Erdös-Ko-Rado Theorems for Permutation Groups

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Overview

Erdös-Ko-Rado

Generalizations

EKR for permutation groups

2-transitive groups

EKR-module property

Erdös-Ko-Rado Theorem (1961)

Let X be a set of n elements. What is the maximum size M of a family of subsets of size k such that any two subsets in the family have nonempty intersection?

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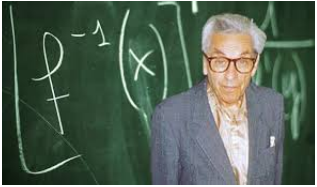
Theorem

$M = \binom{n-1}{k-1}$. Furthermore, if $k < n/2$, then a maximum familiy must be the family of k-subsets containing a fixed element.

EKR



Paul Erdös, Chao Ko and Richard Rado

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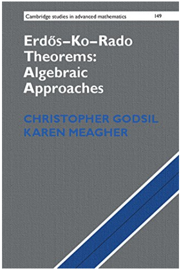

Karen Meagher, Chris Godsil

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- Perfect matchings (K. Meagher, L. Moura, 2005)
- k-tuples (M. Livingston, 1979)
- Permutations: View a permutation g as a set of pairs $(i, g(i))$. Two permutations g and h intersect iff for some i we have $g(i) = h(i)$ iff $g^{-1}h$ lies in the stabilizer of a point. Thus the elements of a point stabilizer form an intersecting set.

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We say (G, X) has the **strict EKR property** if it has the EKR property and the only intersecting sets of size $|G|/n$ are the canonical ones.

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- An intersecting set is the same thing as a coclique of the derangement graph.

Clique-coclique bound

Γ vertex-transitive graph on v vertices

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So $\alpha c \leq cv/\omega$ $\square$.

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(Burnside). If (G, X) is a 2-transitive permutation group then one of the following holds.

(a) *$|X|$ is a prime power p^r, and G has a normal, elementary abelian subgroup N acting regularly on X.*

(b) *G is an almost simple nonabelian group.*

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Example: $(G, X) = (\mathbb{F}_q \rtimes \mathbb{F}_q^\times, \mathbb{F}_q)$. Here Γ_G is a disjoint union of cliques. Any choice of one vertex from each clique gives a maximum coclique. So G does not have the strict EKR property.

All 2-transitive groups have EKR property

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Proof requires the classification of finite simple groups.

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Example of almost simple group not satisfying strict EKR property: $(G, X) = (\mathrm{PGL}(3, q), \mathrm{PG}(2, q))$, line-stabilizers are also maximum intersecting sets.

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P. Spiga (2019): In $(\mathrm{PGL}(n, q), \mathrm{PG}(n-1, q)$ Maximum intersecting sets must be cosets of point stabilizers or hyperplane stabilizers.

Strict EKR property for 2-transitive groups

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- $\mathrm{PSL}(2, q)$ (Long-Plaza-Sin-Xiang)

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- Strict EKR property $\implies$ EKR-module property $\implies$ EKR property
- EKR-module property is used to prove Strict EKR using the **Module Method**

The Module Method (B. Ahmadi, K. Meagher)

Let M be the $(0,1)$-matrix whose rows are indexed by the derangements and whose columns are indexed by ordered pairs (x, y) of distinct elements of X.

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Theorem

If (G, X) satsisfies the EKR-module property and $\operatorname{rank} M = (|X| - 1)(|X| - 2)$ then (G, X) satisfies the strict EKR property.

All 2-transitive groups have the EKR-module property

Theorem

(Meagher-Sin, 2020) Every 2-transitive permutation group satisfies the EKR-module property.

Noncanonical maximum intersecting sets

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Lemma

If H is a nonstandard complement to N and each p-element of H is conjugate to an element of G_x, then H is a noncanonical maximum intersecting set.

Example

$G = N \rtimes \mathrm{SL}(2,4)$, $N = \mathbb{F}_4^2$, viewed as matrices $\begin{pmatrix} L & v \\ 0 & 1 \end{pmatrix}$, where $L \in \mathrm{SL}_2(4)$ and $v \in \mathbb{F}_4^2$.

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The elements

$$t = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad u = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad s = \begin{pmatrix} 0 & 1 & 0 \\ 1 & \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(α a primitive root) have orders 2, 2 and 5. The subgroup $\langle t, s\rangle$ is a point stabilizer, while $\langle tu, s\rangle$ is a nonstandard complement, satisfying the hypothesis of the Lemma.

The End. Thank you!