Please write your proofs carefully and in complete English sentences. If you wish to use theorems from the text, make it clear which theorem you are using, by stating or describing it. Be careful to avoid using mathematical notation incorrectly. When in doubt, use English. Anything that the grader cannot understand may receive no credit.

Name: ______________________________

1. Let σ be the permutation
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 3 & 4 & 5 & 2 & 1 & 6 \end{pmatrix}.$$

   (a) (5 points) Write σ as a product of disjoint cycles.

   Solution:
   $$\sigma = (176)(2345)$$

   (b) (5 points) Write σ as a product of 2-cycles.

   Solution:
   $$\sigma = (16)(17)(25)(24)(23)$$

2. (10 points) Prove that $\mathrm{A}ut(Z_8) \cong U(8)$.

Solution: First, since Z_8 is cyclic, it follows from the operation-preserving property of automorphisms that an automorphism α of Z_8 is completely determined by where it maps the generator $1 \in Z_8$. The image $\alpha(1)$ must be a generator, for otherwise α would not be surjective. Then since the generators are precisely the elements 1, 3, 5 and 7, i.e. the elements of $U(8)$, we have a map: $\theta : \mathrm{A}ut(Z_8) \to U(8)$, sending α to $\alpha(1)$. If α, $\beta \in \mathrm{A}ut(Z_8)$, then

$$(\alpha\beta)(1) = \alpha(\beta(1)) = \beta(1)\alpha(1),$$

where the last equality is by the operation-preserving property of α. This shows that θ has the operation-perserving property. Now θ is injective, since every automorphism is determined by where it maps 1. To see that θ is surjective, let $m \in U(8)$ and consider the mapping $\mu_m : x \mapsto mx \pmod 8$ from Z_8 to itself. By mod 8 arithmetic, we see that μ_m is an automorphism of Z_8, and that $\mu_m(1) = m$, i.e. $\theta(\mu_m) = m$.

3. True or false? If you think the statement is true, give a proof. If false, provide a concrete counterexample.

   (a) (3 points) A finite group G has a subgroup of order d if and only if d divides the order of G.

   Solution: False. The alternating group A_4 has order 12, but no subgroup of order 6. This example was studied in class, so please look at it.

   (b) (3 points) If a group G has an element of order p and one of order q, where p and q are distinct primes, then G has an element of order pq.

   Solution: False. The symmetric group S_3 has the elements (12) and (123) of orders 2 and 3 repsectively, but no element of order 6 (since the group has order 6 but is not cyclic).

   (c) (4 points) Let $\mathrm{I}nn(G)$ be the group of inner automorphisms of the group G. Then $|\mathrm{I}nn(G)| = 1$ if and only if G is abelian.

   Solution: True. Denote the identity automorphism by 1. Let $g \in G$ and $\phi_g : x \mapsto gxg^{-1}$ be the inner automorphism induced by g. Then $\phi_g = 1$ if and only if for all x, we have $gxg^{-1} = x$, which holds if and only if for all x, we have $gx = xg$, or in other words, if and only if g lies in the center of G. Thus, $|\mathrm{I}nn(G)| = 1$ if and only if all elements of G belong to the center, which is equivalent to G being abelian.

4. (5 points) (Extra credit) Prove that if a subgroup H of S_n has odd order, then in fact $H \leq A_n$.

Solution: First Proof.

We have a result in the text that for every subgroup H of S_n, either all elements of H are even or else exactly half of them are even. If H has odd order, the second alternative is not possible.

Second Proof.

By Lagrange, each element of H must have odd order. Therefore if $\sigma \in H$ is written as a product of disjoint cycles, there can only be cycles of odd length. Since cycles of odd length are even permutations, so is σ.