Finite Group Theory. By I. Martin Isaacs. American Mathematical Society, Providence, Rhode Island, 2008, xi+350pp., ISBN 978-0-8218-4344-4, \$???.

Reviewed by **Peter Sin**

In the preface of Finite Group Theory the author, I. Martin Isaacs, states that his principal reason for writing the book was to expose students to the beauty of the subject. Finite group theory is indeed a subject which has both beautiful theory and beautiful examples. The simplicity and elegance of the group axioms have made group theory an almost universal choice as a starting point in the teaching of abstract mathematics. But Isaacs had more than this in mind. Throughout the history of the subject there have been many examples of theorems with proofs which are ingenious, highly original, or which establish an important new general principle, proofs of great aesthetic value.

Everyone who has taken a graduate algebra course is aware of Sylow's theorems, and of the noticable increase in depth of the discussion which follows them. A slightly less well known example is Frattini's proof that the intersection of all maximal subgroups (now called the Fratttini subgroup in his honor) must be nilpotent. This short proof introduced two basic mathematical principles. One is the key algebraic concept of a radical, which also underlies many fundamental results such as Nakayama's Lemma on commutative rings. The other is the importance of transitive group actions and, specifically, the method of applying Sylow's theorems which has become known as the Frattini Argument. Another classical theorem whose proof has an almost magical quality is Burnside's $p^a q^b$ theorem, which states that a group whose order is divisible by at most two primes must be solvable. Its wonderful proof was one of the earliest major applications of character theory. We will return to it later.

The first half of the twentieth century witnessed the definitive work of P. Hall, whose generalizations of Sylow's theorems illuminated the structure of solvable finite groups.

The modern era of finite group theory began around 1959, when a number of startlingly original and powerful ideas were introduced by Thompson, first to prove a conjecture of Frobenius, then the Feit-Thompson Odd Order Theorem and the classification of the N-groups, which include all simple groups with the property that every proper subgroup is solvable. These results caused an explosion of research leading eventually to the classification of the simple finite groups. The classification is a composite of deep results by many contributors, including major achievements by Aschbacher. Much of this vast body of research is technically very challenging and well beyond the scope of most graduate courses. A large portion of it has been redacted with great care and skill in the series by Gorenstein, Lyons and Solomon [3],[4],[5],[6],[7],[8].

In writing his book, Isaacs has selected some the the gems of the theory, including Thompson's proof of Frobenius's [1] conjecture on the nilpotence of Frobenius kernels, and made them accessible to beginning graduate students.

Great beauty is also to be found in the finite groups themselves. We can all admire the perfect symmetry of the platonic solids, and we have been puzzled by Rubik's cube. These tiny glimpses of the multifaceted world of finite groups give a hint of the treasure to be found within. Even if we restrict ourselves to simple groups, there is a bewildering array, including many strange and exotic groups acting as symmetries of geometric objects of all dimensions. Among the simple groups there are some which belong to infinite families. The alternating groups make up the most familiar series of simple groups and often the only one encountered in introductory courses. There are other families which are parametrized by finite fields and root systems, each with its own geometry. These families, constructed by Chevalley from Lie algebras, are finite analogs of simple Lie groups. A good way to understand the structure of these groups is to study their geometries. The theory of buildings was developed by Tits for this purpose. To give an example, we can consider the projective special linear groups, discussed in Chapter 8 of Finite Group Theory. The building of such a group is the simplicial complex whose simplices are the chains of subspaces in the "standard" vector space on which the special linear group acts. Tits' classification of buildings is a key element in the classification theorem for simple groups. Then there are 26 sporadic simple groups which do not belong to the infinite families, many of which are still quite mysterious. The largest of these groups, discovered by Fischer and Griess, is the now famous Monster group. Work of McKay, Thompson, Conway, Norton and Borcherds has revealed amazing connections between the Monster, modular forms and vertex operator algebras. Who knows what other surprises may be in store?

Because the simple groups are so absorbing and since they connect finite

[1]In the published version of this review the conjecture was erroneously attributed to Burnside.

group theory to other parts of mathematics such as Lie algebras, geometry, number theory and combinatorics, a large part of current research in finite groups is directed towards obtaining explicit information about the subgroup structure and representation theory of simple groups.

In the 1980s a huge amount of detailed information about a lot of groups was computed and compiled into the Atlas of Finite Groups [2] in an appropriately oversized volume. This has proved to be a very valuable source of information, especially for testing conjectures.

The gathering of detailed information about simple groups was also an integral part of the classification program. In order to classify the simple groups it is necessary prove theorems which state that a simple group with certain properties must belong to a list of known examples. One considers a minimal counterexample and attempts to reach a contradiction. By minimality, every composition factor of every proper subgroup is a known simple group. At this point delicate properties of the known simple groups are often needed in a detailed analysis to reach a contradiction.

Enthusiasts of finite groups are fortunate to live in the age of computers. With programs such as the freely available GAP (Groups, Algorithms, Programming), students can easily gain hands-on experience with a rich collection of groups, including some sporadic ones. The increasing role of computers has also brought many benefits to researchers. It is now much easier to test conjectures and to avoid following false leads, and a wealth of information, including a version of the Atlas, is accessible online. Computational finite group theory has grown rapidly into a flourishing research area in its own right.

Isaacs' book is based on an intermediate level graduate course. The constraints of such a course force an instructor to be selective about content. Isaacs has chosen to emphasize the general principles of finite group theory and its beautiful arguments rather than to delve into the fascinating examples or the computational aspects. The book is well crafted with close attention paid to precise exposition while maintaining a friendly conversational style. Those who have attended the author's lectures will also recognize the drawings used to depict the inclusions and intersections of subgroups in groups, which have almost become Isaacs' trademarks. A beginning student will learn useful general principles of finite group theory from this book and gain an appreciation of the elegance of the field. There are hardly any prerequisites and the pace is moderate enough that with a little guidance an advanced undergraduate could study it. The exercises are well composed and have the nice property that their solutions can be deduced from the statements in the book. A few are challenging and many are fun to solve. Although introductory in nature, the book contains several items not commonly covered. The chapters on subnormal subgroups treat this topic very clearly and more thoroughly than other texts and include several interesting old results which this reviewer had not seen before.

For me, the highlight of the book is the account of Burnside's $p^a q^b$ theorem mentioned earlier, which includes some interesting history as well as mathematics. The question was raised long ago whether Burnside's result could be deduced without the use of algebraic integers through character theory, but instead by purely combinatorial arguments derived from Sylow's theorems. In the seventies, such a proof was obtained by combining work of Goldschmidt, Bender and Matsyuama. However, as the entire proof was not in a single paper it was not particularly easy to read. In this book we see the whole proof, reworked to some extent by Isaacs. His presentation is very smooth, bringing the important ideas to the fore. Although the proof is "elementary" in the sense of not using characters, it is actually quite an advanced argument which illustrates a number of useful techniques. There are arguments about involutions, which are reminiscent of Brauer's early contributions, and standard arguments on generation of groups by centralizers of elements. Moreover, the proof is based on a method of Bender, similar to one by which Bender succeeded in simplifying part of the proof of the Odd Order Theorem.

A small quirk of the book is the conspicuous absence of references to original papers and further reading. The addition of a bibliography to any future edition would be helpful to readers without access to MathSciNet.

Finite Group Theory was designed to provide the necessary group-theoretical background for the author's students of representation theory. For this purpose it makes a suitable companion for the excellent text on character theory by the same author. Readers who wish to pursue the subject of finite groups further will be ready to progress to more advanced texts such as the book by Aschbacher [1] of the same title or the second volume of the series by Gorenstein, Lyons and Solomon [4].

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