The Divisor Matrix, Dirichlet Series and $\mathrm{SL}(2, \mathbf{Z})$

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Overview

- $\mathcal{A} :=$ the ring of matrices $A = (a_{i,j})_{i,j \in \mathbf{N}}$, with rational entries, such that each column has only finitely many nonzero entries.
- $\mathcal{A}$ operates by left multiplication on the space E of finitely supported column vectors.
- $\mathcal{A}$ operates by right multiplication on $Q^{\mathbf{N}} \cong E^*$, the space of sequences of rational numbers.
- $$(fA)(n) = \sum_{m \in \mathbf{N}} a_{m,n} f(m), \qquad f \in \mathbf{Q}^{\mathbf{N}}, A \in \mathcal{A}.$$

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Dirichlet Space and Dirichlet Ring

- $$\mathcal{DS} := \{f \in \mathbf{Q}^{\mathbf{N}} \mid (\exists C, c > 0)(\forall n)(|f(n)| \leq Cn^c)\}$$
- $f \in \mathcal{DS}$ if and only if $\sum_n f(n)n^{-s}$ converges for some complex number s.
- $\mathcal{DR} :=$ the subring of $\mathcal{A}$ consisting of all elements which leave $\mathcal{DS}$ invariant.

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The divisor matrix $D = (d_{i,j})_{i,j \in \mathbf{N}}$ defined by

$$d_{i,j} = \begin{cases} 1, & \text{if } i \text{ divides } j, \\ 0 & \text{otherwise.} \end{cases}$$

$$G := \mathrm{SL}(2, \mathbf{Z}) = \langle S, R \mid S^4, R^6, S^2 = R^3 \rangle$$

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \text{and} \quad T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Main Theorem

There exists a representation $\rho : \mathrm{SL}(2, \mathbf{Z}) \to \mathcal{A}^\times$ with the following properties.

(a) *The space E (finitely supported columns) has an ascending filtration*

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots$$

of $\mathbf{Q}\,\mathrm{SL}(2, \mathbf{Z})$-submodules such that for each $i \in \mathbf{N}$, the quotient module E_i/E_{i-1} is isomorphic to the standard 2-dimensional $\mathbf{Q}\,\mathrm{SL}(2, \mathbf{Z})$-module.

(b) $\rho(T) = D$.

(c) *$\rho(Y)$ is an integer matrix for every $Y \in \mathrm{SL}(2, \mathbf{Z})$.*

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- Find the transition matrices explicitly.
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- For $m, k \in \mathbf{N}$, let

$$A_k(m) := \{(m_1, m_2, \ldots, m_k) \in (\mathbf{N}\backslash\{1\})^k \mid m_1 m_2 \cdots m_k = m\}$$

- $\alpha_k(m) := |A_k(m)|$
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By considering the first $k-1$ entries of elements of $A_k(m)$, we see that for $k > 1$, we have

Counting Lemma

- $$\alpha_k(m) = \left(\sum_{d|m} \alpha_{k-1}(d)\right) - \alpha_{k-1}(m).$$

- $$\sum_{i=1}^{k-1} (-1)^{k-1-i} \sum_{d|m} \alpha_i(d) = \alpha_k(m) + (-1)^k \alpha_1(m).$$

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Relation to D

- The $(1, m)$ entry of $(D - I)^k$ is equal to $\alpha_k(m)$
- Proof: Let $D - I = (t_{i,j})_{i,j \in \mathbf{N}}$. Then

$$t_{1,j_1} t_{j_1,j_2} \cdots t_{j_{k-1},m} = 1 \iff (j_1, j_2/j_1, \ldots, m/j_{k-1}) \in A_k(m)$$

- More generally,

$$(d, m) \text{ entry of } (D - I)^k = \begin{cases} 0 & \text{if } d \nmid m, \\ \alpha_k(m/d) = (1, m/d) \text{ entry, if } d \mid m. \end{cases}$$

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- Think of J as being the direct sum of infinite Jordan blocks, one for each odd integer.
- Let $Z := (\alpha(i,j))_{i,j\in\mathbf{N}}$ be the matrix described in the following way.
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The matrix Z has the following properties:

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(b) *If $i = d2^k$, where d is odd and $k \geq 1$, then*

$$\alpha(i,j) = \begin{cases} \alpha_k(j/d) & \text{if } d \mid j, \\ 0 & \text{otherwise.} \end{cases}$$

(c) $\alpha(im, jm) = \alpha(i,j)$ *whenever m is odd.*

(d) *Z is upper unitriangular.*

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$$Z(D - I) = (J - I)Z$$

(Proof: Look at the i-th row of both sides, $i = 2^k d$.)

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- We still need to find an explicit formula for Z^{-1}.
- Notation: For each prime p and each integer m let $v_p(m)$ denote the exponent of the highest power of p which divides m and let $v(m) := \sum_p v_p(m)$.
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- It suffices to show that

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or, equivalently,

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- $XDX = (d'_{i,j})_{i,j\in\mathbf{N}}$, $XJX = (c'_{i,j})_{i,j\in\mathbf{N}}$

$$d'_{i,j} = \begin{cases} (-1)^{v_2(i)+v_2(j)}, & \text{if } i \mid j, \\ 0 & \text{otherwise.} \end{cases}, \qquad c'_{i,j} = \begin{cases} 1, & \text{if } j = i, \\ -1, & \text{if } j = 2i, \\ 0 & \text{otherwise.} \end{cases}$$

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- Thus, we aim for the equation:

$$\sum_{m\geq 1}(-1)^{v_2(m)}\alpha(im,j) = \begin{cases}\alpha(i,j), & \text{if } j \text{ is odd,}\\ \alpha(i,j)-\alpha(i,j/2), & \text{if } j \text{ is even.}\end{cases}$$

- Only need to consider the case $i = 2^k$, for $k \geq 1$.
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Proof of claim:

$$(\forall j) \qquad (-1)^k \sum_{d|j} \sum_{r=k}^{v(j/d)} (-1)^r \alpha_r(j/d) = \alpha_k(j).$$

- We can assume $j > 1$.

Alternating Sum Lemma

We have for all $m \in \mathbf{N}$,

$$\sum_{r=1}^{v(m)} (-1)^r \alpha_r(m) = \begin{cases} (-1)^{v(m)}, & \text{if } m \text{ is squarefree and } m > 1, \\ 0 & \text{otherwise.} \end{cases}$$

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- Finally, rewrite the above expression as

$$(-1)^{k-1} + \sum_{r=1}^{k-1} (-1)^{k-1-r} \sum_{d|j} \alpha_r(d),$$

which, by the Counting Lemma is equal to $\alpha_k(j)$. This proves the claim. □

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- The case $m = 1$ is trivial.
- $m = p_1^{\mu_1} p_2^{\mu_2} \cdots p_r^{\mu_r}$, with $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r \geq 1$. Let $n = v(m) = \mu_1 + \cdots + \mu_r$.
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- $N := \{1, \ldots, n\}$.
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- A k-decomposition of n is a k-tuple $(n_1, \ldots, n_k)$ of integers $n_i \geq 1$ such that $n_1 + n_2 + \cdots + n_k = n$.
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- For each pair $((n_1, \ldots, n_k), h)$ consisting of a k-decomposition and a function $h \in F_\mu$, we obtain an element $(m_1, \ldots, m_k) \in A_k(m)$ by setting $m_i := \prod_{j \in N_i} h(j)$.
- Every element of $A_k(m)$ arises in this way and two pairs define the same element of $A_k(m)$ if and only if the k-decompositions are equal and the corresponding functions are in the same orbit under the action of the parabolic subgroup of the k-decomposition.
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- L. Solomon's formula:

$$\sum_{K\subseteq\Pi}(-1)^{|K|}1_{W_K}^{S_n}=\epsilon,$$

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$$\begin{aligned}\sum_{k=1}^{n}(-1)^k\alpha_k(m) &= (-1)^n\langle\sum_{K\subseteq\Pi}(-1)^{|K|}1_{W_K}^{S_n},1_{S_\mu}^{S_n}\rangle\\ &= (-1)^n\langle\epsilon,1_{S_\mu}^{S_n}\rangle\\ &= (-1)^n\langle\epsilon,1\rangle_{S_\mu}\\ &= \begin{cases}(-1)^n, & \text{if } \mu=1^n,\\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

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