No Calculators. Answer the questions in the spaces provided on the question sheets.**Please write your answers in full detail.**

Name: ______________________________

1. (7 points) Find the Laplace transform $F(s)$ of the function

$$f(t) = \begin{cases} \cos t, & \text{if } 0 < t < \pi \\ 0, & \text{if } \pi < t. \end{cases}$$

**Solution:** We can use the step function to write

$$f(t) = \cos t - u(t-\pi)\cos t.$$

To take Laplace transforms, we apply formula (4) to the second term, with $g(t) = \cos t$ and $a = \pi$. To use (4), we need to find $g(t+a)$. We have $g(t+a) = \cos(t+\pi) = -\cos t$. We can now compute

$$F(s)) = \frac{s}{s^2+1} - (e^{-\pi s}\frac{-s}{s^2+1}) = \frac{s(1+e^{-\pi s})}{s^2+1}.$$

2. (6 points) Let $\mathcal{L}\{y(t)\}(s) = Y(s)$. Find

$$\mathcal{L}\{e^{5t} t y'(t)\}(s).$$

Be sure to explain clearly which general properties of the Laplace transform you use. You may refer to formulae on the last page by numbers.

**Solution:** By the $t$ differentiation formula (5), we have

$$\mathcal{L}\{y'(t)\}(s) = sY(s) - y(0).$$

Then by the $s$ differentiation formula (6),

$$\mathcal{L}\{ty'(t)\}(s) = -\frac{d}{ds}(sY(s) - y(0)) = -sY'(s) - Y(s).$$

Finally, by formula (2),

$$\mathcal{L}\{e^{5t} t y'(t)\}(s) = -(s-5)Y'(s-5) - Y(s-5).$$

3. (7 points) Solve the following IVP by the method of Laplace transforms.

$$y'' - 4y = 4t - 8e^{-2t}, \quad y(0) = 0, \quad y'(0) = 5.$$

**Solution:** We have

$$\mathcal{L}\{y'\}(s) = sY, \quad \mathcal{L}\{y''\}(s) = s^2Y - 5.$$

So the transformed equation is

$$s^2Y - 5 - 4Y = \frac{4}{s^2} - \frac{8}{s+2}.$$

This simplifies to

$$(s^2 - 4)Y = \frac{4}{s^2} - \frac{8}{s+2} + 5 = \frac{5s^3 + 2s^2 + 4s + 8}{s^2(s+2)}.$$

Since $(s^2 - 4) = (s+2)(s-2)$, we get

$$Y = \frac{5s^3 + 2s^2 + 4s + 8}{s^2(s+2)^2(s-2)}.$$

Expanding in partial fractions yields

$$Y = \frac{-1}{s^2} + \frac{-1}{s+2} + \frac{2}{(s+2)^2} + \frac{1}{(s-2)}$$

so on taking inverse Laplace transforms we get

$$y(t) = -t - e^{-2t} + 2te^{-2t} + e^{2t}.$$

4. Let $f(t) = \cos t$ and $g(t) = \sin t$.
   (a) (4 points) Find the convolution $(f * g)(t)$. Hint: $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$.
   (b) (3 points) Use your answer in (a) to find the inverse Laplace transform of

$$\frac{s}{(s^2+1)^2}$$

**Solution:**

(a) There are several ways to do this. Here is a quick one. We have

$$(f * g)(t) = \int_0^t \cos(t-v) \sin v \, dv$$

and

$$(g * f)(t) = \int_0^t \sin(t-v) \cos v \, dv.$$

Since $(f * g) = (g * f)(t)$, we may add these equations to get

$$2(f * g)(t) = \int_0^t \cos(t-v) \sin v + \sin(t-v) \cos v \, dv.$$

By the trig identity with $A = t - v$ and $B = v$ this becomes

$$(f * g)(t) = \frac{1}{2} \int_0^t \sin t \, dv = \frac{t}{2} \sin t.$$

(b) By the Convolution Theorem, $\mathcal{L}\{(f * g)(t)\}(s) = F(s)G(s)$. Applying this to $f(t) = \cos t$ and $g(t) = \sin t$ shows that

$$\mathcal{L}\{\cos t * \sin t\}(s) = \frac{s}{s^2+1} . \frac{1}{s^2+1} = \frac{s}{(s^2+1)^2}$$

Hence the inverse Laplace transform of $\frac{s}{(s^2+1)^2}$ is $\cos t * \sin t = \frac{t}{2} \sin t$, by the answer in (a).

**Formulae**

$$\mathcal{L}\{\sin bt\}(s) = \frac{b}{s^2 + b^2} \tag{1}$$

$$\mathcal{L}\{e^{at} f(t)\}(s) = F(s-a) \tag{2}$$

$$\mathcal{L}\{f(t-a)u(t-a)\}(s) = e^{-as}F(s) \tag{3}$$

$$\mathcal{L}\{g(t)u(t-a)\}(s) = e^{-as}\mathcal{L}\{g(t+a)\}(s) \tag{4}$$

$$\mathcal{L}\{f'(t)\}(s) = sF(s) - f(0) \tag{5}$$

$$\mathcal{L}\{tf(t)\}(s) = -\frac{d}{ds}F(s) \tag{6}$$