SOME WEYL MODULES OF THE ALGEBRAIC GROUPS OF TYPE E_6

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Abstract. Let G be a simple algebraic group of type E_6 over an algebraically closed field of characteristic $p > 0$. We determine the submodule structure of the Weyl modules with highest weight $r\omega_1$ for $0 \leq r \leq p-1$, where ω_1 is the fundamental weight of the standard 27-dimensional module. In the process, the structures of other Weyl modules with highest weights linked to $r\omega_1$ are also found.

1. Introduction

In this note we study certain Weyl modules for a simple, simply connected algebraic group G of type E_6 over an algebraically closed field of characteristic $p > 0$. The modules we consider are for highest weights which are of the form $r\omega_1$, $0 \leq r \leq p-1$, where ω_1 is the highest weight of the "standard" 27-dimensional module, and we will give a full description of their G-submodules. If P is the maximal parabolic subgroup stabilizing the highest weight vector in the 27-dimensional module $H^0(\omega_1)^*$, then the embedding of the projective variety G/P for the associated line bundle is projectively normal [3], so the homogeneous coordinate ring is $\bigoplus_{r\geq 0} H^0(r\omega_1)$.

As a consequence of Steinberg's Tensor Product Theorem [5], our results also describe the simple G-socles of the modules $H^0(r\omega_1)$ for all $r \geq 0$.

Our labelling of the fundamental roots and weights is according to Figure 1. We describe the E_6 root system as follows. Let e_i, $i = 1, \ldots, 8$ be an orthonomal basis of an 8-dimensional Euclidean space. Then, in coordinates, our root system R is the union of the set

$$\{\pm e_i \pm e_j \mid 4 \leq i < j \leq 8\}$$

with the set

$$\{\pm\frac{1}{2}[(e_1 - e_2 - e_3) + \sum_{i=5}^{8} \pm e_i] \mid \text{number of minus signs is even}\}.$$

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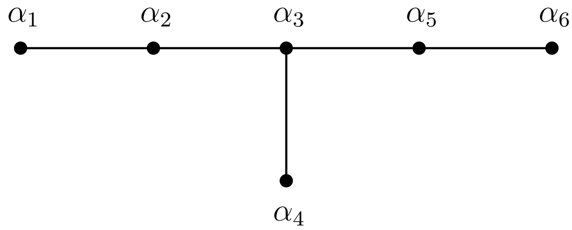

FIGURE 1

A set of fundamental roots is

$$S = \{\alpha_1 = e_4 - e_5, \alpha_2 = e_5 - e_6, \alpha_3 = e_6 - e_7, \alpha_4 = e_7 + e_8,$$
$$\alpha_5 = e_7 - e_8, \alpha_6 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8)\}.$$

The fundamental dominant weights have coordinates

$$\omega_1 = \frac{1}{3}(1, -1, -1, 3, 0, 0, 0, 0), \quad \omega_2 = \frac{1}{3}(2, -2, -2, 3, 3, 0, 0, 0), \quad \omega_3 = (1, -1, -1, 1, 1, 1, 0, 0),$$
$$\omega_4 = \frac{1}{2}(1, -1, -1, 1, 1, 1, 1, 1), \quad \omega_5 = \frac{1}{6}(5, -5, -5, 3, 3, 3, 3, -3), \quad \omega_6 = \frac{1}{3}(2, -2, -2, 0, 0, 0, 0, 0).$$

Our notation will be standard, following [2]. In particular we denote the Weyl module with highest weight λ by $V(\lambda)$ and its simple quotient by $L(\lambda)$. By definition, $V(\lambda) = H^0(-w_0\lambda)^*$, where w_0 is the longest element of the Weyl group [2, II.2.13]. Also, $V(\lambda) \cong {}^{\tau}H^0(\lambda)$, for a certain anti-automorphism τ of G that induces the identity map on characters [2, II. 2.12]. As $-w_0\omega_1 = \omega_6$, the submodule structure of $V(r\omega_1)$ will yield the submodule structures of $V(r\omega_6)$, $H^0(r\omega_1)$ and $H^0(r\omega_6)$ by applying τ and duality.

Theorem 1.1. *Let G be a simply connected, semisimple algebraic group of type E_6 over an algebraically closed field of characteristic p. The following statements give a complete description of the submodule structure of the module $V(r\omega_1)$, for $0 \leq r \leq p-1$.*

(a) *For $0 \leq r \leq p-4$ the Weyl module $V(r\omega_1)$ is simple.*

(b) *($r = p-3$)*

  (i) *If $p = 3$, then $V((p-3)\omega_1) = V(0)$ is simple.*

  (ii) *If $p = 5$, there is an exact sequence*

$$0 \to V(\omega_6) \to V(2\omega_1) \to L(2\omega_1) \to 0.$$

  (iii) *If $p = 7$, there is an exact sequence*

$$0 \to V(\omega_1 + \omega_4) \to V(2\omega_1 + \omega_6) \to V(4\omega_1) \to L(4\omega_1) \to 0.$$

(iv) *For $p \geq 11$, there is an exact sequence*

$$0 \to V((p-9)\omega_1) \to V((p-8)\omega_1 + \omega_6) \\ \to V((p-8)\omega_1 + \omega_2) \to V((p-6)\omega_1 + \omega_4) \\ \to V((p-5)\omega_1 + \omega_6) \to V((p-3)\omega_1) \to L((p-3)\omega_1) \to 0$$

(c) *($r = p-2$)*

(i) *If $p = 2$ or $p = 3$ the Weyl module $V((p-2)\omega_1)$ is simple.*

(ii) *If $p = 5$ there is an exact sequence*

$$0 \to V(0) \to V(3\omega_1) \to L(3\omega_1) \to 0.$$

(iii) *If $p = 7$, there is an exact sequence*

$$0 \to V(\omega_4 + \omega_6) \to V(\omega_1 + 2\omega_6) \to V(5\omega_1) \to L(5\omega_1) \to 0.$$

(iv) *For $p \geq 11$ there is an exact sequence*

$$0 \to V((p-10)\omega_1 + \omega_2) \to V((p-9)\omega_1 + \omega_5) \\ \to V((p-8)\omega_1 + \omega_3) \to V((p-7)\omega_1 + \omega_4 + \omega_6) \\ \to V((p-6)\omega_1 + 2\omega_6) \to V((p-2)\omega_1) \to L((p-2)\omega_1) \to 0$$

(d) *($r = p-1$)*

(i) *If $p \leq 5$ the the Weyl module $V((p-1)\omega_1)$ is simple.*

(ii) *If $p = 7$, there is an exact sequence*

$$0 \to V(3\omega_6) \to V(6\omega_1) \to L(6\omega_1) \to 0.$$

(iii) *For $p \geq 11$ there is an exact sequence*

$$0 \to V((p-11)\omega_1 + 2\omega_2) \to V((p-10)\omega_1 + \omega_2 + \omega_5) \\ \to V((p-9)\omega_1 + \omega_3 + \omega_6) \to V((p-8)\omega_1 + \omega_4 + 2\omega_6) \\ \to V((p-7)\omega_1 + 3\omega_6) \to V((p-1)\omega_1) \to L((p-1)\omega_1) \to 0$$

(e) *In each of the above sequences the first and last nonzero terms are simple modules and the other terms have two composition factors.*

We shall apply the *Jantzen Sum Formula* [2, II.8.19] [1] : The Weyl module $V(\lambda)$ has a descending filtration, of submodules $V(\lambda)^i$, $i > 0$, such that

$$V(\lambda)^1 = \mathrm{rad}(V(\lambda)), \quad \text{(so that } V(\lambda)/V(\lambda)^1 \cong L(\lambda))$$

and

$$J(\lambda) := \sum_{i>0} \mathrm{Ch}(V(\lambda)^i) = -\sum_{\alpha>0} \sum_{\{m:0<mp<\langle \lambda+\rho, \alpha^\vee\rangle\}} v_p(mp)\chi(\lambda - mp\alpha)$$

[1] The validity of the sum formula for all p was proved by Andersen.

We shall refer to the quantity $J(\lambda)$ as the Jantzen sum for λ (or for $V(\lambda)$). We recall that the weight ρ is the half-sum of the positive roots and $v_p(m)$ denotes the exponent of p in the prime factorization of m. Finally, the formal character $\chi(\mu)$ is the so-called Weyl character, defined in [2, II.5.7], which has the following concrete description. There is a unique dominant weight of the form $w(\mu+\rho)$, where $w \in W$. Let $\mu' = w(\mu+\rho)-\rho$. Then $\chi(\mu)$ is equal to $\mathrm{sign}(w)\,\mathrm{Ch}(V(\mu'))$ if μ' is dominant, and zero otherwise. In particular $\chi(\mu) = \mathrm{Ch}(V(\mu))$ if μ is dominant and $\chi(\mu) = 0$ if and only if $\lambda+\rho-mp\alpha$ is orthogonal to some root.

To aid our computation, when p and $\lambda \in X_+$ have been fixed, we shall say that a multiple $m\alpha$ of a positive root is *relevant* if $0 < mp < \langle \lambda+\rho, \alpha^\vee \rangle\}$ and that $m\alpha$ is a *contributor* if $m\alpha$ is relevant and $\chi(\lambda - mp\alpha) \neq 0$. We will call the quantity $v_p(mp)\chi(\lambda - mp\alpha) = \mathrm{sign}(w)\chi(\mu')$ a *contribution* and the dominant weight $w(\lambda+\rho-mp\alpha)-\rho$ a *contributing weight.*

Thus, in computing the Jantzen sums we can begin by determining the relevant root multiples, then determine which among them is a contributor and finally add up the contributions.

We note that if α_0 is the highest root, then $\langle r\omega_1+\rho, \alpha_0^\vee \rangle = r + 11 < 2p$, when $p \geq 11$, so the only relevant root multiples are actually roots. For the primes $p = 2$, 3, 5 and 7, we have to take into account higher multiples.

1.1. **Discussion of the proof of Theorem 1.1.** The proof is by means of computations, whose results are compiled in the tables below. The tables all have the same form. In the first column are dominant weights λ. In the second column, for each λ we list all the relevant root multiples. A root multiple is given by the tuple of coefficients in its expression as a sum of fundamental roots. Since we are dealing throughout with a single root system E_6, the relevant root multiples for any given weight are easily computed. The relevant root multiples for each weight λ are divided into non-contributors and contributors. For those root multiples $m\alpha$ which we claim to be noncontributors, we must exhibit a (co)root β that is orthogonal to $\lambda+\rho-pm\alpha$. Note that β is not necessarily unique up to a sign. The third column gives the coordinate tuple of the weight $\lambda+\rho-pm\alpha$ with respect to the fundamental weights and the fourth columns gives the coordinate tuple of a suitable β with respect to the fundamental (co)roots. The reader can immediately verify that $\lambda+\rho-pm\alpha$ and β are orthogonal, hence that $m\alpha$ is indeed a non-contributor.

For a contributing root multiple $m\alpha$, the third column has the Weyl group element w such that $w(\lambda+\rho-pm\alpha)$ is dominant and fourth column has the contributing weight $w(\lambda+\rho-pm\alpha)-\rho$. The element w is given as a tuple of indices $[i_1, i_2, \ldots, i_r]$, where $w = w_{i_1}w_{i_2}\cdots w_{i_r}$ as a word in the fundamental reflections. The weight $w(\lambda+\rho-pm\alpha)-\rho$ is given by its tuple of coefficients with respect to the fundamental weights. It is visually obvious, from the fact that all entries in the fourth column tuples are nonnegative, that $m\alpha$ is a contributor. The sign of the contribution is given by the length of w.

We have discussed the immediate verifiability of the tables except for checking that for each w entry the weight $w(\lambda + \rho - pm\alpha) - \rho$ is as given. This can be carried out by longer but routine computations (which can easily be automated).

In order to prove the theorem for a particular weight $r\omega_1$ and in characteristic p, we first find all the relevant root multiples and contributions for this weight, which may depend on p. Then we repeat the procedure for all contributing weights in a second iteration of the sum formula. In principle, further iterations of this process might be expected, but for the weights being considered here they turn out not to be necessary; two iterations provide enough information to deduce Theorem 1.1.

1.2. **Proof of Theorem 1.1 in detail.**
(a). We may assume that $p \geq 5$. Now $V(0)$ is trivially simple, and it is well known that $V(\omega_1)$ is simple for all p. (This can also be can be checked from our tables.) For $p = 7$, we have to check also that $V(2\omega_1)$, and $V(3\omega_1)$ are simple. From Table 14 we can see that no relevant root multiples are contributors. Therefore the Jantzen sums are zero. Assume then that $p \geq 11$. For $r < p-10$, there are no relevant roots for $r\omega_1$, so $V(r\omega_1)$ is simple. For $r = p-10,\dots,p-4$, Table 1 lists the relevant roots and shows that none is a contributor, so $V(r\omega_1)$ is simple.
(b). Part (i) is obvious. For (ii), the starting point is the simplicity of $V(\omega_1)$. By the graph automorphism from the symmetry of the Dynkin diagram this implies that $V(\omega_6)$ is also simple. Also, in Table 12 the only contributor for $2\omega_1$ is $\alpha = \alpha = \sum_{i=1}^{5} \alpha_i$, and the contribution is $-\chi(\omega_6)$. Hence, $\mathrm{rad}(V(2\omega_1)) \cong L(\omega_6)$. To prove (iii), we examine the rows of Table 15 corresponding to $4\omega_1$, We see that the only two contributions are $\chi(\omega_1 + \omega_4)$ (from $\alpha = \sum_{i=1}^{6} \alpha_i$) and $-\chi(2\omega_1 + \omega_6)$ (from $\alpha = \sum_{i=1}^{5} \alpha_i$). We then consider the Jantzen sums for the highest weights of these two contributions, by looking at Table 17. There, we see that $V(\omega_1 + \omega_4)$ is simple. Also, the only contribution to the Jantzen sum for $2\omega_1+\omega_6$ is $-\chi(\omega_1+\omega_4)$, and this means that $\mathrm{rad}(V(2\omega_1+\omega_6)) \cong V(\omega_1+\omega_4)$, and the proof of (iii) is complete. (iv) When $p \geq 11$, the relevant roots and contributions for $(p-3)\omega_1$ are given by Table 2. The contributing weights are $(p-9)\omega_1$, $(p-6)\omega_1 + \omega_4$, $(p-8)\omega_1 + \omega_2$, $(p-8)\omega_1 + \omega_6$, and $(p-5)\omega_1 + \omega_6$. The data for the Jantzen sums for these highest weights is given in Table 3. From Table 3 we see first that $V((p-9)\omega_1)$ is simple. Then we see that $J((p-8)\omega_1 + \omega_6) = \chi((p-9)\omega_1)$, which implies that $\mathrm{rad}(V((p-8)\omega_1 + \omega_6)) \cong$

$L((p-9)\omega_1)$. Next, we have

$$\begin{aligned} J((p-8)\omega_1+\omega_2) &= \chi((p-8)\omega_1+\omega_6) - \chi((p-9)\omega_1) \\ &= \chi((p-8)\omega_1+\omega_6) - \mathrm{Ch}(\mathrm{rad}(V((p-8)\omega_1+\omega_6))). \end{aligned} \tag{1}$$

Hence $\mathrm{rad}(V((p-8)\omega_1+\omega_2))$ is a simple module isomorphic to $L((p-8)\omega_1+\omega_6)$. Next, we have

(2)

$$\begin{aligned} J((p-6)\omega_1+\omega_4) &= \chi((p-8)\omega_1+\omega_2) - \chi((p-8)\omega_1+\omega_6) + \chi((p-9)\omega_1) \\ &= \chi((p-8)\omega_1+\omega_2) - J((p-8)\omega_1+\omega_2) \\ &= \chi((p-8)\omega_1+\omega_2) - \mathrm{Ch}(\mathrm{rad}(V((p-8)\omega_1+\omega_2))), \end{aligned}$$

which implies that $\mathrm{rad}(V((p-6)\omega_1+\omega_4)) \cong L((p-8)\omega_1+\omega_2)$. Continuing in the same way, we see that $\mathrm{rad}(V((p-5)\omega_1+\omega_6)) \cong L((p-6)\omega_1+\omega_4)$ and that $\mathrm{rad}(V((p-3)\omega_1)) \cong L((p-5)\omega_1+\omega_6)$. This completes the proof of (iv) and of (b).

(c). (i) The result is clear for $p = 2$ and $p = 3$. (ii)For $p = 5$, it is immediate from the $3\omega_1$ rows of Table 13 that the only contribution is $-\chi(0)$, so $\mathrm{rad}(V(\omega_3))$ is a one-dimensional trivial module. (iii)When $p = 7$, the contributions for $5\omega_1$ can be found from Table 15. The contributions are $\chi(\omega_4+\omega_6)$ and $-\chi(\omega_1+2\omega_6)$. The data for a second iteration of the sum formula, applied to the contributing weights are given in Table 18. We see that $V(\omega_4+\omega_6)$ is simple and that $\mathrm{rad}(V(\omega_1+2\omega_6)) \cong V(\omega_4+\omega_6)$. Thus, (iii) holds.

(iv)Table 4 gives the contributions for $(p-2)\omega_1$. They are $-\chi((p-10)\omega_1+\omega_2)$, $\chi((p-9)\omega_1+\omega_5)$, $-\chi((p-8)\omega_1+\omega_3)$, $\chi((p-7)\omega_1+\omega_4+\omega_6)$ and $-\chi((p-6)\omega_1+2\omega_6)$. For each of the contributing weights, the data for a second iteration of the sum formula are given in Table 5. From Table 5 we see first that $V((p-10)\omega_1+\omega_2)$ is simple. Then we see that $J((p-9)\omega_1+\omega_5) = \chi((p-10)\omega_1+\omega_2)$, which implies that $\mathrm{rad}(V((p-9)\omega_1+\omega_5)) \cong V((p-10)\omega_1+\omega_2)$. Next, we have

$$\begin{aligned} J((p-8)\omega_1+\omega_3) &= \chi((p-9)\omega_1+\omega_5) - \chi((p-10)\omega_1+\omega_2) \\ &= \chi((p-9)\omega_1+\omega_5) - \mathrm{Ch}(\mathrm{rad}(V((p-9)\omega_1+\omega_5))). \end{aligned} \tag{3}$$

Hence $\mathrm{rad}(V((p-8)\omega_1+\omega_3))$ is a simple module isomorphic to $L((p-9)\omega_1+\omega_5)$. Next, we have

(4)

$$\begin{aligned} J((p-7)\omega_1+\omega_4+\omega_6) &= \chi((p-8)\omega_1+\omega_3) - \chi((p-9)\omega_1+\omega_5) + \chi((p-10)\omega_1+\omega_2) \\ &= \chi((p-8)\omega_1+\omega_3) - J((p-8)\omega_1+\omega_3) \\ &= \chi((p-8)\omega_1+\omega_3) - \mathrm{Ch}(\mathrm{rad}(V((p-8)\omega_1+\omega_3))), \end{aligned}$$

which implies that $\mathrm{rad}(V((p-7)\omega_1+\omega_4+\omega_6)) \cong L((p-8)\omega_1+\omega_3)$. Continuing in the same way, we see that $\mathrm{rad}(V((p-6)\omega_1+2\omega_6)) \cong L((p-7)\omega_1+\omega_4+\omega_6)$

and that $\mathrm{rad}(V((p-2)\omega_1)) \cong L((p-6)\omega_1 + 2\omega_6)$, and the proof of (iv), and of (c) is complete.

(d). The procedure for verifying (d) follows the same pattern as in (b) and (c), and was sketched in [4]. The steps are as follows. Tables 8, 9, 11 and 13 yield (i). For part (ii) Table 16 shows that $J(6\omega_1) = \chi(3\omega_6)$. The simplicity of $V(3\omega_6)$ can be checked by a further sum formula calculation or by observing that $V(3\omega_1)$ is simple by Table 14 and the two modules are conjugate under the graph automorphism of G. Finally, to prove (iii) we find the contributions for $V((p-1)\omega_1)$ in Table 6 and consider a second iteration the sum formula for each of the Weyl modules for contributing weights. The necessary data appear in Table 7. In a manner completely analogous to the proof of (c)(iv), we can make the the following sequence of inferences from Table 7.

(1) $V((p-11)\omega_1 + 2\omega_2)$ is simple.
(2) $\mathrm{rad}(V((p-10)\omega_1 + \omega_2 + \omega_5)) \cong L((p-11)\omega_1 + 2\omega_2)$.
(3) $\mathrm{rad}(V((p-9)\omega_1 + \omega_3 + \omega_6)) \cong L((p-10)\omega_1 + \omega_2 + \omega_5)$.
(4) $\mathrm{rad}(V((p-8)\omega_1 + \omega_4 + 2\omega_6)) \cong L((p-9)\omega_1 + \omega_3 + \omega_6)$.
(5) $\mathrm{rad}(V((p-7)\omega_1 + 3\omega_6)) \cong L((p-8)\omega_1 + \omega_4 + 2\omega_6)$.
(6) $\mathrm{rad}(V((p-1)\omega_1)) \cong L((p-7)\omega_1 + 3\omega_6)$.

(e). It is clear from the above discussion that (e) holds.

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λ	$m\alpha$		
		$\lambda+\rho-pm\alpha$	β
$(p-10)\omega_1$	$[1,2,3,2,2,1]$	$[p-9,1,1,-p+1,1,1]$	$[1,2,3,1,2,1]$
		$\lambda+\rho-pm\alpha$	β
$(p-9)\omega_1$	$[1,2,3,2,2,1]$	$[p-8,1,1,-p+1,1,1]$	$[1,2,2,1,2,1]$
	$[1,2,3,1,2,1]$	$[p-8,1,-p+1,p+1,1,1]$	$[1,2,2,1,2,1]$
		$\lambda+\rho-pm\alpha$	β
$(p-8)\omega_1$	$[1,2,3,2,2,1]$	$[p-7,1,1,-p+1,1,1]$	$[1,2,2,1,1,1]$
	$[1,2,2,1,2,1]$	$[p-7,-p+1,p+1,1,-p+1,1]$	$[1,2,2,1,1,1]$
	$[1,2,3,1,2,1]$	$[p-7,1,-p+1,p+1,1,1]$	$[1,2,2,1,1,1]$
		$\lambda+\rho-pm\alpha$	β
$(p-7)\omega_1$	$[1,2,3,2,2,1]$	$[p-6,1,1,-p+1,1,1]$	$[1,1,2,1,1,1]$
	$[1,2,2,1,1,1]$	$[p-6,-p+1,1,1,p+1,-p+1]$	$[1,1,2,1,1,1]$
	$[1,1,2,1,2,1]$	$[-6,p+1,1,1,-p+1,1]$	$[1,1,2,1,1,1]$
	$[1,2,2,1,2,1]$	$[p-6,-p+1,p+1,1,-p+1,1]$	$[1,2,2,1,1,0]$
	$[1,2,3,1,2,1]$	$[p-6,1,-p+1,p+1,1,1]$	$[1,1,2,1,1,1]$
		$\lambda+\rho-pm\alpha$	β
$(p-6)\omega_1$	$[1,2,3,2,2,1]$	$[p-5,1,1,-p+1,1,1]$	$[1,1,1,1,1,1]$
	$[1,1,2,1,1,1]$	$[-5,p+1,-p+1,1,p+1,-p+1]$	$[1,1,1,1,1,1]$
	$[1,2,2,1,1,1]$	$[p-5,-p+1,1,1,p+1,-p+1]$	$[1,1,1,1,1,1]$
	$[1,1,2,1,2,1]$	$[-5,p+1,1,1,-p+1,1]$	$[1,1,1,1,1,1]$
	$[1,2,2,1,2,1]$	$[p-5,-p+1,p+1,1,-p+1,1]$	$[1,1,1,1,1,1]$
	$[1,2,3,1,2,1]$	$[p-5,1,-p+1,p+1,1,1]$	$[1,1,2,1,1,0]$
	$[1,2,2,1,1,0]$	$[p-5,-p+1,1,1,1,p+1]$	$[1,1,2,1,1,0]$
		$\lambda+\rho-pm\alpha$	β
$(p-5)\omega_1$	$[1,2,3,2,2,1]$	$[p-4,1,1,-p+1,1,1]$	$[1,1,1,1,1,0]$
	$[1,1,1,1,1,1]$	$[-4,1,p+1,-p+1,1,-p+1]$	$[1,1,1,0,1,1]$
	$[1,1,2,1,1,1]$	$[-4,p+1,-p+1,1,p+1,-p+1]$	$[1,1,1,0,1,1]$
	$[1,2,2,1,1,1]$	$[p-4,-p+1,1,1,p+1,-p+1]$	$[1,1,1,0,1,1]$
	$[1,1,2,1,2,1]$	$[-4,p+1,1,1,-p+1,1]$	$[1,1,1,0,1,1]$
	$[1,2,2,1,2,1]$	$[p-4,-p+1,p+1,1,-p+1,1]$	$[1,1,1,0,1,1]$
	$[1,2,3,1,2,1]$	$[p-4,1,-p+1,p+1,1,1]$	$[1,1,1,0,1,1]$
	$[1,2,2,1,1,0]$	$[p-4,-p+1,1,1,1,p+1]$	$[1,1,1,1,1,0]$
	$[1,1,2,1,1,0]$	$[-4,p+1,-p+1,1,1,p+1]$	$[1,1,1,1,1,0]$
		$\lambda+\rho-pm\alpha$	β
$(p-4)\omega_1$	$[1,2,3,2,2,1]$	$[p-3,1,1,-p+1,1,1]$	$[1,1,1,1,0,0]$
	$[1,1,1,1,1,1]$	$[-3,1,p+1,-p+1,1,-p+1]$	$[1,1,1,1,0,0]$
	$[1,1,2,1,1,1]$	$[-3,p+1,-p+1,1,p+1,-p+1]$	$[1,1,1,1,0,0]$
	$[1,2,2,1,1,1]$	$[p-3,-p+1,1,1,p+1,-p+1]$	$[1,1,1,1,0,0]$
	$[1,1,2,1,2,1]$	$[-3,p+1,1,1,-p+1,1]$	$[1,1,1,0,1,0]$
	$[1,2,2,1,2,1]$	$[p-3,-p+1,p+1,1,-p+1,1]$	$[1,1,1,0,1,0]$
	$[1,2,3,1,2,1]$	$[p-3,1,-p+1,p+1,1,1]$	$[1,1,1,0,1,0]$
	$[1,1,1,0,1,1]$	$[-3,1,1,p+1,1,-p+1]$	$[1,1,1,0,1,0]$
	$[1,2,2,1,1,0]$	$[p-3,-p+1,1,1,1,p+1]$	$[1,1,1,0,1,0]$
	$[1,1,2,1,1,0]$	$[-3,p+1,-p+1,1,1,p+1]$	$[1,1,1,0,1,0]$
	$[1,1,1,1,1,0]$	$[-3,1,p+1,-p+1,-p+1,p+1]$	$[1,1,1,0,1,0]$

TABLE 1. $r\omega_1$, $p-10\le r\le p-4$, $p\ge 11$

λ	$m\alpha$		
		$\lambda+\rho-pm\alpha$	β
$(p-3)\omega_1$	$[1,1,2,1,1,1]$	$[-2,p+1,-p+1,1,p+1,-p+1]$	$[1,1,1,0,0,0]$
	$[1,2,2,1,1,1]$	$[p-2,-p+1,1,1,p+1,-p+1]$	$[1,1,1,0,0,0]$
	$[1,2,3,1,2,1]$	$[p-2,1,-p+1,p+1,1,1]$	$[1,1,1,0,0,0]$
	$[1,1,1,0,1,1]$	$[-2,1,1,p+1,1,-p+1]$	$[1,1,1,0,0,0]$
	$[1,2,2,1,1,0]$	$[p-2,-p+1,1,1,1,p+1]$	$[1,1,1,0,0,0]$
	$[1,1,2,1,1,0]$	$[-2,p+1,-p+1,1,1,p+1]$	$[1,1,1,0,0,0]$
	$[1,1,1,0,1,0]$	$[-2,1,1,p+1,-p+1,p+1]$	$[1,1,1,0,0,0]$
	$[1,1,1,1,0,0]$	$[-2,1,1,-p+1,p+1,1]$	$[1,1,1,0,0,0]$
		w	$w(\lambda+\rho-pm\alpha)-\rho$
	$[1,2,3,2,2,1]$	$[1,2,3,5,4,3,2,6,5,3,4]$	$[p-9,0,0,0,0,0]$
	$[1,1,1,1,1,1]$	$[1,2,3,5,6,4,2,1]$	$[p-6,0,0,1,0,0]$
	$[1,1,2,1,2,1]$	$[1,2,3,5,4,3,6,5,1]$	$[p-8,1,0,0,0,0]$
	$[1,2,2,1,2,1]$	$[1,2,3,5,4,3,1,6,5,2]$	$[p-8,0,0,0,0,1]$
	$[1,1,1,1,1,0]$	$[1,2,3,5,4,2,1]$	$[p-5,0,0,0,0,1]$

TABLE 2. $(p-3)\omega_1$, $p\geq 11$

λ	$m\alpha$		
$(p-9)\omega_1$		$\lambda+\rho-pm\alpha$	β
	$[1,2,3,2,2,1]$	$[p-8,1,1,-p+1,1,1]$	$[1,2,2,1,2,1]$
	$[1,2,3,1,2,1]$	$[p-8,1,-p+1,p+1,1,1]$	$[1,2,2,1,2,1]$
$(p-8)\omega_1+\omega_6$		$\lambda+\rho-pm\alpha$	β
	$[1,2,3,2,2,1]$	$[p-7,1,1,-p+1,1,2]$	$[1,1,2,1,1,1]$
	$[1,2,2,1,1,1]$	$[p-7,-p+1,1,1,p+1,-p+2]$	$[1,1,2,1,1,1]$
	$[1,1,2,1,2,1]$	$[-7,p+1,1,1,-p+1,2]$	$[1,1,2,1,1,1]$
	$[1,2,3,1,2,1]$	$[p-7,1,-p+1,p+1,1,2]$	$[1,1,2,1,1,1]$
		w	$w(\lambda+\rho-pm\alpha)-\rho$
	$[1,2,2,1,2,1]$	$[1,3,2,6,5,4,3,2,5,4,3,1,6,5,2]$	$[p-9,0,0,0,0,0]$
$(p-8)\omega_1+\omega_2$		$\lambda+\rho-pm\alpha$	β
	$[1,2,3,2,2,1]$	$[p-7,2,1,-p+1,1,1]$	$[1,1,2,1,1,1]$
	$[1,2,2,1,1,1]$	$[p-7,-p+2,1,1,p+1,-p+1]$	$[1,1,2,1,1,1]$
	$[1,1,2,1,2,1]$	$[-7,p+2,1,1,-p+1,1]$	$[1,1,2,1,1,1]$
	$[1,2,3,1,2,1]$	$[p-7,2,-p+1,p+1,1,1]$	$[1,1,2,1,1,1]$
		w	$w(\lambda+\rho-pm\alpha)-\rho$
	$[1,2,2,1,2,1]$	$[1,3,2,5,4,3,2,5,4,3,1,6,5,2]$	$[p-9,0,0,0,0,0]$
	$[1,2,2,1,1,0]$	$[2,1,3,2,5,4,3,2,1,5,4,3,2]$	$[p-8,0,0,0,0,1]$
$(p-6)\omega_1+\omega_4$		$\lambda+\rho-pm\alpha$	β
	$[1,2,3,2,2,1]$	$[p-5,1,1,-p+2,1,1]$	$[1,1,1,1,1,0]$
	$[1,1,1,1,1,1]$	$[-5,1,p+1,-p+2,1,-p+1]$	$[1,1,1,1,1,0]$
	$[1,1,2,1,2,1]$	$[-5,p+1,1,2,-p+1,1]$	$[1,1,1,1,1,0]$
	$[1,2,2,1,2,1]$	$[p-5,-p+1,p+1,2,-p+1,1]$	$[1,1,1,1,1,0]$
	$[1,2,2,1,1,0]$	$[p-5,-p+1,1,2,1,p+1]$	$[1,1,1,1,1,0]$
	$[1,1,2,1,1,0]$	$[-5,p+1,-p+1,2,1,p+1]$	$[1,1,1,1,1,0]$
		w	$w(\lambda+\rho-pm\alpha)-\rho$
	$[1,1,2,1,1,1]$	$[1,2,4,3,5,3,2,6,4,3,1]$	$[p-8,1,0,0,0,0]$
	$[1,2,2,1,1,1]$	$[1,2,4,3,5,3,2,1,6,4,3,2]$	$[p-8,0,0,0,0,1]$
	$[1,2,3,1,2,1]$	$[1,2,4,3,2,1,5,4,3,2,6,5,3]$	$[p-9,0,0,0,0,0]$
$(p-5)\omega_1+\omega_6$		$\lambda+\rho-pm\alpha$	β
	$[1,2,3,2,2,1]$	$[p-4,1,1,-p+1,1,2]$	$[1,1,1,1,1,0]$
	$[1,1,1,1,1,1]$	$[-4,1,p+1,-p+1,1,-p+2]$	$[1,1,1,1,1,0]$
	$[1,1,2,1,2,1]$	$[-4,p+1,1,1,-p+1,2]$	$[1,1,1,1,1,0]$
	$[1,2,2,1,2,1]$	$[p-4,-p+1,p+1,1,-p+1,2]$	$[1,1,1,1,1,0]$
	$[1,2,2,1,1,0]$	$[p-4,-p+1,1,1,1,p+2]$	$[1,1,1,1,1,0]$
	$[1,1,2,1,1,0]$	$[-4,p+1,-p+1,1,1,p+2]$	$[1,1,1,1,1,0]$
		w	$w(\lambda+\rho-pm\alpha)-\rho$
	$[1,1,2,1,1,1]$	$[1,2,3,5,3,2,6,4,3,1]$	$[p-8,1,0,0,0,0]$
	$[1,2,2,1,1,1]$	$[1,2,3,5,3,2,1,6,4,3,2]$	$[p-8,0,0,0,0,1]$
	$[1,2,3,1,2,1]$	$[1,2,3,2,1,5,4,3,2,6,5,3]$	$[p-9,0,0,0,0,0]$
	$[1,1,1,0,1,1]$	$[1,2,3,5,6,5,3,2,1]$	$[p-6,0,0,1,0,0]$

TABLE 3. $(p-3)\omega_1$, $p \geq 11$, second iteration.

λ	$m\alpha$		
		$\lambda+\rho-pm\alpha$	β
$(p-2)\omega_1$	$[1,1,1,1,1,1]$	$[-1,1,p+1,-p+1,1,-p+1]$	$[1,1,0,0,0,0]$
	$[1,2,2,1,1,1]$	$[p-1,-p+1,1,1,p+1,-p+1]$	$[1,1,0,0,0,0]$
	$[1,2,2,1,2,1]$	$[p-1,-p+1,p+1,1,-p+1,1]$	$[1,1,0,0,0,0]$
	$[1,1,1,0,1,1]$	$[-1,1,1,p+1,1,-p+1]$	$[1,1,0,0,0,0]$
	$[1,2,2,1,1,0]$	$[p-1,-p+1,1,1,1,p+1]$	$[1,1,0,0,0,0]$
	$[1,1,1,0,0,0]$	$[-1,1,-p+1,p+1,p+1,1]$	$[1,1,0,0,0,0]$
	$[1,1,1,1,1,0]$	$[-1,1,p+1,-p+1,-p+1,p+1]$	$[1,1,0,0,0,0]$
	$[1,1,1,0,1,0]$	$[-1,1,1,p+1,-p+1,p+1]$	$[1,1,0,0,0,0]$
	$[1,1,1,1,0,0]$	$[-1,1,1,-p+1,p+1,1]$	$[1,1,0,0,0,0]$
		w	$w(\lambda+\rho-pm\alpha)-\rho$
	$[1,2,3,2,2,1]$	$[1,2,3,5,4,3,2,6,5,3,4]$	$[p-10,1,0,0,0,0]$
	$[1,1,2,1,1,1]$	$[1,2,3,5,6,4,3,1]$	$[p-7,0,0,1,0,1]$
	$[1,1,2,1,2,1]$	$[1,2,3,5,4,3,6,5,1]$	$[p-8,0,1,0,0,0]$
	$[1,2,3,1,2,1]$	$[1,2,3,5,4,3,2,6,5,3]$	$[p-9,0,0,0,1,0]$
	$[1,1,2,1,1,0]$	$[1,2,3,5,4,3,1]$	$[p-6,0,0,0,0,2]$

TABLE 4. $(p-2)\omega_1$, $p\geq 11$

λ	$m\alpha$		
$(p-10)\omega_1+\omega_2$		$\lambda+\rho-pm\alpha$	β
	$[1,2,3,2,2,1]$	$[p-9,2,1,-p+1,1,1]$	$[1,2,2,1,1,1]$
	$[1,2,2,1,2,1]$	$[p-9,-p+2,p+1,1,-p+1,1]$	$[1,2,2,1,1,1]$
	$[1,2,3,1,2,1]$	$[p-9,2,-p+1,p+1,1,1]$	$[1,2,2,1,1,1]$
$(p-9)\omega_1+\omega_5$		$\lambda+\rho-pm\alpha$	β
	$[1,2,3,2,2,1]$	$[p-8,1,1,-p+1,2,1]$	$[1,2,2,1,1,1]$
	$[1,2,2,1,2,1]$	$[p-8,-p+1,p+1,1,-p+2,1]$	$[1,2,2,1,1,1]$
	$[1,2,3,1,2,1]$	$[p-8,1,-p+1,p+1,2,1]$	$[1,2,2,1,1,1]$
		w	$w(\lambda+\rho-pm\alpha)-\rho$
	$[1,1,2,1,2,1]$	$[1,5,3,2,6,5,4,3,2,5,4,3,6,5,1]$	$[p-10,1,0,0,0,0]$
$(p-8)\omega_1+\omega_3$		$\lambda+\rho-pm\alpha$	β
	$[1,2,3,2,2,1]$	$[p-7,1,2,-p+1,1,1]$	$[1,1,2,1,1,0]$
	$[1,1,2,1,1,1]$	$[-7,p+1,-p+2,1,p+1,-p+1]$	$[1,1,2,1,1,0]$
	$[1,1,2,1,2,1]$	$[-7,p+1,2,1,-p+1,1]$	$[1,1,2,1,1,0]$
	$[1,2,3,1,2,1]$	$[p-7,1,-p+2,p+1,1,1]$	$[1,1,2,1,1,0]$
	$[1,2,2,1,1,0]$	$[p-7,-p+1,2,1,1,p+1]$	$[1,1,2,1,1,0]$
		w	$w(\lambda+\rho-pm\alpha)-\rho$
	$[1,2,2,1,1,1]$	$[1,3,2,4,3,5,3,2,1,6,4,3,2]$	$[p-9,0,0,0,1,0]$
	$[1,2,2,1,2,1]$	$[1,3,2,5,4,3,2,5,4,3,1,6,5,2]$	$[p-10,1,0,0,0,0]$
$(p-7)\omega_1+\omega_4+\omega_6$		$\lambda+\rho-pm\alpha$	β
	$[1,2,3,2,2,1]$	$[p-6,1,1,-p+2,1,2]$	$[1,1,2,1,1,0]$
	$[1,1,2,1,1,1]$	$[-6,p+1,-p+1,2,p+1,-p+2]$	$[1,1,2,1,1,0]$
	$[1,1,2,1,2,1]$	$[-6,p+1,1,2,-p+1,2]$	$[1,1,2,1,1,0]$
	$[1,2,3,1,2,1]$	$[p-6,1,-p+1,p+2,1,2]$	$[1,1,2,1,1,0]$
	$[1,2,2,1,1,0]$	$[p-6,-p+1,1,2,1,p+2]$	$[1,1,2,1,1,0]$
		w	$w(\lambda+\rho-pm\alpha)-\rho$
	$[1,1,1,1,1,1]$	$[1,2,4,3,6,5,3,6,4,2,1]$	$[p-8,0,1,0,0,0]$
	$[1,2,2,1,1,1]$	$[1,2,4,3,5,3,2,1,6,4,3,2]$	$[p-9,0,0,0,1,0]$
	$[1,2,2,1,2,1]$	$[1,2,5,4,3,2,5,4,3,1,6,5,2]$	$[p-10,1,0,0,0,0]$
$(p-6)\omega_1+2\omega_6$		$\lambda+\rho-pm\alpha$	β
	$[1,2,3,2,2,1]$	$[p-5,1,1,-p+1,1,3]$	$[1,1,2,1,1,0]$
	$[1,1,2,1,1,1]$	$[-5,p+1,-p+1,1,p+1,-p+3]$	$[1,1,2,1,1,0]$
	$[1,1,2,1,2,1]$	$[-5,p+1,1,1,-p+1,3]$	$[1,1,2,1,1,0]$
	$[1,2,3,1,2,1]$	$[p-5,1,-p+1,p+1,1,3]$	$[1,1,2,1,1,0]$
	$[1,2,2,1,1,0]$	$[p-5,-p+1,1,1,1,p+3]$	$[1,1,2,1,1,0]$
		w	$w(\lambda+\rho-pm\alpha)-\rho$
	$[1,1,1,1,1,1]$	$[1,2,3,6,5,3,6,4,2,1]$	$[p-8,0,1,0,0,0]$
	$[1,2,2,1,1,1]$	$[1,2,3,5,3,2,1,6,4,3,2]$	$[p-9,0,0,0,1,0]$
	$[1,2,2,1,2,1]$	$[1,2,5,3,2,5,4,3,1,6,5,2]$	$[p-10,1,0,0,0,0]$
	$[1,1,1,0,1,1]$	$[1,2,3,5,6,5,3,2,1]$	$[p-7,0,0,1,0,1]$

TABLE 5. $(p-2)\omega_1$, $p\geq 11$, second iteration

λ	$m\alpha$		
		$\lambda+\rho-pm\alpha$	β
$(p-1)\omega_1$	$[1,1,1,1,1,1]$	$[0,1,p+1,-p+1,1,-p+1]$	$[1,0,0,0,0,0]$
	$[1,1,2,1,1,1]$	$[0,p+1,-p+1,1,p+1,-p+1]$	$[1,0,0,0,0,0]$
	$[1,1,2,1,2,1]$	$[0,p+1,1,1,-p+1,1]$	$[1,0,0,0,0,0]$
	$[1,1,1,0,1,1]$	$[0,1,1,p+1,1,-p+1]$	$[1,0,0,0,0,0]$
	$[1,1,0,0,0,0]$	$[0,-p+1,p+1,1,1,1]$	$[1,0,0,0,0,0]$
	$[1,1,2,1,1,0]$	$[0,p+1,-p+1,1,1,p+1]$	$[1,0,0,0,0,0]$
	$[1,1,1,0,0,0]$	$[0,1,-p+1,p+1,p+1,1]$	$[1,0,0,0,0,0]$
	$[1,1,1,1,1,0]$	$[0,1,p+1,-p+1,-p+1,p+1]$	$[1,0,0,0,0,0]$
	$[1,1,1,0,1,0]$	$[0,1,1,p+1,-p+1,p+1]$	$[1,0,0,0,0,0]$
	$[1,1,1,1,0,0]$	$[0,1,1,-p+1,p+1,1]$	$[1,0,0,0,0,0]$
		w	$w(\lambda+\rho-pm\alpha)-\rho$
	$[1,2,3,2,2,1]$	$[1,2,3,5,4,3,2,6,5,3,4]$	$[p-11,2,0,0,0,0]$
	$[1,2,2,1,1,1]$	$[1,2,3,5,6,4,3,2]$	$[p-8,0,0,1,0,2]$
	$[1,2,2,1,2,1]$	$[1,2,3,5,4,3,6,5,2]$	$[p-9,0,1,0,0,1]$
	$[1,2,3,1,2,1]$	$[1,2,3,5,4,3,2,6,5,3]$	$[p-10,1,0,0,1,0]$
	$[1,2,2,1,1,0]$	$[1,2,3,5,4,3,2]$	$[p-7,0,0,0,0,3]$

TABLE 6. $(p-1)\omega_1$, $p \geq 11$

λ	$m\alpha$		
$(p-11)\omega_1+2\omega_2$		$\lambda+\rho-pm\alpha$	β
	$[1,2,3,2,2,1]$	$[p-10,3,1,-p+1,1,1]$	$[1,2,2,1,1,0]$
	$[1,2,2,1,1,1]$	$[p-10,-p+3,1,1,p+1,-p+1]$	$[1,2,2,1,1,0]$
	$[1,2,2,1,2,1]$	$[p-10,-p+3,p+1,1,-p+1,1]$	$[1,2,2,1,1,0]$
	$[1,2,3,1,2,1]$	$[p-10,3,-p+1,p+1,1,1]$	$[1,2,2,1,1,0]$
$(p-10)\omega_1+\omega_2+\omega_5$		$\lambda+\rho-pm\alpha$	β
	$[1,2,3,2,2,1]$	$[p-9,2,1,-p+1,2,1]$	$[1,2,2,1,1,0]$
	$[1,2,2,1,1,1]$	$[p-9,-p+2,1,1,p+2,-p+1]$	$[1,2,2,1,1,0]$
	$[1,2,2,1,2,1]$	$[p-9,-p+2,p+1,1,-p+2,1]$	$[1,2,2,1,1,0]$
	$[1,2,3,1,2,1]$	$[p-9,2,-p+1,p+1,2,1]$	$[1,2,2,1,1,0]$
		w	$w(\lambda+\rho-pm\alpha)-\rho$
	$[1,1,2,1,2,1]$	$[1,5,3,2,6,5,4,3,2,5,4,3,6,5,1]$	$[p-11,2,0,0,0,0]$
$(p-9)\omega_1+\omega_3+\omega_6$		$\lambda+\rho-pm\alpha$	β
	$[1,2,3,2,2,1]$	$[p-8,1,2,-p+1,1,2]$	$[1,2,2,1,1,0]$
	$[1,2,2,1,1,1]$	$[p-8,-p+1,2,1,p+1,-p+2]$	$[1,2,2,1,1,0]$
	$[1,2,2,1,2,1]$	$[p-8,-p+1,p+2,1,-p+1,2]$	$[1,2,2,1,1,0]$
	$[1,2,3,1,2,1]$	$[p-8,1,-p+2,p+1,1,2]$	$[1,2,2,1,1,0]$
		w	$w(\lambda+\rho-pm\alpha)-\rho$
	$[1,1,2,1,1,1]$	$[1,3,2,4,3,6,5,3,2,6,4,3,1]$	$[p-10,1,0,0,1,0]$
	$[1,1,2,1,2,1]$	$[1,3,2,6,5,4,3,2,5,4,3,6,5,1]$	$[p-11,2,0,0,0,0]$
$(p-8)\omega_1+\omega_4+2\omega_6$		$\lambda+\rho-pm\alpha$	β
	$[1,2,3,2,2,1]$	$[p-7,1,1,-p+2,1,3]$	$[1,2,2,1,1,0]$
	$[1,2,2,1,1,1]$	$[p-7,-p+1,1,2,p+1,-p+3]$	$[1,2,2,1,1,0]$
	$[1,2,2,1,2,1]$	$[p-7,-p+1,p+1,2,-p+1,3]$	$[1,2,2,1,1,0]$
	$[1,2,3,1,2,1]$	$[p-7,1,-p+1,p+2,1,3]$	$[1,2,2,1,1,0]$
		w	$w(\lambda+\rho-pm\alpha)-\rho$
	$[1,1,1,1,1,1]$	$[1,2,4,3,6,5,3,6,4,2,1]$	$[p-9,0,1,0,0,1]$
	$[1,1,2,1,1,1]$	$[1,2,4,3,6,5,3,2,6,4,3,1]$	$[p-10,1,0,0,1,0]$
	$[1,1,2,1,2,1]$	$[1,2,6,5,4,3,2,5,4,3,6,5,1]$	$[p-11,2,0,0,0,0]$
$(p-7)\omega_1+3\omega_6$		$\lambda+\rho-pm\alpha$	β
	$[1,2,3,2,2,1]$	$[p-6,1,1,-p+1,1,4]$	$[1,2,2,1,1,0]$
	$[1,2,2,1,1,1]$	$[p-6,-p+1,1,1,p+1,-p+4]$	$[1,2,2,1,1,0]$
	$[1,2,2,1,2,1]$	$[p-6,-p+1,p+1,1,-p+1,4]$	$[1,2,2,1,1,0]$
	$[1,2,3,1,2,1]$	$[p-6,1,-p+1,p+1,1,4]$	$[1,2,2,1,1,0]$
		w	$w(\lambda+\rho-pm\alpha)-\rho$
	$[1,1,1,1,1,1]$	$[1,2,3,6,5,3,6,4,2,1]$	$[p-9,0,1,0,0,1]$
	$[1,1,2,1,1,1]$	$[1,2,3,6,5,3,2,6,4,3,1]$	$[p-10,1,0,0,1,0]$
	$[1,1,2,1,2,1]$	$[1,2,6,5,3,2,5,4,3,6,5,1]$	$[p-11,2,0,0,0,0]$
	$[1,1,1,0,1,1]$	$[1,2,3,5,6,5,3,2,1]$	$[p-8,0,0,1,0,2]$

TABLE 7. $(p-1)\omega_1$, $p\geq 11$, second iteration

λ	$m\alpha$		
		$\lambda+\rho-pm\alpha$	β
ω_1	$[1,2,3,2,2,1]$	$[2,1,1,-1,1,1]$	$[0,0,1,1,0,0]$
	$[2,4,6,4,4,2]$	$[2,1,1,-3,1,1]$	$[0,0,1,1,1,1]$
	$[3,6,9,6,6,3]$	$[2,1,1,-5,1,1]$	$[1,2,3,2,2,1]$
	$[4,8,12,8,8,4]$	$[2,1,1,-7,1,1]$	$[1,1,2,1,1,1]$
	$[5,10,15,10,10,5]$	$[2,1,1,-9,1,1]$	$[1,2,2,1,2,1]$
	$[0,0,1,1,1,1]$	$[2,3,1,-1,1,-1]$	$[0,0,1,1,1,1]$
	$[0,1,1,1,1,1]$	$[4,-1,3,-1,1,-1]$	$[0,0,0,0,1,1]$
	$[0,2,2,2,2,2]$	$[6,-3,5,-3,1,-3]$	$[0,0,1,1,1,1]$
	$[1,1,1,1,1,1]$	$[0,1,3,-1,1,-1]$	$[0,0,0,0,1,1]$
	$[2,2,2,2,2,2]$	$[-2,1,5,-3,1,-3]$	$[0,0,1,1,1,1]$
	$[3,3,3,3,3,3]$	$[-4,1,7,-5,1,-5]$	$[1,1,1,0,1,1]$
	$[0,1,2,1,1,1]$	$[4,1,-1,1,3,-1]$	$[0,1,1,0,0,0]$
	$[0,2,4,2,2,2]$	$[6,1,-3,1,5,-3]$	$[0,0,1,1,1,1]$
	$[1,1,2,1,1,1]$	$[0,3,-1,1,3,-1]$	$[1,0,0,0,0,0]$
	$[2,2,4,2,2,2]$	$[-2,5,-3,1,5,-3]$	$[0,0,1,1,1,1]$
	$[3,3,6,3,3,3]$	$[-4,7,-5,1,7,-5]$	$[0,1,2,1,1,1]$
	$[1,2,2,1,1,1]$	$[2,-1,1,1,3,-1]$	$[0,1,1,0,0,0]$
	$[2,4,4,2,2,2]$	$[2,-3,1,1,5,-3]$	$[0,1,1,0,1,1]$
	$[3,6,6,3,3,3]$	$[2,-5,1,1,7,-5]$	$[0,1,2,1,1,1]$
	$[4,8,8,4,4,4]$	$[2,-7,1,1,9,-7]$	$[1,1,2,1,1,1]$
	$[0,1,2,1,2,1]$	$[4,1,1,1,-1,1]$	$[0,0,0,0,1,1]$
	$[0,2,4,2,4,2]$	$[6,1,1,1,-3,1]$	$[0,0,1,1,1,1]$
	$[0,3,6,3,6,3]$	$[8,1,1,1,-5,1]$	$[0,1,2,1,1,1]$
	$[1,1,2,1,2,1]$	$[0,3,1,1,-1,1]$	$[0,0,0,0,1,1]$
	$[2,2,4,2,4,2]$	$[-2,5,1,1,-3,1]$	$[0,0,1,1,1,1]$
	$[3,3,6,3,6,3]$	$[-4,7,1,1,-5,1]$	$[1,1,1,0,1,1]$
	$[4,4,8,4,8,4]$	$[-6,9,1,1,-7,1]$	$[1,1,2,1,1,1]$
	$[1,2,2,1,2,1]$	$[2,-1,3,1,-1,1]$	$[0,0,0,0,1,1]$
	$[2,4,4,2,4,2]$	$[2,-3,5,1,-3,1]$	$[0,1,1,0,1,1]$
	$[3,6,6,3,6,3]$	$[2,-5,7,1,-5,1]$	$[1,1,1,0,1,1]$
	$[4,8,8,4,8,4]$	$[2,-7,9,1,-7,1]$	$[1,2,2,1,1,0]$
	$[1,2,3,1,2,1]$	$[2,1,-1,3,1,1]$	$[0,1,1,0,0,0]$
	$[2,4,6,2,4,2]$	$[2,1,-3,5,1,1]$	$[0,1,1,0,1,1]$
	$[3,6,9,3,6,3]$	$[2,1,-5,7,1,1]$	$[0,1,2,1,1,1]$
	$[4,8,12,4,8,4]$	$[2,1,-7,9,1,1]$	$[1,1,2,1,1,1]$
	$[5,10,15,5,10,5]$	$[2,1,-9,11,1,1]$	$[1,2,2,1,2,1]$

TABLE 8. $p=2$, relevant root multiples for ω_1, Part 1.

λ	$m\alpha$		
		$\lambda+\rho-pm\alpha$	β
ω_1	$[0,0,1,0,1,1]$	$[2,3,-1,3,1,-1]$	$[0,0,0,0,1,1]$
	$[0,1,1,0,1,1]$	$[4,-1,1,3,1,-1]$	$[0,0,0,0,1,1]$
	$[1,1,1,0,1,1]$	$[0,1,1,3,1,-1]$	$[0,0,0,0,1,1]$
	$[2,2,2,0,2,2]$	$[-2,1,1,5,1,-3]$	$[0,1,1,0,1,1]$
	$[1,2,2,1,1,0]$	$[2,-1,1,1,1,3]$	$[0,1,1,0,0,0]$
	$[2,4,4,2,2,0]$	$[2,-3,1,1,1,5]$	$[1,2,2,1,1,0]$
	$[3,6,6,3,3,0]$	$[2,-5,1,1,1,7]$	$[1,1,1,1,1,0]$
	$[1,1,0,0,0,0]$	$[0,-1,3,1,1,1]$	$[1,0,0,0,0,0]$
	$[1,1,2,1,1,0]$	$[0,3,-1,1,1,3]$	$[1,0,0,0,0,0]$
	$[2,2,4,2,2,0]$	$[-2,5,-3,1,1,5]$	$[1,1,1,0,0,0]$
	$[3,3,6,3,3,0]$	$[-4,7,-5,1,1,7]$	$[1,1,1,1,1,0]$
	$[1,1,1,0,0,0]$	$[0,1,-1,3,3,1]$	$[1,0,0,0,0,0]$
	$[1,1,1,1,1,0]$	$[0,1,3,-1,-1,3]$	$[1,0,0,0,0,0]$
	$[2,2,2,2,2,0]$	$[-2,1,5,-3,-3,5]$	$[0,1,1,1,1,0]$
	$[1,1,1,0,1,0]$	$[0,1,1,3,-1,3]$	$[1,0,0,0,0,0]$
	$[2,2,2,0,2,0]$	$[-2,1,1,5,-3,5]$	$[1,1,1,0,0,0]$
	$[1,1,1,1,0,0]$	$[0,1,1,-1,3,1]$	$[1,0,0,0,0,0]$
	$[2,2,2,2,0,0]$	$[-2,1,1,-3,5,1]$	$[1,1,1,0,0,0]$
	$[0,1,2,1,1,0]$	$[4,1,-1,1,1,3]$	$[0,1,1,0,0,0]$
	$[0,2,4,2,2,0]$	$[6,1,-3,1,1,5]$	$[0,1,1,1,1,0]$
	$[0,1,1,1,1,0]$	$[4,-1,3,-1,-1,3]$	$[0,1,1,1,1,0]$
	$[0,1,1,0,1,0]$	$[4,-1,1,3,-1,3]$	$[0,1,1,0,0,0]$
	$[0,1,1,1,0,0]$	$[4,-1,1,-1,3,1]$	$[0,1,1,0,0,0]$
	$[0,0,1,1,1,0]$	$[2,3,1,-1,-1,3]$	$[0,0,1,0,1,0]$

TABLE 9. $p=2$, relevant root multiples for ω_1, Part 2.

λ	$m\alpha$		
		$\lambda+\rho-pm\alpha$	β
ω_1	$[1,2,3,2,2,1]$	$[2,1,1,-2,1,1]$	$[0,1,1,1,0,0]$
	$[2,4,6,4,4,2]$	$[2,1,1,-5,1,1]$	$[1,2,3,2,2,1]$
	$[3,6,9,6,6,3]$	$[2,1,1,-8,1,1]$	$[1,2,2,1,1,1]$
	$[0,0,1,1,1,1]$	$[2,4,1,-2,1,-2]$	$[0,0,1,0,1,1]$
	$[0,1,1,1,1,1]$	$[5,-2,4,-2,1,-2]$	$[0,1,1,1,0,0]$
	$[1,1,1,1,1,1]$	$[-1,1,4,-2,1,-2]$	$[1,1,0,0,0,0]$
	$[2,2,2,2,2,2]$	$[-4,1,7,-5,1,-5]$	$[1,1,1,0,1,1]$
	$[0,1,2,1,1,1]$	$[5,1,-2,1,4,-2]$	$[0,1,2,1,1,1]$
	$[1,1,2,1,1,1]$	$[-1,4,-2,1,4,-2]$	$[0,0,1,0,1,1]$
	$[2,2,4,2,2,2]$	$[-4,7,-5,1,7,-5]$	$[0,1,2,1,1,1]$
	$[1,2,2,1,1,1]$	$[2,-2,1,1,4,-2]$	$[1,1,0,0,0,0]$
	$[2,4,4,2,2,2]$	$[2,-5,1,1,7,-5]$	$[0,1,2,1,1,1]$
	$[0,1,2,1,2,1]$	$[5,1,1,1,-2,1]$	$[0,0,1,0,1,1]$
	$[0,2,4,2,4,2]$	$[8,1,1,1,-5,1]$	$[0,1,2,1,1,1]$
	$[1,1,2,1,2,1]$	$[-1,4,1,1,-2,1]$	$[0,0,1,0,1,1]$
	$[2,2,4,2,4,2]$	$[-4,7,1,1,-5,1]$	$[1,1,1,0,1,1]$
	$[1,2,2,1,2,1]$	$[2,-2,4,1,-2,1]$	$[1,1,0,0,0,0]$
	$[2,4,4,2,4,2]$	$[2,-5,7,1,-5,1]$	$[1,1,1,0,1,1]$
	$[3,6,6,3,6,3]$	$[2,-8,10,1,-8,1]$	$[1,2,2,1,1,1]$
	$[1,2,3,1,2,1]$	$[2,1,-2,4,1,1]$	$[0,0,1,0,1,1]$
	$[2,4,6,2,4,2]$	$[2,1,-5,7,1,1]$	$[0,1,2,1,1,1]$
	$[3,6,9,3,6,3]$	$[2,1,-8,10,1,1]$	$[1,2,2,1,1,1]$
	$[0,1,1,0,1,1]$	$[5,-2,1,4,1,-2]$	$[0,0,1,0,1,1]$
	$[1,1,1,0,1,1]$	$[-1,1,1,4,1,-2]$	$[0,0,1,0,1,1]$
	$[1,2,2,1,1,0]$	$[2,-2,1,1,1,4]$	$[1,1,0,0,0,0]$
	$[2,4,4,2,2,0]$	$[2,-5,1,1,1,7]$	$[1,1,1,1,1,0]$
	$[1,1,2,1,1,0]$	$[-1,4,-2,1,1,4]$	$[0,0,1,1,1,0]$
	$[2,2,4,2,2,0]$	$[-4,7,-5,1,1,7]$	$[1,1,1,1,1,0]$
	$[1,1,1,0,0,0]$	$[-1,1,-2,4,4,1]$	$[1,1,0,0,0,0]$
	$[1,1,1,1,1,0]$	$[-1,1,4,-2,-2,4]$	$[1,1,0,0,0,0]$
	$[1,1,1,0,1,0]$	$[-1,1,1,4,-2,4]$	$[1,1,0,0,0,0]$
	$[1,1,1,1,0,0]$	$[-1,1,1,-2,4,1]$	$[1,1,0,0,0,0]$
	$[0,1,2,1,1,0]$	$[5,1,-2,1,1,4]$	$[0,1,1,0,1,0]$
	$[0,1,1,1,1,0]$	$[5,-2,4,-2,-2,4]$	$[0,1,1,0,1,0]$

TABLE 10. $p=3$, relevant root multiples for ω_1

λ	$m\alpha$		
		$\lambda+\rho-pm\alpha$	β
$2\omega_1$	$[1,2,3,2,2,1]$	$[3,1,1,-2,1,1]$	$[0,1,1,1,0,0]$
	$[2,4,6,4,4,2]$	$[3,1,1,-5,1,1]$	$[0,1,2,1,1,1]$
	$[3,6,9,6,6,3]$	$[3,1,1,-8,1,1]$	$[1,1,2,1,1,1]$
	$[4,8,12,8,8,4]$	$[3,1,1,-11,1,1]$	$[1,2,3,1,2,1]$
	$[0,0,1,1,1,1]$	$[3,4,1,-2,1,-2]$	$[0,0,1,0,1,1]$
	$[0,1,1,1,1,1]$	$[6,-2,4,-2,1,-2]$	$[0,1,1,1,0,0]$
	$[1,1,1,1,1,1]$	$[0,1,4,-2,1,-2]$	$[1,0,0,0,0,0]$
	$[2,2,2,2,2,2]$	$[-3,1,7,-5,1,-5]$	$[1,1,1,1,0,0]$
	$[0,1,2,1,1,1]$	$[6,1,-2,1,4,-2]$	$[0,1,2,1,1,1]$
	$[1,1,2,1,1,1]$	$[0,4,-2,1,4,-2]$	$[0,0,1,0,1,1]$
	$[2,2,4,2,2,2]$	$[-3,7,-5,1,7,-5]$	$[0,1,2,1,1,1]$
	$[1,2,2,1,1,1]$	$[3,-2,1,1,4,-2]$	$[0,1,1,1,0,0]$
	$[2,4,4,2,2,2]$	$[3,-5,1,1,7,-5]$	$[0,1,2,1,1,1]$
	$[3,6,6,3,3,3]$	$[3,-8,1,1,10,-8]$	$[1,1,2,1,1,1]$
	$[0,1,2,1,2,1]$	$[6,1,1,1,-2,1]$	$[0,0,1,0,1,1]$
	$[0,2,4,2,4,2]$	$[9,1,1,1,-5,1]$	$[0,1,2,1,1,1]$
	$[1,1,2,1,2,1]$	$[0,4,1,1,-2,1]$	$[0,0,1,0,1,1]$
	$[2,2,4,2,4,2]$	$[-3,7,1,1,-5,1]$	$[1,1,1,0,1,0]$
	$[3,3,6,3,6,3]$	$[-6,10,1,1,-8,1]$	$[1,1,2,1,1,1]$
	$[1,2,2,1,2,1]$	$[3,-2,4,1,-2,1]$	$[0,1,1,0,1,0]$
	$[2,4,4,2,4,2]$	$[3,-5,7,1,-5,1]$	$[1,1,1,0,1,0]$
	$[3,6,6,3,6,3]$	$[3,-8,10,1,-8,1]$	$[1,2,2,1,1,0]$
	$[1,2,3,1,2,1]$	$[3,1,-2,4,1,1]$	$[0,0,1,0,1,1]$
	$[2,4,6,2,4,2]$	$[3,1,-5,7,1,1]$	$[0,1,2,1,1,1]$
	$[3,6,9,3,6,3]$	$[3,1,-8,10,1,1]$	$[1,1,2,1,1,1]$
	$[0,1,1,0,1,1]$	$[6,-2,1,4,1,-2]$	$[0,0,1,0,1,1]$
	$[1,1,1,0,1,1]$	$[0,1,1,4,1,-2]$	$[0,0,1,0,1,1]$
	$[2,2,2,0,2,2]$	$[-3,1,1,7,1,-5]$	$[1,1,1,0,1,0]$
	$[1,2,2,1,1,0]$	$[3,-2,1,1,1,4]$	$[0,1,1,0,1,0]$
	$[2,4,4,2,2,0]$	$[3,-5,1,1,1,7]$	$[1,1,1,0,1,0]$
	$[1,1,0,0,0,0]$	$[0,-2,4,1,1,1]$	$[1,0,0,0,0,0]$
	$[1,1,2,1,1,0]$	$[0,4,-2,1,1,4]$	$[1,0,0,0,0,0]$
	$[2,2,4,2,2,0]$	$[-3,7,-5,1,1,7]$	$[1,1,1,0,1,0]$
	$[1,1,1,0,0,0]$	$[0,1,-2,4,4,1]$	$[1,0,0,0,0,0]$
	$[1,1,1,1,1,0]$	$[0,1,4,-2,-2,4]$	$[1,0,0,0,0,0]$
	$[2,2,2,2,2,0]$	$[-3,1,7,-5,-5,7]$	$[1,1,1,0,1,0]$
	$[1,1,1,0,1,0]$	$[0,1,1,4,-2,4]$	$[1,0,0,0,0,0]$
	$[1,1,1,1,0,0]$	$[0,1,1,-2,4,1]$	$[1,0,0,0,0,0]$
	$[0,1,2,1,1,0]$	$[6,1,-2,1,1,4]$	$[0,1,1,0,1,0]$
	$[0,1,1,1,1,0]$	$[6,-2,4,-2,-2,4]$	$[0,1,1,0,1,0]$

TABLE 11. $p=3$, relevant root multiples for $2\omega_1$

λ	$m\alpha$		
		$\lambda+\rho-pm\alpha$	β
ω_1	$[1,2,3,2,2,1]$	$[2,1,1,-4,1,1]$	$[0,1,1,1,1,1]$
	$[2,4,6,4,4,2]$	$[2,1,1,-9,1,1]$	$[1,2,2,1,2,1]$
	$[1,1,1,1,1,1]$	$[-3,1,6,-4,1,-4]$	$[0,1,1,1,1,1]$
	$[0,1,2,1,1,1]$	$[7,1,-4,1,6,-4]$	$[0,1,1,1,1,1]$
	$[1,1,2,1,1,1]$	$[-3,6,-4,1,6,-4]$	$[1,1,1,1,0,0]$
	$[1,2,2,1,1,1]$	$[2,-4,1,1,6,-4]$	$[0,1,1,1,1,1]$
	$[0,1,2,1,2,1]$	$[7,1,1,1,-4,1]$	$[0,1,1,1,1,1]$
	$[1,1,2,1,2,1]$	$[-3,6,1,1,-4,1]$	$[1,1,1,0,1,0]$
	$[1,2,2,1,2,1]$	$[2,-4,6,1,-4,1]$	$[0,1,1,1,1,1]$
	$[1,2,3,1,2,1]$	$[2,1,-4,6,1,1]$	$[1,1,1,0,1,0]$
	$[2,4,6,2,4,2]$	$[2,1,-9,11,1,1]$	$[1,2,2,1,2,1]$
	$[1,1,1,0,1,1]$	$[-3,1,1,6,1,-4]$	$[1,1,1,0,1,0]$
	$[1,2,2,1,1,0]$	$[2,-4,1,1,1,6]$	$[1,1,1,0,1,0]$
	$[1,1,2,1,1,0]$	$[-3,6,-4,1,1,6]$	$[1,1,1,0,1,0]$
	$[1,1,1,1,1,0]$	$[-3,1,6,-4,-4,6]$	$[1,1,1,0,1,0]$
		$\lambda+\rho-pm\alpha$	β
$2\omega_1$	$[1,2,3,2,2,1]$	$[3,1,1,-4,1,1]$	$[0,1,1,1,1,1]$
	$[2,4,6,4,4,2]$	$[3,1,1,-9,1,1]$	$[1,2,2,1,1,1]$
	$[1,1,1,1,1,1]$	$[-2,1,6,-4,1,-4]$	$[0,1,1,1,1,1]$
	$[0,1,2,1,1,1]$	$[8,1,-4,1,6,-4]$	$[0,1,1,1,1,1]$
	$[1,1,2,1,1,1]$	$[-2,6,-4,1,6,-4]$	$[1,1,1,0,0,0]$
	$[1,2,2,1,1,1]$	$[3,-4,1,1,6,-4]$	$[0,1,1,1,1,1]$
	$[0,1,2,1,2,1]$	$[8,1,1,1,-4,1]$	$[0,1,1,1,1,1]$
	$[1,1,2,1,2,1]$	$[-2,6,1,1,-4,1]$	$[1,1,2,1,2,1]$
	$[1,2,2,1,2,1]$	$[3,-4,6,1,-4,1]$	$[0,1,1,1,1,1]$
	$[2,4,4,2,4,2]$	$[3,-9,11,1,-9,1]$	$[1,2,2,1,1,1]$
	$[1,2,3,1,2,1]$	$[3,1,-4,6,1,1]$	$[1,1,1,0,0,0]$
	$[2,4,6,2,4,2]$	$[3,1,-9,11,1,1]$	$[1,2,2,1,1,1]$
	$[1,1,1,0,1,1]$	$[-2,1,1,6,1,-4]$	$[1,1,1,0,0,0]$
	$[1,2,2,1,1,0]$	$[3,-4,1,1,1,6]$	$[1,1,1,0,0,0]$
	$[1,1,2,1,1,0]$	$[-2,6,-4,1,1,6]$	$[1,1,1,0,0,0]$
	$[1,1,1,0,1,0]$	$[-2,1,1,6,-4,6]$	$[1,1,1,0,0,0]$
	$[1,1,1,1,0,0]$	$[-2,1,1,-4,6,1]$	$[1,1,1,0,0,0]$
		w	$w(\lambda+\rho-pm\alpha)-\rho$
	$[1,1,1,1,1,0]$	$[1,2,3,5,4,2,1]$	$[0,0,0,0,0,1]$

TABLE 12. $p=5$, relevant root multiples for ω_1 and $2\omega_1$.

λ	$m\alpha$		
		$\lambda+\rho-pm\alpha$	β
$3\omega_1$	$[1,2,3,2,2,1]$	$[4,1,1,-4,1,1]$	$[0,1,1,1,1,1]$
	$[2,4,6,4,4,2]$	$[4,1,1,-9,1,1]$	$[1,1,2,1,1,1]$
	$[1,1,1,1,1,1]$	$[-1,1,6,-4,1,-4]$	$[0,1,1,1,1,1]$
	$[0,1,2,1,1,1]$	$[9,1,-4,1,6,-4]$	$[0,1,1,1,1,1]$
	$[1,1,2,1,1,1]$	$[-1,6,-4,1,6,-4]$	$[1,1,2,1,1,1]$
	$[1,2,2,1,1,1]$	$[4,-4,1,1,6,-4]$	$[0,1,1,1,1,1]$
	$[2,4,4,2,2,2]$	$[4,-9,1,1,11,-9]$	$[1,1,2,1,1,1]$
	$[0,1,2,1,2,1]$	$[9,1,1,1,-4,1]$	$[0,1,1,1,1,1]$
	$[2,2,4,2,4,2]$	$[-6,11,1,1,-9,1]$	$[1,1,2,1,1,1]$
	$[1,2,2,1,2,1]$	$[4,-4,6,1,-4,1]$	$[0,1,1,1,1,1]$
	$[2,4,4,2,4,2]$	$[4,-9,11,1,-9,1]$	$[1,2,2,1,1,0]$
	$[1,2,3,1,2,1]$	$[4,1,-4,6,1,1]$	$[0,1,2,1,1,0]$
	$[2,4,6,2,4,2]$	$[4,1,-9,11,1,1]$	$[1,1,2,1,1,1]$
	$[1,1,1,0,1,1]$	$[-1,1,1,6,1,-4]$	$[1,1,0,0,0,0]$
	$[1,2,2,1,1,0]$	$[4,-4,1,1,1,6]$	$[1,2,2,1,1,0]$
	$[1,1,2,1,1,0]$	$[-1,6,-4,1,1,6]$	$[0,1,2,1,1,0]$
	$[1,1,1,0,0,0]$	$[-1,1,-4,6,6,1]$	$[1,1,0,0,0,0]$
	$[1,1,1,1,1,0]$	$[-1,1,6,-4,-4,6]$	$[1,1,0,0,0,0]$
	$[1,1,1,0,1,0]$	$[-1,1,1,6,-4,6]$	$[1,1,0,0,0,0]$
	$[1,1,1,1,0,0]$	$[-1,1,1,-4,6,1]$	$[1,1,0,0,0,0]$
		w	$w(\lambda+\rho-pm\alpha)-\rho$
	$[1,1,2,1,2,1]$	$[3,5,4,3,6,5,1]$	$[0,0,0,0,0,0]$
		$\lambda+\rho-pm\alpha$	β
$4\omega_1$	$[1,2,3,2,2,1]$	$[5,1,1,-4,1,1]$	$[0,1,1,1,1,1]$
	$[2,4,6,4,4,2]$	$[5,1,1,-9,1,1]$	$[1,1,1,1,1,1]$
	$[1,1,1,1,1,1]$	$[0,1,6,-4,1,-4]$	$[0,1,1,1,1,1]$
	$[0,1,2,1,1,1]$	$[10,1,-4,1,6,-4]$	$[0,1,1,1,1,1]$
	$[1,1,2,1,1,1]$	$[0,6,-4,1,6,-4]$	$[1,0,0,0,0,0]$
	$[2,2,4,2,2,2]$	$[-5,11,-9,1,11,-9]$	$[1,1,1,1,1,1]$
	$[1,2,2,1,1,1]$	$[5,-4,1,1,6,-4]$	$[0,1,1,1,1,1]$
	$[2,4,4,2,2,2]$	$[5,-9,1,1,11,-9]$	$[1,1,1,1,1,1]$
	$[0,1,2,1,2,1]$	$[10,1,1,1,-4,1]$	$[0,1,1,1,1,1]$
	$[1,1,2,1,2,1]$	$[0,6,1,1,-4,1]$	$[1,0,0,0,0,0]$
	$[2,2,4,2,4,2]$	$[-5,11,1,1,-9,1]$	$[1,1,1,1,1,1]$
	$[1,2,2,1,2,1]$	$[5,-4,6,1,-4,1]$	$[0,1,1,1,1,1]$
	$[2,4,4,2,4,2]$	$[5,-9,11,1,-9,1]$	$[1,1,1,1,1,1]$
	$[1,2,3,1,2,1]$	$[5,1,-4,6,1,1]$	$[0,1,2,1,1,0]$
	$[2,4,6,2,4,2]$	$[5,1,-9,11,1,1]$	$[1,1,2,1,1,0]$
	$[1,1,1,0,1,1]$	$[0,1,1,6,1,-4]$	$[1,0,0,0,0,0]$
	$[1,2,2,1,1,0]$	$[5,-4,1,1,1,6]$	$[0,1,2,1,1,0]$
	$[2,4,4,2,2,0]$	$[5,-9,1,1,1,11]$	$[1,1,2,1,1,0]$
	$[1,1,0,0,0,0]$	$[0,-4,6,1,1,1]$	$[1,0,0,0,0,0]$
	$[1,1,2,1,1,0]$	$[0,6,-4,1,1,6]$	$[1,0,0,0,0,0]$
	$[1,1,1,0,0,0]$	$[0,1,-4,6,6,1]$	$[1,0,0,0,0,0]$
	$[1,1,1,1,1,0]$	$[0,1,6,-4,-4,6]$	$[1,0,0,0,0,0]$
	$[1,1,1,0,1,0]$	$[0,1,1,6,-4,6]$	$[1,0,0,0,0,0]$
	$[1,1,1,1,0,0]$	$[0,1,1,-4,6,1]$	$[1,0,0,0,0,0]$

TABLE 13. $p=5$, relevant root multiples for $3\omega_1$ and $4\omega_1$.

λ	$m\alpha$		
		$\lambda+\rho-pm\alpha$	β
ω_1	$[1,2,3,2,2,1]$	$[2,1,1,-6,1,1]$	$[1,1,1,1,1,1]$
	$[1,1,2,1,1,1]$	$[-5,8,-6,1,8,-6]$	$[1,1,1,1,1,1]$
	$[1,2,2,1,1,1]$	$[2,-6,1,1,8,-6]$	$[1,1,1,1,1,1]$
	$[1,1,2,1,2,1]$	$[-5,8,1,1,-6,1]$	$[1,1,1,1,1,1]$
	$[1,2,2,1,2,1]$	$[2,-6,8,1,-6,1]$	$[1,1,1,1,1,1]$
	$[1,2,3,1,2,1]$	$[2,1,-6,8,1,1]$	$[0,1,2,1,2,1]$
	$[1,2,2,1,1,0]$	$[2,-6,1,1,1,8]$	$[1,1,2,1,1,0]$
		$\lambda+\rho-pm\alpha$	β
$2\omega_1$	$[1,2,3,2,2,1]$	$[3,1,1,-6,1,1]$	$[0,1,2,1,2,1]$
	$[1,1,1,1,1,1]$	$[-4,1,8,-6,1,-6]$	$[1,1,1,0,1,1]$
	$[1,1,2,1,1,1]$	$[-4,8,-6,1,8,-6]$	$[1,1,1,0,1,1]$
	$[1,2,2,1,1,1]$	$[3,-6,1,1,8,-6]$	$[1,1,1,0,1,1]$
	$[1,1,2,1,2,1]$	$[-4,8,1,1,-6,1]$	$[0,1,2,1,2,1]$
	$[1,2,2,1,2,1]$	$[3,-6,8,1,-6,1]$	$[0,1,2,1,2,1]$
	$[1,2,3,1,2,1]$	$[3,1,-6,8,1,1]$	$[0,1,2,1,2,1]$
	$[1,2,2,1,1,0]$	$[3,-6,1,1,1,8]$	$[1,1,1,1,1,0]$
	$[1,1,2,1,1,0]$	$[-4,8,-6,1,1,8]$	$[1,1,1,1,1,0]$
		$\lambda+\rho-pm\alpha$	β
$3\omega_1$	$[1,2,3,2,2,1]$	$[4,1,1,-6,1,1]$	$[1,2,3,2,2,1]$
	$[1,1,1,1,1,1]$	$[-3,1,8,-6,1,-6]$	$[1,1,1,1,0,0]$
	$[1,1,2,1,1,1]$	$[-3,8,-6,1,8,-6]$	$[1,1,1,1,0,0]$
	$[1,2,2,1,1,1]$	$[4,-6,1,1,8,-6]$	$[1,1,1,1,0,0]$
	$[1,1,2,1,2,1]$	$[-3,8,1,1,-6,1]$	$[0,1,2,1,2,1]$
	$[1,2,2,1,2,1]$	$[4,-6,8,1,-6,1]$	$[0,1,2,1,2,1]$
	$[1,2,3,1,2,1]$	$[4,1,-6,8,1,1]$	$[0,1,2,1,2,1]$
	$[1,1,1,0,1,1]$	$[-3,1,1,8,1,-6]$	$[1,1,1,0,1,0]$
	$[1,2,2,1,1,0]$	$[4,-6,1,1,1,8]$	$[1,1,1,0,1,0]$
	$[1,1,2,1,1,0]$	$[-3,8,-6,1,1,8]$	$[1,1,1,0,1,0]$
	$[1,1,1,1,1,0]$	$[-3,1,8,-6,-6,8]$	$[1,1,1,0,1,0]$

TABLE 14. $p=7$, relevant root multiples for $r\omega_1$, $1\leq r\leq 3$.

λ	$m\alpha$		
		$\lambda+\rho-pm\alpha$	β
$4\omega_1$	[1, 2, 3, 2, 2, 1]	[5, 1, 1, −6, 1, 1]	[0, 1, 2, 1, 2, 1]
	[2, 4, 6, 4, 4, 2]	[5, 1, 1, −13, 1, 1]	[1, 2, 3, 1, 2, 1]
	[1, 1, 2, 1, 1, 1]	[−2, 8, −6, 1, 8, −6]	[1, 1, 1, 0, 0, 0]
	[1, 2, 2, 1, 1, 1]	[5, −6, 1, 1, 8, −6]	[1, 1, 1, 0, 0, 0]
	[1, 1, 2, 1, 2, 1]	[−2, 8, 1, 1, −6, 1]	[0, 1, 2, 1, 2, 1]
	[1, 2, 2, 1, 2, 1]	[5, −6, 8, 1, −6, 1]	[0, 1, 2, 1, 2, 1]
	[1, 2, 3, 1, 2, 1]	[5, 1, −6, 8, 1, 1]	[0, 1, 2, 1, 2, 1]
	[1, 1, 1, 0, 1, 1]	[−2, 1, 1, 8, 1, −6]	[1, 1, 1, 0, 0, 0]
	[1, 2, 2, 1, 1, 0]	[5, −6, 1, 1, 1, 8]	[1, 1, 1, 0, 0, 0]
	[1, 1, 2, 1, 1, 0]	[−2, 8, −6, 1, 1, 8]	[1, 1, 1, 0, 0, 0]
	[1, 1, 1, 0, 1, 0]	[−2, 1, 1, 8, −6, 8]	[1, 1, 1, 0, 0, 0]
	[1, 1, 1, 1, 0, 0]	[−2, 1, 1, −6, 8, 1]	[1, 1, 1, 0, 0, 0]
		w	$w(\lambda+\rho-pm\alpha)-\rho$
	[1, 1, 1, 1, 1, 1]	[1, 2, 3, 5, 6, 4, 2, 1]	[1, 0, 0, 1, 0, 0]
	[1, 1, 1, 1, 1, 0]	[1, 2, 3, 5, 4, 2, 1]	[2, 0, 0, 0, 0, 1]
		$\lambda+\rho-pm\alpha$	β
$5\omega_1$	[1, 2, 3, 2, 2, 1]	[6, 1, 1, −6, 1, 1]	[0, 1, 2, 1, 2, 1]
	[2, 4, 6, 4, 4, 2]	[6, 1, 1, −13, 1, 1]	[1, 2, 2, 1, 2, 1]
	[1, 1, 1, 1, 1, 1]	[−1, 1, 8, −6, 1, −6]	[1, 1, 0, 0, 0, 0]
	[1, 2, 2, 1, 1, 1]	[6, −6, 1, 1, 8, −6]	[1, 1, 0, 0, 0, 0]
	[1, 1, 2, 1, 2, 1]	[−1, 8, 1, 1, −6, 1]	[0, 1, 2, 1, 2, 1]
	[1, 2, 2, 1, 2, 1]	[6, −6, 8, 1, −6, 1]	[0, 1, 2, 1, 2, 1]
	[1, 2, 3, 1, 2, 1]	[6, 1, −6, 8, 1, 1]	[0, 1, 2, 1, 2, 1]
	[2, 4, 6, 2, 4, 2]	[6, 1, −13, 15, 1, 1]	[1, 2, 2, 1, 2, 1]
	[1, 1, 1, 0, 1, 1]	[−1, 1, 1, 8, 1, −6]	[1, 1, 0, 0, 0, 0]
	[1, 2, 2, 1, 1, 0]	[6, −6, 1, 1, 1, 8]	[1, 1, 0, 0, 0, 0]
	[1, 1, 1, 0, 0, 0]	[−1, 1, −6, 8, 8, 1]	[1, 1, 0, 0, 0, 0]
	[1, 1, 1, 1, 1, 0]	[−1, 1, 8, −6, −6, 8]	[1, 1, 0, 0, 0, 0]
	[1, 1, 1, 0, 1, 0]	[−1, 1, 1, 8, −6, 8]	[1, 1, 0, 0, 0, 0]
	[1, 1, 1, 1, 0, 0]	[−1, 1, 1, −6, 8, 1]	[1, 1, 0, 0, 0, 0]
		w	$w(\lambda+\rho-pm\alpha)-\rho$
	[1, 1, 2, 1, 1, 1]	[1, 2, 3, 5, 6, 4, 3, 1]	[0, 0, 0, 1, 0, 1]
	[1, 1, 2, 1, 1, 0]	[1, 2, 3, 5, 4, 3, 1]	[1, 0, 0, 0, 0, 2]

TABLE 15. $p = 7$, relevant root multiples for $r\omega_1$, $4 \le r \le 5$.

λ	$m\alpha$		
		$\lambda+\rho-pm\alpha$	β
$6\omega_1$	$[1,2,3,2,2,1]$	$[7,1,1,-6,1,1]$	$[0,1,2,1,2,1]$
	$[2,4,6,4,4,2]$	$[7,1,1,-13,1,1]$	$[1,2,2,1,1,1]$
	$[1,1,1,1,1,1]$	$[0,1,8,-6,1,-6]$	$[1,0,0,0,0,0]$
	$[1,1,2,1,1,1]$	$[0,8,-6,1,8,-6]$	$[1,0,0,0,0,0]$
	$[1,2,2,1,1,1]$	$[7,-6,1,1,8,-6]$	$[1,2,2,1,1,1]$
	$[1,1,2,1,2,1]$	$[0,8,1,1,-6,1]$	$[0,1,2,1,2,1]$
	$[1,2,2,1,2,1]$	$[7,-6,8,1,-6,1]$	$[0,1,2,1,2,1]$
	$[2,4,4,2,4,2]$	$[7,-13,15,1,-13,1]$	$[1,2,2,1,1,1]$
	$[1,2,3,1,2,1]$	$[7,1,-6,8,1,1]$	$[0,1,2,1,2,1]$
	$[2,4,6,2,4,2]$	$[7,1,-13,15,1,1]$	$[1,2,2,1,1,1]$
	$[1,1,1,0,1,1]$	$[0,1,1,8,1,-6]$	$[1,0,0,0,0,0]$
	$[1,1,0,0,0,0]$	$[0,-6,8,1,1,1]$	$[1,0,0,0,0,0]$
	$[1,1,2,1,1,0]$	$[0,8,-6,1,1,8]$	$[1,0,0,0,0,0]$
	$[1,1,1,0,0,0]$	$[0,1,-6,8,8,1]$	$[1,0,0,0,0,0]$
	$[1,1,1,1,1,0]$	$[0,1,8,-6,-6,8]$	$[1,0,0,0,0,0]$
	$[1,1,1,0,1,0]$	$[0,1,1,8,-6,8]$	$[1,0,0,0,0,0]$
	$[1,1,1,1,0,0]$	$[0,1,1,-6,8,1]$	$[1,0,0,0,0,0]$
		w	$w(\lambda+\rho-pm\alpha)-\rho$
	$[1,2,2,1,1,0]$	$[1,2,3,5,4,3,2]$	$[0,0,0,0,0,3]$

TABLE 16. $p=7$, relevant root multiples for $6\omega_1$.

λ	$m\alpha$		
		$\lambda+\rho-pm\alpha$	β
$2\omega_1+\omega_6$	$[1,2,3,2,2,1]$	$[3,1,1,-6,1,2]$	$[1,2,3,2,2,1]$
	$[1,1,1,1,1,1]$	$[-4,1,8,-6,1,-5]$	$[1,1,1,1,1,0]$
	$[1,1,2,1,1,1]$	$[-4,8,-6,1,8,-5]$	$[0,1,2,1,1,1]$
	$[1,2,2,1,1,1]$	$[3,-6,1,1,8,-5]$	$[0,1,2,1,1,1]$
	$[0,1,2,1,2,1]$	$[10,1,1,1,-6,2]$	$[0,1,2,1,1,1]$
	$[1,1,2,1,2,1]$	$[-4,8,1,1,-6,2]$	$[1,1,1,1,1,0]$
	$[1,2,2,1,2,1]$	$[3,-6,8,1,-6,2]$	$[1,1,1,1,1,0]$
	$[1,2,3,1,2,1]$	$[3,1,-6,8,1,2]$	$[0,1,2,1,1,1]$
	$[1,2,2,1,1,0]$	$[3,-6,1,1,1,9]$	$[1,1,1,1,1,0]$
	$[1,1,2,1,1,0]$	$[-4,8,-6,1,1,9]$	$[1,1,1,1,1,0]$
		w	$w(\lambda+\rho-pm\alpha)-\rho$
	$[1,1,1,0,1,1]$	$[1,2,3,5,6,5,3,2,1]$	$[1,0,0,1,0,0]$
		$\lambda+\rho-pm\alpha$	β
$\omega_1+\omega_4$	$[1,2,3,2,2,1]$	$[2,1,1,-5,1,1]$	$[1,2,3,2,2,1]$
	$[1,1,1,1,1,1]$	$[-5,1,8,-5,1,-6]$	$[1,1,1,1,1,0]$
	$[1,1,2,1,1,1]$	$[-5,8,-6,2,8,-6]$	$[0,1,2,1,1,1]$
	$[1,2,2,1,1,1]$	$[2,-6,1,2,8,-6]$	$[0,1,2,1,1,1]$
	$[0,1,2,1,2,1]$	$[9,1,1,2,-6,1]$	$[0,1,2,1,1,1]$
	$[1,1,2,1,2,1]$	$[-5,8,1,2,-6,1]$	$[1,1,1,1,1,0]$
	$[1,2,2,1,2,1]$	$[2,-6,8,2,-6,1]$	$[1,1,1,1,1,0]$
	$[1,2,3,1,2,1]$	$[2,1,-6,9,1,1]$	$[0,1,2,1,1,1]$
	$[1,2,2,1,1,0]$	$[2,-6,1,2,1,8]$	$[1,1,1,1,1,0]$
	$[1,1,2,1,1,0]$	$[-5,8,-6,2,1,8]$	$[1,1,1,1,1,0]$

TABLE 17. $p=7$, second iteration relevant root multiples for $4\omega_1$.

λ	$m\alpha$		
		$\lambda+\rho-pm\alpha$	β
$\omega_1+2\omega_6$	$[1,2,3,2,2,1]$	$[2,1,1,-6,1,3]$	$[1,2,3,2,2,1]$
	$[1,1,1,1,1,1]$	$[-5,1,8,-6,1,-4]$	$[0,1,1,1,1,1]$
	$[0,1,2,1,1,1]$	$[9,1,-6,1,8,-4]$	$[0,1,1,1,1,1]$
	$[1,1,2,1,1,1]$	$[-5,8,-6,1,8,-4]$	$[1,1,2,1,1,0]$
	$[1,2,2,1,1,1]$	$[2,-6,1,1,8,-4]$	$[0,1,1,1,1,1]$
	$[0,1,2,1,2,1]$	$[9,1,1,1,-6,3]$	$[0,1,1,1,1,1]$
	$[1,1,2,1,2,1]$	$[-5,8,1,1,-6,3]$	$[1,1,2,1,1,0]$
	$[1,2,2,1,2,1]$	$[2,-6,8,1,-6,3]$	$[0,1,1,1,1,1]$
	$[1,2,3,1,2,1]$	$[2,1,-6,8,1,3]$	$[1,1,2,1,1,0]$
	$[1,2,2,1,1,0]$	$[2,-6,1,1,1,10]$	$[1,1,2,1,1,0]$
		w	$w(\lambda+\rho-pm\alpha)-\rho$
	$[1,1,1,0,1,1]$	$[1,2,3,5,6,5,3,2,1]$	$[0,0,0,1,0,1]$
		$\lambda+\rho-pm\alpha$	β
$\omega_4+\omega_6$	$[1,2,3,2,2,1]$	$[1,1,1,-5,1,2]$	$[1,2,3,2,2,1]$
	$[1,1,1,1,1,1]$	$[-6,1,8,-5,1,-5]$	$[0,1,1,1,1,1]$
	$[0,1,2,1,1,1]$	$[8,1,-6,2,8,-5]$	$[0,1,1,1,1,1]$
	$[1,1,2,1,1,1]$	$[-6,8,-6,2,8,-5]$	$[1,1,2,1,1,0]$
	$[1,2,2,1,1,1]$	$[1,-6,1,2,8,-5]$	$[0,1,1,1,1,1]$
	$[0,1,2,1,2,1]$	$[8,1,1,2,-6,2]$	$[0,1,1,1,1,1]$
	$[1,1,2,1,2,1]$	$[-6,8,1,2,-6,2]$	$[1,1,2,1,1,0]$
	$[1,2,2,1,2,1]$	$[1,-6,8,2,-6,2]$	$[0,1,1,1,1,1]$
	$[1,2,3,1,2,1]$	$[1,1,-6,9,1,2]$	$[1,1,2,1,1,0]$
	$[1,2,2,1,1,0]$	$[1,-6,1,2,1,9]$	$[1,1,2,1,1,0]$

TABLE 18. $p=7$, second iteration relevant root multiples for $5\omega_1$.