> Write your proofs using *complete English sentences* as well as mathematical formulae.
> Bonus points may be awarded for particularly well-argued proofs.
> In this exam F denotes a field and $\mathbb{R}$ denotes the field of real numbers.

Name: ______________________________

1. Let V be the vector space of 2×2 matrices with real entries and let A be a fixed element of V.
   (a) (2 points) Show that the mapping $T : V \to V$ given by
   $$T(X) = AX - XA, \quad X \in V,$$
   is a linear mapping. (You may freely use rules of matrix algebra.)
   (b) (4 points) Let $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Find a basis for $R(T)$.

Solution: (a) Let X and Y be elements of V and c a scalar. Then

$$\begin{aligned} T(X+Y) &= A(X+Y) - (X+Y)A = AX + AY - XA + YA \\ &= (AX - XA) + (AY - YA) = T(X) + T(Y) \end{aligned}$$

and

$$T(cX) = A(cX) - (cX)A = cAX - cXA = c(AX - XA) = cT(X).$$

We have checked that T is a linear map.

(b) We compute:

$$T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} = \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix}.$$

Let $B = \{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\}$.

The first element of B is $T\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)$ and the second element is $T\left(\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}\right)$, so both lie in $R(T)$. To see that they are linearly independent, notice that they each have an entry 1 where the other has entry 0. The computation above shows that every element in $R(T)$ is a linear combination of the vectors in B. Thus, B is a basis of $R(T)$.

2. (6 points) In $\mathbb{R}^2$, find the image of the triangle with vertices $(1,2)$, $(2,6)$ and $(3,0)$ under an anticlockwise rotation through an angle of $\frac{\pi}{6}$ radians about the origin. You may assume that the rotation is a linear map.

Solution: Call the rotation T. We compute the matrix of T with respect to the standard basis (call it β) of $\mathbb{R}^2$. Under T, the vector $\begin{pmatrix}1\\0\end{pmatrix}$ is mapped to $\begin{pmatrix}\cos(\frac{\pi}{6})\\ \sin(\frac{\pi}{6})\end{pmatrix} = \begin{pmatrix}\frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ is mapped to $\begin{pmatrix}-\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{pmatrix}$ (as one can see by drawing a simple picture). So we have computed that

$$[T]_\beta^\beta = \begin{pmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}.$$

Now using the formula $[T(v)]_\beta = [T]_\beta^\beta [v]_\beta$, and noting that $[\begin{pmatrix}a\\b\end{pmatrix}]_\beta = \begin{pmatrix}a\\b\end{pmatrix}$ when β is the standard basis, we have

$$T(\begin{pmatrix}1\\2\end{pmatrix}) = \begin{pmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}\frac{\sqrt{3}}{2} - 1\\ \frac{1}{2} + \sqrt{3}\end{pmatrix}$$
$$T(\begin{pmatrix}2\\6\end{pmatrix}) = \begin{pmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}\begin{pmatrix}2\\6\end{pmatrix} = \begin{pmatrix}\sqrt{3} - 3\\ 1 + 3\sqrt{3}\end{pmatrix}$$
$$T(\begin{pmatrix}3\\0\end{pmatrix}) = \begin{pmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}\begin{pmatrix}3\\0\end{pmatrix} = \begin{pmatrix}\frac{3\sqrt{3}}{2}\\ \frac{3}{2}\end{pmatrix}.$$

So the image of the triangle is the triangle with vertices at $(\frac{\sqrt{3}}{2} - 1, \frac{1}{2} + \sqrt{3})$, $(\sqrt{3} - 3, 1 + 3\sqrt{3})$, and $(3\frac{\sqrt{3}}{2}, \frac{3}{2})$.

3. (a) (4 points) Prove that there is a linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T\left(\begin{pmatrix}1\\1\end{pmatrix}\right) = \begin{pmatrix}\frac{1}{9}\\-1\end{pmatrix}$ and $T\left(\begin{pmatrix}1\\-1\end{pmatrix}\right) = \begin{pmatrix}\frac{\pi}{5}\\3\end{pmatrix}$.

Solution: In $\mathbb{R}^2$, two vectors form a basis as long as neither is a scalar multiple of the other. In particular $\beta := \{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\}$ is a basis. Then by the fundamental theorem of linear algebra, there exists a linear map T such that $T\left(\begin{pmatrix}1\\1\end{pmatrix}\right) = \begin{pmatrix}\frac{1}{9}\\-1\end{pmatrix}$ and $T\left(\begin{pmatrix}1\\-1\end{pmatrix}\right) = \begin{pmatrix}\frac{\pi}{5}\\3\end{pmatrix}$.

(b) (4 points) Is the linear map T invertible? Justify your answer.

Solution: Yes. One way to see this is to note that $\gamma := \{\begin{pmatrix}\frac{1}{9}\\-1\end{pmatrix}, \begin{pmatrix}\frac{\pi}{5}\\3\end{pmatrix}\}$ is also a basis of $\mathbb{R}^2$, by our initial remark, so the fundamental theorem gives us a linear map U sending $\begin{pmatrix}\frac{1}{9}\\-1\end{pmatrix}$ to $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}\frac{\pi}{5}\\3\end{pmatrix}$ to $\begin{pmatrix}1\\-1\end{pmatrix}$. Then UT sends each element of β to itself, so it must be the identity map. Likewise, TU sends each element of γ to itself, so is also the identity map.