin{pmatrix} \rho_m(g) & \tau(g) \\ 0 & \rho_n(g) \end{pmatrix}. \tag{1}$$

The representation is determined by function τ, which satisfies the cocycle condition

$$\tau(gh) = \rho_m(g)\tau(h) + \tau(g)\rho_n(h) \ \forall g,\, h \in \Gamma. \tag{2}$$

Conversely, if a function $\tau : \Gamma \to M_{m+1,n+1}(\mathbf{C})$ satisfies (2) then the function ϕ in (1) is a homomomorphism, so it lies in $\hom_{m,n}(\Gamma)$.

For $M \in M_{m+1,n+1}(\mathbf{C})$ we set

$$A(M) = \begin{pmatrix} I_{m+1} & M \\ 0 & I_{n+1}. \end{pmatrix}.$$

There is an equivalence relation on $\hom_{m,n}(\Gamma)$ where representations ϕ and ϕ' are equivalent if for some M we have $\phi'(g) = A(M)^{-1}\phi(g)A(M)$, for all $g \in \Gamma$. The set of equivalence classes is $\mathrm{Ext}^1_{\mathbf{C}\Gamma}(S^m(V), S^n(V))$.

By Maschke's Theorem, each equivalence class contains a representation of the form (1) for which $\tau(R) = 0$. Let $\hom(m, n, R, \Gamma)$ denote this subset of $\hom_{m,n}(\Gamma)$. From the defining relations of Γ, we see that a representation $\phi \in \hom(m, n, R, \Gamma)$ is determined uniquely by the choice of

$$\phi(S) = \begin{pmatrix} \rho_m(S) & \tau(S) \\ 0 & \rho_m(S) \end{pmatrix}.$$

The matrix $\phi(S)$ must satisfy

$$\phi(S)^2 = \begin{pmatrix} \rho_m(S)^2 & 0 \\ 0 & \rho_n(S)^2 \end{pmatrix}$$

but is otherwise unrestricted. It is immediate that this relation is equivalent to the condition that $\tau(S) \in S(m, n)$. In this way, we can identify $\hom(m, n, R, \Gamma)$ with $S(m, n)$.

Now suppose that ϕ and $\phi' \in \hom(m, n, R, \Gamma)$ correspond to $\tau(S)$ and $\tau'(S) \in S(m, n)$, respectively, and that they are equivalent. Thus, for some matrix $M \in M_{m+1,n+1}(\mathbf{C})$, we have

(3)

$$\phi'(R) = A(M)^{-1}\phi(R)A(M) \qquad \text{and} \qquad \phi'(S) = A(M)^{-1}\phi(S)A(M).$$

These equations are equivalent to the conditions that $M \in I(\rho_m|_{\langle R\rangle}, \rho_n|_{\langle R\rangle})$ and $\tau'(S) = \tau(S) + \rho_m(S)M - M\rho_n(S)$. Therefore, if $T(m, n)$ is the set of elements X in $S(m, n)$ which have the form $X = \rho_m(S)M - M\rho_n(S)$ for some $M \in M_{m+1,n+1}(\mathbf{C})$, then

$$\mathrm{Ext}^1_{\mathbf{C}\Gamma}(S^m(V), S^n(V)) \cong S(m, n)/T(m, n).$$

Let $\lambda : I(\rho_m|_{\langle R\rangle}, \rho_n|_{\langle R\rangle}) \to M_{m+1,n+1}(\mathbf{C})$ be defined by $\lambda(M) = \rho_m(S)M - M\rho_n(S)$. The kernel of this linear map is the set of matrices M which interwine $\rho_m(g)$ and $\rho_n(g)$, for $g = T$ and $g = S$, hence for all $g \in \Gamma$. Thus, the kernel is $\mathbf{C} \cdot I_{m+1}$ if $m = n$ and zero otherwise. The image of λ is $T(m,n)$. Therefore,

$$\dim_{\mathbf{C}} T(m,n) = i(\rho_m|_{\langle R\rangle}, \rho_m|_{\langle R\rangle}) - \delta_{m,n},$$

and the theorem now follows. □

Example 3.5. We have $\dim_{\mathbf{C}} \mathrm{Ext}^1_{\mathbf{C}\Gamma}(S^2(V), S^2(V)) = 2$. The classes of nonsplit modules are equivalent to one of the following. In all cases R is represented by

$$\begin{bmatrix} \rho_2(R) & 0 \\ 0 & \rho_2(R) \end{bmatrix}$$

and we describe the matrix for S.

If S is represented by

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

the module is uniserial, but not T-indecomposable.

For each $a \in \mathbf{C}$ we have a representation in which S acts as

$$\begin{bmatrix} 0 & 0 & 1 & 1 & a & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & a & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

These modules are T-indecomposable and inequivalent for different values of a.

Remark 3.6. Since $\mathrm{Ext}^1_{\mathbf{C}\Gamma}(S^0(V), S^0(V)) = 0$, it is clear that Theorem 2.1 does not extend without restriction to sequences of nonnegative even integers. Calculations suggest that there may exist T-indecomposable modules corresponding to sequences of nonnegative even integers which do not have two consecutive 0s, but we have not yet found a proof.

4. Uniserial modules for certain special linear groups

Let $\mathcal{O}$ be the ring of integers in a number field. We regard Γ as embedded in $\mathrm{SL}(2, \mathcal{O})$ in the standard way. We consider the existence of countably infinite-dimensional modules E' for $\mathrm{SL}(2,\mathcal{O})$ over $\mathbf{C}$ satisfying the following properties.

(1) T acts indecomposably on E'.
(2) E' has a filtration $0 = E_0 \subset E_1 \subset E_2 \subset \cdots$ such that quotient E_{i+1}/E_i is isomorphic to V.

In particular, by our previous results [1, Theorem 8.1], the restriction of E' to Γ must be isomorphic to the module E in the Introduction.

We recall from [1, Lemma 8.2] that $\mathrm{Ext}^1_{\mathbf{C}\Gamma}(V,V) \cong \mathbf{C}$ and from [1, §5] that there is a nontrivial cocycle $g : \Gamma \to M_{2,2}(\mathbf{C})$ such that

$$g(S) = 0, \qquad g(T) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \tag{4}$$

We introduce matrices

$$X(\eta) = \begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad H(\eta) = \begin{pmatrix} \eta & 0 \\ 0 & \eta^{-1}, \end{pmatrix}$$

where η is a complex number and is nonzero for $H(\eta)$.

Theorem 4.1. *Let G be a subgroup of $\mathrm{SL}(2,\mathbf{C})$ that contains Γ. Suppose that $H(\eta)$ and $X(\eta)$ lie in G for some $\eta \in \mathbf{C}^\times$. Let $f : G \to M_{2,2}(\mathbf{C})$ be a function satisfying the cocycle condition*

$$f(gh) = gf(h) + f(g)h, \quad \text{for } g,\, h \in G.$$

(a) *If f extends the special cocycle (4) on Γ then $\eta^4 = 1$.*
(b) *If f extends the zero cocycle on Γ and η is algebraic, then $f(X(\eta)) = 0$ and $f(H(\eta)) = 0$.*

Proof. The proof is by computations resulting from applying relations in G to the cocycle relation or, equivalently, to the matrix representation

$$\phi(g) = \begin{pmatrix} g & f(g) \\ 0 & g \end{pmatrix}, \quad g \in G.$$

Suppose

$$f(X(\eta)) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \qquad f(H(\eta)) = \begin{pmatrix} p & q \\ r & s \end{pmatrix}. \tag{5}$$

(a) We assume first that f extends the special cocycle. The relation $(SH(\eta))^2 = -1$ implies $r = q$ and $s = -\eta^{-2}p$. The relation $TX(\eta) = X(\eta)T$ implies $\gamma = \eta$ and $\delta = \alpha + \eta$. Thus,

$$f(X(\eta)) = \begin{pmatrix} \alpha & \beta \\ \eta & \alpha+\eta \end{pmatrix}, \qquad f(H(\eta)) = \begin{pmatrix} p & q \\ q & -\eta^{-2}p \end{pmatrix}.$$

The relation $(SH(\eta)^{-1}X(\eta))^3 = -1$ yields, after simplifications, that $\alpha = (-\eta + \eta^2)/2$ and $\beta = p + \eta^3$, so

$$f(X(\eta)) = \begin{pmatrix} \frac{-\eta+\eta^2}{2} & p + \eta^3 \\ \eta & \frac{\eta+\eta^2}{2} \end{pmatrix}, \qquad f(H(\eta)) = \begin{pmatrix} p & q \\ q & -\eta^{-2}p \end{pmatrix}. \tag{6}$$

We compute

$$f(H(\eta^2)) = H(\eta)f(H(\eta) + f(H(\eta))H(\eta) = \begin{pmatrix} 2\eta p & (\eta + \eta^{-1})q \\ (\eta + \eta^{-1})q & -2\eta^{-3}p. \end{pmatrix}. \tag{7}$$

Now (6) holds for any $\eta \in \mathbf{C}^\times$ such that $H(\eta)$ and $X(\eta) \in G$. Since $H(\eta^2) = H(\eta)^2$ and $X(\eta^2) = H(\eta)TH(\eta)^{-1}$, we can apply (6) with η^2 in place of η, replacing (p, q, η) with $(2\eta p, (\eta + \eta^{-1})q, \eta^2)$ from (7). Then we have

$$f(X(\eta^2)) = \begin{pmatrix} \frac{-\eta^2+\eta^4}{2} & 2\eta p + \eta^6 \\ \eta^2 & \frac{\eta^2+\eta^4}{2} \end{pmatrix}.$$

On the other hand we have $X(\eta^2) = H(\eta)TH(\eta)^{-1}$, which yields

$$f(X(\eta^2)) = \begin{pmatrix} -\eta q & 2\eta p + \eta^2 \\ \eta^{-2} & \eta q + 1 \end{pmatrix}.$$

From the last two equations we conclude that $\eta^4 = 1$, which proves (a).

For (b) we assume that $f(S) = 0$ and $f(T) = 0$. Starting from (5) and using the same relations in G as were used in (a), we deduce in this case that

$$f(X(\eta)) = \begin{pmatrix} 0 & -\eta^2 s \\ 0 & 0 \end{pmatrix}, \qquad f(H(\eta)) = \begin{pmatrix} -\eta^2 s & 0 \\ 0 & s \end{pmatrix}. \tag{8}$$

Since η is algebraic, let the minimal polynomial of η^2 with integer coefficients be $P(x) = \sum_{n=0}^{d} a_n x^n$. Since $X(\eta^{2n}) = H(\eta)^n TH(\eta)^{-n}$ we obtain

$$f(X(\eta^{2n})) = \begin{pmatrix} 0 & -2n\eta^{2n+1}s \\ 0 & 0 \end{pmatrix}, \quad (n \geq 1).$$

Then since $X(\eta^{2n})^{a_n} = X(a_n\eta^{2n})$ and $T^{a_0} = X(a_0)$ we can take products to obtain

$$0 = f(X(P(\eta^2))) = -\sum_{n=1}^{d} a_n 2n\eta^{2n+1}s.$$

Since the right hand side is of the form $\eta^3 s$ times a polynomial in η^2 of degree $\leq d - 1$, it follows from the minimality of d that $s = 0$. □

The only number fields without units of infinite order are $\mathbf{Q}$ and the imaginary quadratic fields $\mathbf{Q}(\sqrt{-m})$, where m is a squarefree positive integer. Let $\mathcal{O}_{-m}$ denotes the ring of integers in $\mathbf{Q}(\sqrt{-m})$. The groups $\mathrm{SL}(2, \mathcal{O}_{-m})$ are known as *Bianchi groups.* Generators and relations were obtained by Swan [2],[3].

4.1. **Representations over the ring of power series.** Let $\mathbf{C}[[t]]$ be the ring of formal complex power series in the indeterminate t. Let $\mathcal{U}$ denote the ring of matrices of the form

$$U = \left[\begin{array}{c:c:c:c:c} X^{(0)} & X^{(1)} & X^{(2)} & X^{(3)} & \dots \\ \hdashline 0 & X^{(0)} & X^{(1)} & X^{(2)} & \dots \\ \hdashline 0 & 0 & X^{(0)} & X^{(1)} & \dots \\ \hdashline 0 & 0 & 0 & X^{(0)} & \dots \\ \hdashline \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right], \tag{9}$$

where, for all $n \geq 0$,

$$X^{(n)} = \begin{bmatrix} x_{1,1}^{(n)} & x_{1,2}^{(n)} \\ x_{2,1}^{(n)} & x_{2,2}^{(n)} \end{bmatrix} \in M_2(\mathbf{C})$$

and the $X^{(n)}$ are repeated down the diagonals. The center $Z(\mathcal{U})$ consists of those matrices in which the submatrices $X^{(n)}$ are all scalar matrices. The map $Z(\mathcal{U}) \to \mathbf{C}[[t]]$ sending the matrix with $X^{(n)} = a_n I$, for all $n \geq 0$, to $\sum_{n\geq 0} a_n t^n$ is a $\mathbf{C}$-algebra isomorphism, and extends to a $\mathbf{C}[[t]]$-algebra isomorphism

$$\gamma : \mathcal{U} \to M_2(\mathbf{C}[[t]]), \qquad U \mapsto \begin{bmatrix} x_{1,1}(t) & x_{1,2}(t) \\ x_{2,1}(t) & x_{2,2}(t) \end{bmatrix}, \tag{10}$$

where

$$x_{i,j}(t) = \sum_{n=0}^{\infty} x_{i,j}^{(n)} t^n, \qquad i,\, j \in \{1, 2\}.$$

Thus, any homomorphism of a group G into $\mathcal{U}$ defines, by composition with γ, a representation $G \to \mathrm{GL}(2, \mathbf{C}[[t]])$ and vice versa, using γ^{-1}.

The representation τ_1 of Γ given in [1, Lemma 5.2(c)] has its image in $\mathcal{U}$ and the corresponding representation $\psi : \Gamma \to \mathrm{GL}(2, \mathbf{C}[[t]])$ is given by

$$\psi(S) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \psi(T) = \begin{bmatrix} 1 & 1 + g(t) \\ g(t) & 1 + t \end{bmatrix}. \tag{11}$$

Here, the power series $g(t)$ is defined by

$$1+g(t) = \sum_{k=0}^{\infty} b_k t^k, \qquad b_m = \frac{(-1)^{m-1}}{m}\binom{2m-2}{m-1}, \quad (m \geq 2), \quad b_1 = b_0 = 1.$$

(The b_n are the Catalan numbers, up to signs.) The power series $g(t)$ satisfies the identity

$$g(t)^2 + g(t) = t. \tag{12}$$

We note that the image of ψ is actually in $\mathrm{SL}(2, \mathbf{C}[[t]])$. The T-indecomposability of the representation τ_1 can be seen from $\psi(T)$ by observing that the $(2,1)$ entry $g(t)$ has zero constant term and nonzero coefficient of t.

4.2. **Examples.** We now consider the possiblity of constructing representations of the groups $G = \mathrm{SL}(2, \mathcal{O}_{-m})$ that extend the representation τ_1 (or, equivalently, ψ) of Γ.

Example 4.2. $G = \mathrm{SL}(2, \mathcal{O}_{-1})$. Let $i^2 = -1$. $\mathrm{SL}(2, \mathcal{O}_{-1})$ is generated by S,T and $J = -I$ together with

$$X = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad H = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

The defining relations are

$$\begin{aligned} J^2 = 1, \quad & J \text{ central}, \quad XT = TX, \\ S^2 = J, \quad & (TS)^3 = J, \quad H^2 = J, \quad (TH)^2 = J, \\ (XH)^2 = J, \quad & (SH)^2 = J, \quad (XSH)^3 = J. \end{aligned}$$

Let d and $e \in \mathbf{C}[[t]]$ be defined by $d^2 = t^2 + 4t + 1$, $d(0) = 1$ and $e^2 = 3t^2 + 12t + 4$, $e(0) = 2$. We claim that $\psi : \Gamma \to \mathrm{SL}(2, \mathbf{C}[[t]])$ can be extended to $\psi_{-1} : G \to \mathrm{SL}(2, \mathbf{C}[[t]])$ by setting

$$\psi_{-1}(X) = \begin{bmatrix} \frac{-it+e}{2d} & \frac{i(g+1)}{d} \\ \frac{ig}{d} & \frac{it+e}{2d} \end{bmatrix}, \qquad \psi_{-1}(H) = \begin{bmatrix} \frac{i(1+2g)}{d} & \frac{it}{d} \\ \frac{it}{d} & -\frac{i(1+2g)}{d} \end{bmatrix}.$$

In order to check that the relations hold among these 2×2 matrices, we note first that all matrices have determinant 1. Then, apart from the relations expressing commutativity of elements, every other relation is of the form $M^r = \pm I$, with $r = 2$ or 3, and these can be checked by computing traces, making use of equation (12).

Example 4.3. $G = \mathrm{SL}(2, \mathcal{O}_{-2})$. Let $\eta^2 = -2$. By [3, Theorem 10.1], the group $\mathrm{SL}(2, \mathcal{O}_{-2})$ is generated by matrices S, T and

$$X = \begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix},$$

with defining relations

$$\begin{aligned} &J^2 = 1, \quad J \text{ central}, \quad XT = TX, \\ &S^2 = J, \quad (TS)^3 = J, \quad (SX^{-1}SX)^2 = J. \end{aligned}$$

Let $d \in \mathbf{C}[[t]]$ be defined by $d^2 = t^2 + 4t + 1$, $d(0) = 1$ We claim that $\psi : \Gamma \to \mathrm{SL}(2, \mathbf{C}[[t]])$ can be extended to $\psi_{-2} : G \to \mathrm{SL}(2, \mathbf{C}[[t]])$ by setting

$$\psi_{-2}(X) = \begin{bmatrix} \frac{-\eta t + \sqrt{2}\sqrt{d^2+1}}{2d} & \frac{\eta(g+1)}{d} \\ \frac{\eta g}{d} & \frac{\eta t + \sqrt{2}\sqrt{d^2+1}}{2d} \end{bmatrix}.$$

We leave to the reader the task of verifying that the defining relations among the group generators are satisfied by their images under ψ. As in the previous example, it is helpful to consider traces and to make use of the equation (12).

Example 4.4. $G = \mathrm{SL}(2, \mathcal{O}_{-3})$. The ring $\mathcal{O}_{-3}$ is the ring of cyclotomic integers of order 3. By [3, Theorem 6.1], G is generated by Γ together with the two elements

$$X = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix},$$

where $\omega = \frac{1}{2}(-1 + \sqrt{-3})$. We will show that $\mathrm{Ext}^1_{\mathbf{C}\,\mathrm{SL}(2,\mathcal{O}_{-3})}(V, V) = 0$. Now $\mathrm{Ext}^1_{\mathbf{C}\Gamma}(V, V) = \mathbf{C}$ and a nontrivial class is represented by the special cocycle (4). Theorem 4.1(a) implies that the special cocycle cannot be extended to a cocycle on G and Theorem 4.1 (b) shows that any extension of the zero cocycle to must vanish at X and at H, hence on all of G.

Example 4.5. $G = \mathrm{SL}(2, \mathcal{O}_{-5})$. Let $\eta^2 = -5$. The group $\mathrm{SL}(2, \mathcal{O}_{-5})$ is generated by matrices S, T, together with:

$$X = \begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} -\eta & 2 \\ 2 & \eta \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} -\eta - 4 & -2\eta \\ 2\eta & \eta - 4 \end{pmatrix}.$$

The defining relations (from [3]) are :

$$\begin{aligned} J^2 &= I, \quad J \text{ central}, \quad XT = TX, \\ S^2 &= J, \quad B^2 = J, \quad (TS)^3 = J, \\ (SB)^2 &= J, \quad (SXBX^{-1})^2 = J, \quad C(ST) = (ST)C, \\ & \quad (T^{-1}XBX^{-1})C = C(-T^{-1}B^{-1}). \end{aligned}$$

Let g be the power series of our paper with Catalan number coefficients, and let d, f and h be power series defined by:

$$d^2 = 1 + 4t + t^2, \quad f^2 = 4 - 4t - t^2, \quad h^2 = 1 - 16t^2 - 8t^3 - t^4$$

with the constant terms chosen to be positive in all three cases.

Then $\psi : \Gamma \to \mathrm{SL}(2, \mathbf{C}[[t]])$ can be extended to $\psi_{-5} : G \to \mathrm{SL}(2, \mathbf{C}[[t]])$ by setting

$$\psi_{-5}(X) = \begin{bmatrix} p & q \\ r & s \end{bmatrix}, \quad p = \frac{-\eta t + f}{2d}, \quad q = \frac{\eta(1+g)}{d}, \quad r = \frac{\eta g}{d}, \quad s = \frac{\eta t + f}{2d},$$

$$\psi_{-5}(B) = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}, \quad a = \frac{5 - t^2}{\eta(2g+1) - tf}, \quad b = \frac{t(2g+1) + \eta f}{\eta(2g+1) - tf},$$

$$\psi_{-5}(C) = \begin{bmatrix} v & w(1+t) \\ -w & v - (2g+1)w \end{bmatrix}, \quad w = \frac{-2\eta}{h}, \quad v = \frac{-(t+2)f - \eta(2g+1)}{h}.$$

The lengthy checking of the relations is omitted. We carried it out with the aid of the computer algebra system *Macaulay2* [5].

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