Geometry and Representation Theory of $\mathrm{Sp}(4, 2^t)$ and $\mathrm{Sz}(2^t)$

Peter Sin,
University of Florida

Presented at University of Delaware, 17th November, 2006

1. Geometry of symplectic 3-space

We begin with Tits' construction [7].

- V be a 4-diml. vector space, coordinates x_i, $i = 0, 1, 2, 3$.
- W 2-diml. subspace, $\wedge^2 W$ is a point of $\mathbb{P}(\wedge^2 V)$.
- If W is spanned by (a_0, a_1, a_2, a_3) and (b_0, b_1, b_2, b_3) then the "Plücker" coordinates. of W are $(p_{01} : p_{02} : p_{03} : p_{12} : p_{13} : p_{23})$, with $p_{ij} = a_i b_j - a_j b_i$.
- These coordinates satisfy the quadratic form

$$p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 0. \qquad (1)$$

and form the *Klein Quadric* $\widehat{Q}$

Assume that V has a nonsingular alternating bilinear form and x_i are symplectic coordinates so that the matrix of the form is $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$.

- A 2-subspace is t.i. iff $p_{03} + p_{12} = 0$.
- The t.i. 2-subspaces form the intersection $Q = \widehat{Q} \cap H$ of $\widehat{Q}$ with the hyperplane H of the above equation.
- The equation of Q is

$$p_{01}p_{23} - p_{02}p_{13} - p_{03}^2 = 0. \tag{2}$$

2. The isogeny τ

Suppose the field of V is $k = \overline{\mathbf{F}}_2$.

- $z = (0:0:1:1:0:0) \in H \setminus Q$ is the radical of the (alternating) bilinear form associated with (2).
- z is the common point of intersection of every tangent hyperplane to Q in H.
- Projection $H \to V_1 = H/z$ gives a bijection $Q \to \mathbb{P}(V_1)$.
- $$\alpha : \{\text{2-diml tot. isotropic subspaces of } V\} \cong \mathbb{P}(V_1).$$
- The alternating form induced on V_1 is nonsingular.
- $y_0 = \overline{p}_{01},\ y_1 = \overline{p}_{02},\ y_2 = \overline{p}_{13},\ y_3 = \overline{p}_{23}$ are symplectic coords for V_1.

V_1 is a lot like V !

- Identify V with V_1 by their symplectic coordinates.
- This fixes an isomorphism $\mathrm{Sp}(V) \cong \mathrm{Sp}(V_1)$.
- Under this identification, the induced action on V_1 induces an endomorphism τ of $\mathrm{Sp}(V)$.

- $x = (a_0 : a_1 : a_2 : a_3)$. Assume for simplicity $a_0 \neq 0$.
- $x^\perp$ is spanned by x, $(0 : a_0 : 0 : a_2)$ and $(0 : 0 : a_0 : a_1)$.
- The set of t.i. 2-subspaces which contain x form an isotropic line in Q, spanned by $(a_0^2 : 0 : a_0a_2 : a_0a_2 : a_0a_3 + a_1a_2 : a_2^2)$ and $(0 : a_0^2 : a_0a_1 : a_0a_1 : a_1^2 : a_0a_3 + a_1a_2)$.
- This line maps to the t.i. line spanned by $(a_0^2 : 0 : a_0a_3 + a_1a_2 : a_2^2)$ and $(0 : a_0^2 : a_1^2 : a_0a_3 + a_1a_2)$.
- $\beta : \mathbb{P}(V) \to \{\text{t.i. lines of } \mathbb{P}(V)\}$.
- Compute Plučker coordinates: $\alpha(\beta(x)) = (a_0^2 : a_1^2 : a_2^2 : a_3^2)$.
- Conclude that β is a bijection and τ^2 is the Frobenius map, given by squaring all matrix entries.
- τ is an *isogeny* of algebraic groups.

3. The groups $G(n)$

- $G(n)$ = the subgroup of $\mathrm{Sp}(V)$ fixed by τ^n.
- $G(2t) \cong \mathrm{Sp}(4, 2^t)$.
- $G(2m+1) = \mathrm{Sz}(2^{2m+1})$, *Suzuki* groups.

For $x = (a_0 : a_1 : a_2 : a_3)$, set $x^{(2^i)} = ({a_0}^{2^i} : {a_1}^{2^i} : {a_2}^{2^i} : {a_3}^{2^i})$. Then $G(2m+1)$ preserves the set

$$\mathcal{T} = \{x \mid x = x^{(2^{2m+1})}, x^{(2^{m+1})} \in \beta(x)\}.$$

This is called the *Tits ovoid.* It consists of $(0 : 0 : 0 : 1)$ and the points $(1 : x : y : z)$ satisfying

$$z = xy + x^{2^{m+1}+2} + y^{2^{m+1}}. \tag{3}$$

4. Irreducible representations

- $N = \{0, 1, \ldots, n-1\}$.
- For $i \in N$, $V_i = V^{(\tau^i)}$, the $\mathrm{Sp}(4,k)$-module V "twisted" by τ^i, i.e. an element g acts on V_i as $\tau^i(g)$ acts on the standard module V.
- For $I \subset N$, set $V_I = \bigotimes_{i \in I} V_i$.

The 2^n modules V_I are a complete set of nonisomorphic simple modules for $kG(n)$-modules. (Steinberg's Tensor Product Theorem.)

5. Extensions of simple modules

When does there exist short exact sequence

$$0 \to V_J \to E \to V_I \to 0 \tag{4}$$

which does not split?

Theorem. *([5]) Let I, $J \subseteq N$. Then*

$$\mathrm{Ext}^1_{kG(n)}(V_I, V_J) \cong \begin{cases} k, & \text{if } I \triangle J = \{i\},\ i-1 \notin I \cap J; \\ 0, & otherwise. \end{cases} \tag{5}$$

6. The Generalized Quadrangle $W(q)$

Fix $n = 2t$, $q = 2^t$, $V(q) \leq V$, the $\mathbb{F}_q$ – *span* of the given symplectic basis of V, $G = \mathrm{Sp}(V(q)) \cong G(n)$.

- $P = \{$1-diml. subspaces of $V(q)\}$,
- $L = \{$2-diml. tot.isotropic subspaces of $V(q)\}$.
- $W(q)$ is the incidence system (P, L).
- k^P, k^L vector spaces with bases P, L.
- $\eta : k^L \to k^P$, $\ell \mapsto \sum_{p \in \ell} p$, incidence map.
- η is a homomorphism of kG-modules; its image $\mathcal{C} \leq k^P$ is the *code* of $W(q)$.

6.1. Structure of $\mathcal{C}$

Theorem. *([4])*

$$\begin{aligned}\dim \mathcal{C} &= 1 + \left(\frac{1+\sqrt{17}}{2}\right)^{2t} + \left(\frac{1-\sqrt{17}}{2}\right)^{2t} \\ &= 1 + \sum_{I \in \mathcal{N}} 4^{|I|},\end{aligned} \tag{6}$$

where $\mathcal{N}$ is the set of subsets of $N = \mathbf{Z}/2t\mathbf{Z}$ which contain no two consecutive elements.

6.2. Main ideas of the proof

- $\mathrm{Hom}_{kG}(S^2(V), k) \cong k$, spanned by the trace map.
- $U = \ker(\mathrm{trace})$, is a uniserial module with descending series V_1, k, V_2.
- Let $M = \bigotimes_{i=0}^{t-1} U^{(2^i)}$.

Using the theorem on extensions, one shows that M has a submodule R such that the composition factors of R are precisely those composition factors V_I of M with $I \notin \mathcal{N}$, and M/R has composition factors V_J, one for each $J \in \mathcal{N}$. Finally, using results of [3] and [1], it can be proved that

$$\mathcal{C} \cong k \oplus M/R.$$

7. Open Problems.

- Give an analogous construction of ${}^2F_4(2^{2m+1})$.
- Compute the code of the $G_2(3^t)$ generalized hexagon. The geometric construction was given by Tits [7] and the simple module extensions were classified in [6].
- Compute the integral invariants of the incidence matrix of the symplectic generalized quadrangles.
- Compute the 2-ranks of the incidence matrices for 1-subspaces and t.i. subspaces of a fixed dimension in a symplectic vector space (of characteristic 2 and dimension ≥ 6).
- Determine the exact structure of the subcode generated by Tits ovoids in $W(q)$. (Bagchi and Sastry [2] showed that the characteristic function of a Tits ovoid belongs to $\mathcal{C}$.)

References

[1] J. L. Alperin, Projective modules for $\mathrm{SL}(2, 2^n)$, J. Pure and Applied Algebra **15** (1979) 219–234.

[2] B. Bagchi, N. S. N. Sastry, Intersection pattern of classical ovoids in symplectic 3-space of even order, J. Algebra **26** (1989) 147–160.

[3] C. W. Curtis, Modular representations of finite groups with split BN-pairs, pages 57-95 in Seminar on Algebraic Groups and Related Finite Groups, Lecture Notes in Mathematics 131, Springer, Berlin (1969).

[4] N. S. N. Sastry, P. Sin, The code of a regular generalized quadrangle of even order, Proc. Symposia in Pure Mathematics **63** (1998), 485–496.

[5] P. Sin, Extensions of simple modules for $\mathrm{Sp}_4(2^n)$ and $Suz(2^m)$, Bull. Lond. Math. Soc. **24** (1992), 159–164.

[6] P. Sin, Extensions of simple modules for $G_2(3^n)$ and ${}^2G_2(3^m)$, Proc. Lond. Math. Soc. **66** (1993), 327–357.

[7] J. Tits, Les Groupes simples de Suzuki et de Ree, Sem. Bourbaki, Exp. 210, (1961).