> Write your proofs using *complete English sentences* as well as mathematical formulae.
> Bonus points may be awarded for particularly well-argued proofs.
> In this exam F denotes a field and $\mathbb{R}$ denotes the field of real numbers.

Name: ____

1. Let V be the vector space of real polynomials of degree at most 3.

   (a) (2 points) Show that the mapping $T : V \to V$ given by

   $$T(f(x)) = (1+x)f'(x), \quad \text{for } f(x) \in V,$$

   is a linear mapping. (As usual, $f'(x)$ denotes the derivative of $f(x)$.)

   (b) (2 points) Compute the matrix $[T]_\beta^\beta$, where $\beta = \{1, x, x^2, x^3\}$ is the standard ordered basis of powers of x.

   (c) (2 points) Determine the rank and nullity of T. (Rank is the dimension of $R(T)$, nullity is the dimension of $N(T)$.)

2. In R^3 let β be the standard basis and let

$$\beta' = \{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \}.$$

(a) (4 points) Compute the change-of-basis matrices $[1]_{\beta}^{\beta'}$ and $[1]_{\beta'}^{\beta}$.

(b) (4 points) Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$. Find the matrix $[L_A]_{\beta'}^{\beta'}$.

3. (a) (4 points) Prove that there is a unique linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} \frac{1}{5} \\ -1 \end{pmatrix}$ and $T\left(\begin{pmatrix} 1 \\ 2020 \end{pmatrix}\right) = \begin{pmatrix} \frac{-2}{7} \\ \ln 2 \end{pmatrix}$.

   (b) (2 points) Is this linear map an isomorphism?