EXTENSIONS OF SIMPLE MODULES FOR $G_2(3^n)$ AND ${}^2G_2(3^m)$

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[Received 27 August 1991]

ABSTRACT

The group of extensions between any two irreducible 3-modular representations of the groups $G_2(3^n)$ and ${}^2G_2(3^m)$ is determined.

Introduction

This is the fourth paper in a series [**8**, **9**, **10**] with the common goal of computing extensions between simple modules for groups of Lie type of ranks 1 and 2 over fields of small characteristic. The methods employed in these calculations have their origins in the papers [**1**] and [**4**] and are perhaps most clearly illustrated in [**8**], while they are somewhat obscured by complications in [**9**] and not fully developed in [**10**]. Unfortunately, the present paper is the most elaborate of the four because we must deal with very intricate module structures.

The work is based on the empirical observation (for which there is also some theoretical motivation (see [**2**])), that, in the known cases, a non-split extension $\mathscr{E}$ of simple modules can be factorized into the tensor product $\mathscr{E}' \otimes M$ of a simple module M with non-split extension $\mathscr{E}'$ which is (up to twisting by Frobenius) isomorphic to a subquotient of the tensor product $M' \otimes M''$ of two simple modules with *restricted* highest weights. With this as a working hypothesis, the problem of determining all extensions of simple modules falls into two parts; first one must know enough about the submodule structures of the modules $M' \otimes M''$ to be able to determine the extensions $\mathscr{E}'$. Then one needs to find an inductive argument to prove that all extensions really do arise in the way we have described. In order to carry out the first stage of this plan we shall make use of the powerful theory of Weyl modules and good filtrations for representations of algebraic groups. For the inductive part of the proof, we shall rely on the representation theory of the finite groups, where two advantages are that the Frobenius map is an automorphism, not just an endomorphism, and that the Steinberg module is projective. In actual fact the demarcation between these techniques will not be quite as sharp as we have just put it and to some extent they are used in combination throughout.

1. *Preliminaries and statement of the theorem*

Let $\mathbf{G} = G_2(F)$ be a simple algebraic group of type G_2 over an algebraic closure F of $\mathbb{F}_3$. This group is defined over $\mathbb{F}_3$ so the Frobenius map σ: $x \mapsto x^3$ of F induces an endomorphism of $\mathbf{G}$ which we shall also call the Frobenius map σ. In

Financial support for this work was provided by NSF grant number DMS 900-1273 and in part by a postdoctoral fellowship at the Mathematical Sciences Research Institute, Berkeley during the Fall of 1990.

1991 *Mathematics Subject Classification*: 20G40, 20C20, 20G05.

Proc. London Math. Soc. (3) 66 (1993) 327–357.

characteristic 3, the group $\mathbf{G}$ also has a special isogeny τ with $\tau^2 = \sigma$. For each natural number n, the finite group $G_2(3^n)$ is defined to be the subgroup of $\mathbf{G}$ fixed by σ^n. The subgroup fixed by τ^n will of course be $G_2(3^{n/2})$ if n is even, but for each odd value of n we obtain the Ree group ${}^2G_2(3^n)$, which may therefore be considered as the subgroup of $G_2(3^n)$ fixed by the automorphism $\tau|_{G_2(3^n)}$ of order 2. Thus the restriction of τ to ${}^2G_2(3^n)$ is equal to the restriction of $\sigma^{(n+1)/2}$.

We introduce the following notation, which will enable us to treat the two families of finite groups as one. For each natural number m, set

$$G(m) = \mathbf{G}^{\tau^m} = \begin{cases} G_2(3^{m/2}) & \text{if } m \text{ is even,} \\ {}^2G_2(3^m) & \text{if } m \text{ is odd.} \end{cases}$$

The groups $G(m)$ are simple for $m \geq 2$. *Except in the appendix, we shall assume that $m > 2$ throughout this paper.*

When the parameter m is understood we shall abbreviate $G(m)$ to G. We shall deal only with finite-dimensional FG-modules and rational $\mathbf{G}$-modules over F. For any $\mathbf{G}$-module V, we denote by V_i the $\mathbf{G}$-module obtained by composing the representation $\mathbf{G} \to \mathrm{GL}(V)$ with the endomorphism τ^i. We also make the analogous convention for $FG(m)$-modules. Thus V_2 is the first Frobenius twist of V, and V and V_m are isomorphic as $FG(m)$-modules.

Let α_1 be the long fundamental root and α_2 the short one in a base of the root system of type G_2 associated to $\mathbf{G}$ and let λ_1 and λ_2 be the corresponding fundamental dominant weights, so that $(\lambda_i, \alpha_j^\vee) = \delta_{ij}$. Let Λ denote the weight lattice, Λ_+ the set of dominant weights and $\Lambda_r \subset \Lambda$ the set

$$\{a\lambda_1 + b\lambda_2 \mid 0 \leq a, b \leq 3^r - 1\}$$

of 3^r-restricted weights. As usual, for each dominant weight λ, the simple module with highest weight λ is denoted by $L(\lambda)$, while $V(\lambda)$ denotes the Weyl module whose unique simple quotient is isomorphic to $L(\lambda)$, and $H^0(\lambda)$ stands for the dual Weyl module with unique simple submodule isomorphic to $L(\lambda)$. Since $L(\lambda)$ is $\mathbf{G}$-isomorphic to its F-dual, we have $H^0(\lambda) \cong_{\mathbf{G}} \mathrm{Hom}_F(V(\lambda), F)$. For $\lambda = a\lambda_1 + b\lambda_2$ we have $L(\lambda)_1 \cong L(b\lambda_1 + 3a\lambda_2)$.

The module $L(\lambda_2) \cong V(\lambda_2) \cong H^0(\lambda_2)$ is 7-dimensional and may be identified as the space of elements of trace zero in the Cayley algebra $\mathscr{C}$ over F, on which $\mathbf{G}$ operates as algebra automorphisms. The module $V(\lambda_1)$ is the 14-dimensional adjoint module, which may be interpreted as the Lie algebra of derivations of $\mathscr{C}$, on which $\mathbf{G}$ acts by conjugation in $\mathrm{End}_F(\mathscr{C})$. As a $\mathbf{G}$-module, it is a non-split extension of $L(\lambda_1) \cong L(\lambda_2)_1$ by $L(\lambda_2)$, the latter corresponding to the ideal of inner derivations (see [**6**, p. 14]). The module $L(2\lambda_2) \cong V(2\lambda_2) \cong H^0(2\lambda_2)$ is 27-dimensional. We shall write $L(\lambda_2)$ as E and $L(2\lambda_2)$ as S, so $L(\lambda_1) \cong E_1$ and $L(2\lambda_1) \cong S_1$. For any $\mathbf{G}$-module V and any finite set I of natural numbers we define

$$V_I = \bigotimes_{i \in I} V_i \quad (V_\varnothing = F).$$

For $G(m)$, since $V_0 \cong V_m$, we shall always consider only subsets of $N = N(m) = \{0, 1, \ldots, m-1\}$, and indices such as $i+1$ or $i+2$ are to be read modulo m.

By Steinberg's Tensor Product Theorem (a refined version of the usual one; see § 11 of [**11**]), the simple $\mathbf{G}$-modules are precisely the modules $E_I \otimes S_J$, where I and J are disjoint finite sets of natural numbers. Furthermore, the simple

$FG(m)$-modules are the restrictions of those simple $\mathbf{G}$-modules for which I and J are both subsets of N. Thus there are 3^m isomorphism classes of simple $FG(m)$-modules (which for even values of m correspond to the elements of $\Lambda_{m/2}$). The simple modules with 3-restricted highest weights are F, E, E_1, S, $E_{\{0,1\}} \cong L(\lambda_1+\lambda_2)$, S_1, $E_1 \otimes S \cong L(\lambda_1+2\lambda_2)$, $E \otimes S_1 \cong L(2\lambda_1+\lambda_2)$ and $S_{\{0,1\}} \cong L(2\lambda_1+2\lambda_2)$. Thus, $S_{\{0,1\}}$ is the first Steinberg module for $\mathbf{G}$, and S_N is the Steinberg module for $G(m)$, a projective, simple $FG(m)$-module. One might perhaps think of S as the '$\frac{1}{2}$th Steinberg module' for $\mathbf{G}$. All of the simple modules are self-dual.

Our aim is to compute the dimensions of the vector spaces

$$\mathrm{Ext}^1_{FG(m)}(E_I \otimes S_J, E_A \otimes S_B),$$

where (I, J) and (A, B) are pairs of disjoint subsets of N. The self-duality of the simple modules yields:

$$(1.1)\quad \mathrm{Ext}^1_{FG(m)}(E_I \otimes S_J, E_A \otimes S_B) \cong \mathrm{Ext}^1_{FG(m)}(E_{I\cap A} \otimes S_{J\cup B}, E_{I\cup A} \otimes S_{J\cap B}),$$

which reduces the general problem to the case where $I \subseteq A$, $B \subseteq J$. The following statement therefore describes all the extensions between simple modules.

THEOREM. *Let*† $m > 2$ *and let* (I, J) *and* (A, B) *be pairs of disjoint subsets of* N *such that* $I \subseteq A$ *and* $B \subseteq J$. *Then* $\mathrm{Ext}^1_{FG}(E_I \otimes S_J, E_A \otimes S_B) = 0$ *unless one of the following holds:*

(I) $A = I \cup \{i, i+1\}, J = B \ (i, i+1 \notin I)$;

(II) $A = I \cup \{i, i+1\}, J = B \cup \{i+1\} \ (i, i+1 \notin I)$;

(III) $A = I \cup \{i\}, J = B$ *and* $i+1 \in I \ (i \notin I)$;

(IV) $A = I \cup \{i\}, J = B$ *and* $i+1 \in J \ (i \notin I)$.

If one of these conditions holds then the space of extensions is one-dimensional. Furthermore, the same statement holds for $\mathbf{G}$ *if the phrase 'subsets of* N*' is replaced by 'finite sets of natural numbers'.*

The result for $\mathbf{G}$ follows from [**5**, Theorem 7.1], which in our setting states that for $\lambda, \mu \in \Lambda_r$, the restriction map

$$(1.2)\qquad \mathrm{Ext}^1_{\mathbf{G}}(L(\lambda), L(\mu)) \to \mathrm{Ext}^1_{FG(2s)}(L(\lambda), L(\mu))$$

is injective if $s \geq r$ and is an isomorphism for all s sufficiently large compared to r (note that our '$G(2s)$' corresponds to '$G(3^s)$' in the notation of [**5**]. In the course of our proof we shall make use of a slight variation (Lemma 3.1) of the results quoted above, namely that for simple modules labelled by pairs of disjoint sets of natural numbers less than m, the restriction map from $\mathbf{G}$ to $G(m)$ of the group of extensions is injective for odd as well as even values of m. This is what will allow us to derive the submodule structure of certain $FG(m)$-modules from their $\mathbf{G}$-module structures.

The remainder of the paper is organized as follows. Section 2 contains some calculations in the Grothendieck ring of FG-modules, which are straightforward but nevertheless have important consequences for module structures, just as in [**9**]. In § 3, we obtain quite detailed information about several 'small' modules,

†The cases $m \leq 2$ are given in the appendix.

especially tensor products of restricted modules. These are described by means of filtrations by Weyl modules and then the Weyl modules are themselves examined closely. It is this very precise knowledge which will enable us to prove the theorem in § 5, after a little more technical preparation in § 4. For completeness we have included an appendix dealing with the cases $m \leqslant 2$.

2. *Products of characters*

Let η_I denote the class of the simple module E_I in the Grothendieck ring $\mathscr{R}(G)$ of FG and let ψ_J denote the class of S_J. It is clear from our description of the simple FG-modules in § 1 that the multiplication in $\mathscr{R}(G)$ is completely determined by the products η^2, $\eta\psi$ and ψ^2. If we take the classes $\eta_I\psi_J$ of the simple modules as a Z-basis of $\mathscr{R}(G)$, it should be clear what is meant by the simple constituents of an element of $\mathscr{R}(G)$ and by the multiplicity of a simple constituent in an element.

LEMMA 2.1. *We have*

(a) $\eta^2 = \psi + \eta_1 + 2\eta + 1$,

(b) $\eta\psi = \psi + 2\eta_{\{0,1\}} + \eta_2 + 4\eta_1 + 4\eta + 1$,

(c) $\psi^2 = \psi\eta_1 + 2\psi_1 + 2\psi + \eta_{\{0,2\}} + 6\eta_{\{0,1\}} + \eta_2 + 6\eta_1 + 5\eta + 5$.

Proof. Since $E \cong V(\lambda_2)$ and $S \cong V(2\lambda_2)$, we may use the classical formulae (of Weyl and Freudenthal) to decompose the tensor products of the corresponding Weyl modules over the complex numbers into other Weyl modules over the complex numbers. Then, since we may use the same classical formulae to determine the weight multiplicities in these resulting complex Weyl modules and since the discussion of § 1 allows us to determine the weight multiplicities of all simple **G**-modules, it is a completely mechanical process to find the composition factors of the reductions mod 3 of these complex Weyl modules. We omit the details.

The multiplication formulae of Lemma 2.1 suggest that, as in **[9]**, we should define the *mass* of the simple module $E_I \otimes S_J$ (and of $\eta_I\psi_J$) to be $|I| + 2|J|$, and the mass of an arbitrary FG-module to be the maximum of the masses of its composition factors. This coarse numerical invariant, which is clearly preserved under twisting by τ, will be useful in some inductive arguments. The following property, immediate from Lemma 2.1, shows that mass behaves well with respect to tensor products. Let (I, J) and (A, B) be pairs of disjoint subsets of N. Then

$$\text{mass}((E_I \otimes S_J) \otimes (E_A \otimes S_B)) \leqslant \text{mass}(E_I \otimes S_J) + \text{mass}(E_A \otimes S_B) \tag{2.1.1}$$

with equality if and only if $B \cap (I \cup J) = \varnothing = J \cap (A \cup B)$.

LEMMA 2.2. (a) *The class $\eta_i(\eta_J\psi_K)$ has no constituent ψ_T with $|T| > |K| + 1$, and none with $|T| > |K|$ if $i \in K$.*

(b) *The class $\psi_i(\eta_J\psi_K)$ has no constituent ψ_T with $|T| > |K| + 1$.*

Proof. The proof will be by induction on mass$(\eta_J\psi_K)$.

(a) If $i \in J$, then by Lemma 2.1 we have

$$\eta_i(\eta_J\psi_K) = (1 + \psi_i + 2\eta_i + \eta_{i+1})\eta_{J\setminus\{i\}}\psi_K,$$

and the conclusion is clear for all terms except $\eta_{i+1}(\eta_{J\setminus\{i\}}\psi_K)$, to which we may apply induction. If $i \in K$, we have

$$\eta_i(\eta_J\psi_K) = (1 + \psi_i + 4\eta_i + 2\eta_{\{i,i+1\}} + \eta_{i+2} + 4\eta_{i+1})\eta_J\psi_{K\setminus\{i\}}.$$

The conclusion is clear for the first three terms and induction on mass applies to the other terms; for example, the fourth term is $2\eta_{i+1}(\eta_{J\cup\{i\}}\psi_{K\setminus\{i\}})$.

(b) If $i \in J$, we are back in (a). If $i \in K$, we have

$$\psi_i(\eta_J\psi_K) = (\psi_i\eta_{i+1} + 2\psi_{i+1} + 2\psi_i + \eta_{\{i,i+2\}} + 6\eta_{\{i,i+1\}} + \eta_{i+2} + 6\eta_{i+1} + 5\eta_i + 5)\eta_J\psi_{K\setminus\{i\}}.$$

We apply induction on mass to the term $\psi_{i+1}(\eta_J\psi_{K\setminus\{i\}})$ and Part (a) to the other terms which are not (multiples of) simple elements.

LEMMA 2.3. *Let I, A, $B \subset N$, $A \cap B = \varnothing$. Then*

$$\text{(a)}\qquad [\eta_I(\eta_A\psi_B):\psi_N] = \begin{cases} 2^m + 1 & \text{if } I = B = N,\\ 1 & \text{if } A \subseteq I,\ A \cup B = N \text{ and } I \cap B \neq N,\\ 0 & \text{otherwise;}\end{cases}$$

$$\text{(b)}\qquad [\psi_i(\eta_A\psi_B):\psi_N] = \begin{cases} 3 & \text{if } B = N,\\ 1 & \text{if } B = N\setminus\{i\} \text{ and } A \subseteq \{i\},\\ 1 & \text{if } B = N\setminus\{i+1\} \text{ and } A = \{i+1\},\\ 0 & \text{otherwise.}\end{cases}$$

Here the bracket '[?:?]' means the multiplicity of the (simple) element to the right of the colon as a constituent of the element on the left-hand side.

Proof. (a) Let $C = N\setminus(A \cup B)$ and $J = N\setminus I$. Then

$$\begin{aligned}\eta_I\eta_A\psi_B &= \eta_{I\cap A}^2(\eta_{I\cap B}\psi_{I\cap B})\eta_{(I\cap C)\cup(J\cap A)}\psi_{J\cap B}\\ &= \prod_{i\in I\cap A}(1 + \psi_i + 2\eta_i + \eta_{i+1})\\ &\quad\times \prod_{j\in I\cap B}(\psi_j + 2\eta_{\{j,j+1\}} + \eta_{j+2} + 4\eta_{j+1} + 4\eta_j + 1)\eta_{(I\cap C)\cup(J\cap A)}\psi_{J\cap B}.\end{aligned}$$

The sets $I\cap A$, $I \cap B$, $I\cap C$, $J\cap A$ and $J \cap B$ are disjoint, so by (2.1.1) a necessary condition for ψ_N to be a constituent of this product is that for each of these subsets K, the corresponding factor has a term of mass $2^{|K|}$ and that the union of the subsets be equal to N. Thus we must have $I \cap C = \varnothing = J \cap A$, and hence that $A \subseteq I$, and also we must have $A \cup B = N$. Assuming these conditions to be fulfilled, we obtain the term $\psi_{I\cap A}\psi_{I\cap B}\psi_{J\cap B} = \psi_A\psi_B = \psi_N$ by taking the product of the ψ_i in each factor, so these terms yield one ψ_N in the product. The only other terms with enough mass have the form

$$\psi_A\psi_D\Big(\prod_{j\in(I\cap B)\setminus D} 2\eta_{\{j,j+1\}}\Big)\psi_{J\cap B} \quad \text{for } D \subset I \cap B \tag{2.3.1}$$

(where $\subset$ is used in its strict sense). Since $A \cup (J \cap B) \cup D = N\setminus((I \cap B)\setminus D)$, we see from (2.1.1) that there can be a constituent ψ_N in (2.3.1) only if $((I\cap B)\setminus D) + 1 = (I \cap B)\setminus D$. Clearly, this is not possible unless $D = \varnothing$ and $I = B = N$. In this case (2.3.1) becomes

$$\prod_{j\in N}(2\eta_{\{j,j+1\}}) = 2^m \prod_{j\in N}\eta_j^2,$$

of which ψ_N is a constituent with multiplicity 2^m, as is easily seen from Lemma 2.1 and (2.1.1). This proves (a).

(b) Twisting by a power of the automorphism τ, we may assume $i=0$. By (2.1.1) we see that a necessary condition for ψ_n to appear as a constituent of $\psi(\eta_A\psi_B)$ is that $|B|\geqslant m-1$, and that if $A=\emptyset$ then $B=N\backslash\{0\}$, in which case we get one ψ_N. Suppose $A=\{j\}$ and $B=N\backslash\{j\}$. If $j=0$, then by Lemma 2.1 and (2.1.1) we see that ψ_N occurs with multiplicity 1 as a constituent of $(\psi\eta)\psi_{N\backslash\{0\}}$. If $j=1$, then $\psi\eta_1\psi_{N\backslash\{1\}}=\psi^2\eta_1\psi_{N\backslash\{0,1\}}$ and the constituent $\eta_1\psi$ in the expansion of ψ^2 gives a term ψ_N in the product, while no other constituent of ψ^2 has enough mass to contribute a ψ_N. If $j\notin\{0, 1\}$, then (2.1.1) shows that ψ_N is not a constituent of $\psi\eta_j\psi_{N\backslash\{j\}}=\psi^2(\eta_j\psi_{N\backslash\{j\}})$. Finally, if $B=N$, then $\psi\psi_N=\psi^2\psi_{N\backslash\{0\}}$. The terms of mass 2 or greater in the expansion of ψ^2 are 2ψ, $\eta_1\psi$, $2\psi_1$, $6\eta_{\{0,1\}}$ and $\eta_{\{0,2\}}$. By (2.1.1) the last three do not give rise to any constituents ψ_N in the product. The term 2ψ clearly yields $2\psi_N$ and by (a) the product $(\eta_1\psi)\psi_{N\backslash\{0\}}=\eta_1\psi_N$ has ψ_N as a constituent with multiplicity 1. The lemma is proved.

Now S_N is a projective FG-module, so its multiplicity as a composition factor of an FG-module M is equal to $\dim \mathrm{Hom}_{FG}(S_N, M)$. Therefore, since $\mathrm{Hom}_{FG}(S_N, E_I\otimes E_A\otimes S_B)\cong \mathrm{Hom}_{FG}(S_N\otimes E_I, E_A\otimes S_B)$, Lemma 2.3 gives the multiplicity of the projective cover $P(E_A\otimes S_B)$ of the simple FG-module $E_A\otimes S_B$ as a direct summand of the projective FG-module $S_N\otimes E_I$.

Corollary 2.4. *We have*

(a)
$$S_N\otimes E_N\cong(F^{2^m}\otimes S_N)\oplus\bigoplus_{T\subseteq N}P(S_{N\backslash T}\otimes E_T),$$

(b) *for* $I\subset N$,
$$S_N\otimes E_I\cong\bigoplus_{T\subseteq I}P(S_{N\backslash T}\otimes E_T),$$

(c) $S_N\otimes S\cong S_N\oplus S_N\oplus S_N\oplus P(S_{N\backslash\{0\}})\oplus P(E\otimes S_{N\backslash\{0\}})\oplus P(E_1\otimes S_{N\backslash\{1\}})$.

Remark. We may invert (a) and (b) to obtain the following formulae in the Green ring of FG:

(a′) $[P(E_N)]=\sum_{T\subseteq N}(-1)^{|N\backslash T|}[S_N][E_T]-2^m[S_N]$;

(b′) $[P(E_I\otimes S_{N\backslash I})]=\sum_{T\subseteq I}(-1)^{|I\backslash T|}[S_N][E_T]\ (I\neq N)$.

Here $[M]$ denotes the class of the FG-module M.

3. *Some results on modules*

This section will be concerned with obtaining extremely explicit information about the structures of $E\otimes E$, $E\otimes S$ and $S\otimes S$. We begin by recalling the following terminology. Let M be an FG-module or a $\mathbf{G}$-module. By the *socle* of M, $\mathrm{soc}\, M=\mathrm{soc}^1 M$, we mean the largest semisimple submodule. Then $\mathrm{soc}^i M$ is defined recursively by the formula $\mathrm{soc}^i M/\mathrm{soc}^{i-1} M=\mathrm{soc}(M/\mathrm{soc}^{i-1} M)$. We write $\mathrm{soc}^i M/\mathrm{soc}^{i-1} M$ as $(\mathrm{soc}^i/\mathrm{soc}^{i-1})M$ and call this the ith *socle layer* of M. Dual to all this, we define the *radical* of M, JM to be the intersection of all maximal submodules of M, so that the *head* of M, $\mathrm{hd}(M)=M/JM$ is the maximal semisimple quotient of M. Then J^iM is defined as $J(J^{i-1}M)$ and the ith *Loewy layer* of M is $(J^{i-1}/J^i)M=J^{i-1}M/J^iM$.

We shall sometimes write '$M' \leq M$' to mean that the module M' is isomorphic to a submodule of the module M.

The first lemma, an adaptation of the argument of [**3**, Proposition 2.7], tells us that if V is a finite-dimensional **G**-module whose composition factors are all of the form $E_I \otimes S_J$ for $I, J \subseteq N = \{0, 1, \ldots, m-1\}$, then restricting to $G(m)$ does not change its socle and Loewy layers.

We shall need some notation for the proof of this result. If we define $\tau\lambda_2 = \lambda_1$, $\tau\lambda_1 = 3\lambda_2$, etc., so that $L(\tau^i\lambda) = L(\lambda)_i$, then each dominant weight has a unique 'τ-adic' expression

$$\lambda = \sum_{j=0}^{r} a_j \tau^j \lambda_2 \quad \text{for } 0 \leq a_j \leq 2.$$

If the upper limit r in the sum is less than m, we shall say that λ is *τ^m-restricted.* The τ^m-restricted weights parametrize the simple $G(m)$-modules; for λ as above, we have $L(\lambda) = E_I \otimes S_J$ where $I = \{j \mid a_j = 1\}$ and $J = \{j \mid a_j = 2\}$. Let $\tilde{\rho} = 2\sum_{j=0}^{n-1} \tau^j \lambda_2$. Then $L(\tilde{\rho}) = S_N$ and it can be checked using Weyl's dimension formula that in fact $S_N = H^0(\tilde{\rho})$. Finally, we define the linear functional f on Λ by $f(\lambda_2) = 1$, $f(\lambda_1) = \sqrt{3}$, so that if $\lambda \in \Lambda_+$ is given by its τ-adic expression above, then $f(\lambda) = \sum a_j \sqrt{3}^j$. We have $f(\alpha_1) = f(2\lambda_1 - 3\lambda_2) = (2\sqrt{3}) - 3 > 0$ and $f(\alpha_2) = f(2\lambda_2 - \lambda_1) = 2 - \sqrt{3} > 0$. It follows that whenever μ lies below λ in the usual partial ordering on Λ we have $f(\mu) < f(\lambda)$.

LEMMA 3.1. *Let (I, J) and (A, B) be pairs of disjoint subsets of $N = \{0, 1, \ldots, m-1\}$. Then the restriction map*

$$\mathrm{Ext}^1_{\mathbf{G}}(E_I \otimes S_J, E_A \otimes S_B) \to \mathrm{Ext}^1_{FG(m)}(E_I \otimes S_J, E_A \otimes S_B)$$

is injective.

Proof. Let $E_I \otimes S_J = L(\mu)$ and $E_A \otimes S_B = L(\lambda)$ for τ^m-restricted weights μ and λ. There are no **G**-extensions if μ and λ are either equal or incomparable in the partial order. Also, we have $\mathrm{Ext}^1_{\mathbf{G}}(L(\mu), L(\lambda)) \cong \mathrm{Ext}^1_{\mathbf{G}}(L(\lambda), L(\mu))$ and the same holds for $G(m)$. Therefore we may assume $\mu < \lambda$. Suppose X is a non-trivial **G**-extension of $L(\mu)$ by $L(\lambda)$. We shall show that $L(\mu)$ is not a $G(m)$-submodule of X. As in [**3**], Proposition 2.7, we see that there are **G**-module embeddings

$$X \hookrightarrow H^0(\lambda) \hookrightarrow H^0(\tilde{\rho}) \otimes H^0(\tilde{\rho} - \lambda) \cong S_N \otimes H^0(\tilde{\rho} - \lambda).$$

Thus, it suffices to prove that $\mathrm{Hom}_{FG(m)}(L(\mu), S_N \otimes H^0(\tilde{\rho} - \lambda)) \cong \mathrm{Hom}_{FG(m)}(S_N, L(\mu) \otimes H^0(\tilde{\rho} - \lambda)) = 0$. To see this, let $L(\nu)$ be a **G**-composition factor of $L(\mu) \otimes H^0(\tilde{\rho} - \lambda)$. Then $\nu \leq \mu + \tilde{\rho} - \lambda < \tilde{\rho}$. It follows that any $G(m)$-composition factor $L(\omega)$ of $L(\nu)$ must satisfy

$$f(\omega) \leq f(\nu) < f(\tilde{\rho}).$$

Therefore, $\mathrm{Hom}_{FG(m)}(S_N, L(\nu)) = 0$, for all ν as above, which completes the proof.

We sketch briefly why the lemma implies that the socle and Loewy layers are preserved on restriction from **G** to G. By duality, it suffices to consider the socle layers, and by induction on the **G**-socle length, one is reduced to showing that $\mathrm{soc}_{\mathbf{G}} V = \mathrm{soc}_G V$. The induction hypothesis yields $\mathrm{soc}_{\mathbf{G}}(V/\mathrm{soc}_{\mathbf{G}} V) =$

$\mathrm{soc}_G(V/\mathrm{soc}_{\mathbf{G}} V)$ which implies $\mathrm{soc}_G V \subseteq \mathrm{soc}^2_{\mathbf{G}} V$. This reduces the problem to modules of **G**-socle length 2, where Lemma 3.1 may be applied.

These remarks are relevant in particular to the **G**-modules $E \otimes E$, $S \otimes E$, $S \otimes S$ and the Weyl modules we shall discuss, because of our standing assumption that $m > 2$.

Good filtrations and Weyl filtrations

A **G**-module V is said to have a *good filtration* if there exists an ascending chain of submodules $0 = V^0 \subset V^1 \subset V^2 \subset \ldots$ such that $\bigcup_{i\geq 0} V^i = V$ and each factor V^i/V^{i-1} is isomorphic to some $H^0(\lambda_i)$, with $\lambda_i \in \Lambda_+$. We shall need the following facts (see [**7**, pp. 238–240]).

(a) A direct summand of a **G**-module with a good filtration has a good filtration.

(b) If V is a finite-dimensional **G**-module and has a good filtration then the number of filtration factors isomorphic to $H^0(\lambda)$ is equal to $\dim \mathrm{Hom}_{\mathbf{G}}(V(\lambda), V)$

(c) If V and W have good filtrations then so does $V \otimes W$ (this is a theorem of S. Donkin). If the good filtration factors of V and W are known then those of $V \otimes W$ can be computed using the classical formulae for Weyl modules over the complex numbers.

In addition, since $\mathrm{Ext}^1_{\mathbf{G}}(L(\lambda), H^0(\mu)) = 0$ for $\mu \not> \lambda$ (see [**7**, p. 207]), we have

(d) If V has a good filtration and if $H^0(\lambda)$ and $H^0(\mu)$ are two factors, with $\mu \not> \lambda$, then V has a good filtration in which $H^0(\lambda)$ appears higher up than $H^0(\mu)$. In particular, if V is finite-dimensional and λ and μ are respectively maximal and minimal in the set of highest weights of the good filtration factors of V, then $H^0(\lambda)$ is a homomorphic image of V and $H^0(\mu)$ is isomorphic to a submodule of V.

There is a dual notion of a *Weyl filtration* (descending filtration by Weyl modules) and dual versions of (a)–(d) above. If V has a good filtration then of course its dual $\mathrm{Hom}_F(V, F)$ has a Weyl filtration whose factors are the duals of the good filtration factors of V. A useful special case of this is that a self-dual module with a good filtration also has a Weyl filtration.

LEMMA 3.2. (a) *In the decomposition* $E \otimes E \cong S^2(E) \oplus \bigwedge^2(E)$ *we have* $S^2(E) \cong F \oplus S$ *and* $\bigwedge^2(E)$ *is uniserial with series* E, E_1, E.

(b) $\bigwedge^2(E) \otimes E_1$ *has a simple head and a simple socle, both isomorphic to* $E_{\{0,1\}}$, *and*

$$J(\textstyle\bigwedge^2(E) \otimes E_1)/\mathrm{soc}(\textstyle\bigwedge^2(E) \otimes E_1) \cong E_1 \otimes E_1.$$

(c) $E \otimes S \cong S \oplus Z$, *where* Z *is a* **G**-*module with head and socle both isomorphic to* E. *We have*

$$Z \otimes S_{N\setminus\{0\}} \cong P(E \otimes S_{N\setminus\{0\}})$$

as FG-*modules.*

(d) $S \otimes S \cong (E_1 \otimes S) \oplus S \oplus S \oplus X \oplus Y$ *as* **G**-*modules, where* X *has head and socle both isomorphic to* F, *and* Y *has head and socle both isomorphic to* E. *We have* FG-*isomorphisms*

$$Y \otimes S_{N\setminus\{0\}} \cong P(E \otimes S_{N\setminus\{0\}})$$

and

$$X \otimes S_{N\setminus\{0\}} \cong P(S_{N\setminus\{0\}}).$$

Moreover, Y has a good filtration with factors

$$H^0(\lambda_2),\ H^0(\lambda_1),\ H^0(\lambda_1+\lambda_2),\ H^0(3\lambda_2)$$

and X has a good filtration with factors

$$H^0(0),\ H^0(\lambda_1+\lambda_2),\ H^0(2\lambda_1),\ H^0(4\lambda_2).$$

Both X and Y are self-dual.

(e) *We have*

$$S^2(S) \cong X \oplus S \oplus S \quad \text{and} \quad \textstyle\bigwedge^2(S) \cong Y \oplus (E_1 \otimes S).$$

Proof. Recalling that $E \cong H^0(\lambda_2)$ and $S \cong H^0(2\lambda_2)$, it is routine to determine the good filtration factors $H^0(\lambda)$ of $E \otimes E$, $E \otimes S$ and $S \otimes S$, and also to find the composition factors of these dual Weyl modules.

(a) The module $E \otimes E$ has a good filtration with factors $H^0(0) \cong F$, $H^0(\lambda_2) \cong E$, $H^0(\lambda_1)$ and $H^0(2\lambda_2) \cong S$. Since $E \otimes E$ is self-dual, it follows that both the submodule F and the quotient module S are in fact direct summands, because these each occur only once as composition factors. From the structure of $H^0(\lambda_1)$ and the self-duality of $E \otimes E$, it follows that the complementary summand in $E \otimes E$ to $F \oplus S$ must be uniserial with series E, E_1, E. Consideration of the dimensions of $S^2(E)$ and $\bigwedge^2(E)$ now finishes the proof of (a).

(b) By (a) we have

$$\begin{aligned}\mathrm{Hom}_G(\textstyle\bigwedge^2(E) \otimes E_1, \mathrm{hd}(E_1 \otimes E_1)) &\cong \mathrm{Hom}_G(\textstyle\bigwedge^2(E) \otimes E_1, F \oplus S_1 \oplus E_1)\\ &\cong \mathrm{Hom}_G(\textstyle\bigwedge^2(E), E_1 \oplus (E_1 \otimes S_1) \oplus (E_1 \otimes E_1)),\end{aligned}$$

which is zero since $E_1 \oplus (E_1 \otimes S_1) \oplus (E_1 \otimes E_1)$ does not have $E \cong \mathrm{hd} \bigwedge^2(E)$ as a composition factor. It now follows from (a) that $\mathrm{hd}(\bigwedge^2(E) \otimes E_1) \cong E_{\{0,1\}}$ and by self-duality that the socle is isomorphic to the head, proving (b).

(c) The module $E \otimes S$ has a good filtration with factors

$$H^0(\lambda_2),\ H^0(\lambda_1),\ H^0(2\lambda_2),\ H^0(\lambda_1+\lambda_2),\ H^0(3\lambda_2).$$

Since $H^0(2\lambda_2)$ appears once as a factor, we have $\mathrm{Hom}_G(V(2\lambda_2), E \otimes S) \cong F$, which shows that $S \cong V(2\lambda_2)$ is isomorphic to a submodule of $E \otimes S$. Since $E \otimes S$ is self-dual and S occurs only once as a composition factor, it follows that S is isomorphic to a direct summand of $E \otimes S$. Let Z be the complementary summand. Then $E \cong H^0(\lambda_2)$ is isomorphic to a submodule of Z. By Corollary 2.4(b), we have

$$(E \otimes S) \otimes S_{N\setminus\{0\}} \cong S_N \oplus P(E \otimes S_{N\setminus\{0\}})$$

as FG-modules. It follows that $Z \otimes S_{N\setminus\{0\}} \cong P(E \otimes S_{N\setminus\{0\}})$ and hence that $\mathrm{soc}\, Z$ is simple. The assertion about $\mathrm{hd}(Z)$ follows by self-duality.

(d) The module $S \otimes S$ has a good filtration with factors

$$\begin{gathered}H^0(0),\ H^0(\lambda_2),\ H^0(\lambda_1),\ H^0(2\lambda_2),\ H^0(2\lambda_2),\ H^0(\lambda_1+\lambda_2),\\ H^0(\lambda_1+\lambda_2),\ H^0(3\lambda_2),\ H^0(2\lambda_1),\ H^0(\lambda_1+2\lambda_2),\ H^0(4\lambda_2).\end{gathered}$$

From this we see that $\mathrm{Hom}_{\mathbf{G}}(S, S\otimes S)=\mathrm{Hom}_{\mathbf{G}}(V(2\lambda_2), S\otimes S)$ is 2-dimensional and that

$$\dim \mathrm{Hom}_{\mathbf{G}}(E_1\otimes S, S\otimes S)=\dim \mathrm{Hom}_{\mathbf{G}}(V(\lambda_1+2\lambda_2), S\otimes S)=1.$$

Since these dimensions are equal to the multiplicities of S and $E_1\otimes S$ as composition factors of $S\otimes S$ and since $S\otimes S$ is self-dual, it follows that it has a direct summand isomorphic to $S\oplus S\oplus(E_1\otimes S)$. The dimension of this summand is 243. On the other hand, we have the decomposition $S\otimes S\cong S^2(S)\oplus\bigwedge^2(S)$ into direct summands of dimensions 378 and 351. It follows that the $\mathbf{G}$-module $S\otimes S$ decomposes into at least five indecomposable direct summands. But Corollary 2.4(c) shows that the FG-module

$$(S\otimes S)\otimes S_{N\setminus\{0\}}\cong S_N\oplus S_N\oplus S_N\oplus P(S_{N\setminus\{0\}})\oplus P(E\otimes S_{N\setminus\{0\}})\oplus P(E_1\otimes S_{N\setminus\{1\}})$$

has six indecomposable direct summands and Corollary 2.4(a) shows that

$$(3.2.1)\qquad (S\oplus S\oplus(E_1\otimes S))\otimes S_{N\setminus\{0\}}\cong S_N\oplus S_N\oplus S_N\oplus P(E_1\otimes S_{N\setminus\{1\}})$$

accounts for four of these six summands. It follows that $S\otimes S$ has precisely five indecomposable direct summands, both as a $\mathbf{G}$-module and as an FG-module. Let us write

$$S\otimes S\cong S\oplus S\oplus(E_1\otimes S)\oplus X\oplus Y,$$

where according to (3.2.1) we may choose X and Y to be such that $X\otimes S_{N\setminus\{0\}}\cong P(S_{N\setminus\{0\}})$ and $Y\otimes S_{N\setminus\{0\}}\cong P(E\otimes S_{N\setminus\{0\}})$ as FG-modules. Then X and Y have simple heads and socles and since it is easy to see that F and E are both submodules, and hence also quotients, of $S\otimes S$, we must have $\mathrm{soc}\,X\cong F\cong\mathrm{hd}\,X$ and $\mathrm{soc}\,Y\cong E\cong\mathrm{hd}\,Y$. We have shown that for $m>2$ we have $FG(m)$-isomorphisms

$$Y\otimes S_{N(m)\setminus\{0\}}\cong P(E\otimes S_{N(m)\setminus\{0\}})\cong Z\otimes S_{N(m)\setminus\{0\}}.$$

Since the character of the Steinberg module $S_{N(m)}$ of $G(m)$ does not vanish on 3-regular elements, multiplication by $\psi_{N(m)\setminus\{0\}}$ induces an injective endomorphism of $\mathcal{R}(G(m))$. Thus Y and Z have the same composition factors as $FG(m)$-modules. Since m may be arbitrarily large, this means that they have the same $\mathbf{G}$-composition factors, or, equivalently, the same weight multiplicities. Therefore a good filtration of Y has the same factors, counting multiplicities, as one of Z. Since we also know the good filtration factors of $S\otimes S$, we can deduce those of X, and (d) is proved.

(e) We see that X has a good filtration factor $H^0(4\lambda_2)$. Since $4\lambda_2$ is the highest weight of $S\otimes S$, it is a weight of $S^2(S)$. Then (e) is forced by consideration of dimensions.

Remark. One can show that in fact $Y\cong Z$, by showing that an isomorphism between the bottom factors $V(3\lambda_2)$ of Weyl filtrations of Y and Z can be extended. We shall not need this, however.

The good filtrations of X, Y, and Z give us the following information about maps.

COROLLARY 3.3. *The following equalities hold:*

(i) $\dim \operatorname{Hom}_{\mathbf{G}}(V(\lambda_1+\lambda_2), X) = 1$;

(ii) $\dim \operatorname{Hom}_{\mathbf{G}}(V(2\lambda_1), X) = 1$;

(iii) $\dim \operatorname{Hom}_{\mathbf{G}}(V(4\lambda_2), X) = 1$;

(iv) $\dim \operatorname{Hom}_{\mathbf{G}}(V(\lambda_1), Y) = \dim \operatorname{Hom}_{\mathbf{G}}(V(\lambda_1), Z) = 1$;

(v) $\dim \operatorname{Hom}_{\mathbf{G}}(V(\lambda_1+\lambda_2), Y) = \dim \operatorname{Hom}_{\mathbf{G}}(V(\lambda_1+\lambda_2), Z) = 1$;

(vi) $\dim \operatorname{Hom}_{\mathbf{G}}(V(3\lambda_2), Y) = \dim \operatorname{Hom}_{\mathbf{G}}(V(3\lambda_2), Z) = 1$.

In order to exploit this new information, we next study the structures of the Weyl modules appearing in Corollary 3.3. Here, we shall need the general fact (see [7, p. 207]) that if $\lambda, \mu \in \Lambda_+$, with $\mu < \lambda$, then $\operatorname{Ext}^1_{\mathbf{G}}(L(\lambda), L(\mu)) \cong \operatorname{Hom}_{\mathbf{G}}(JV(\lambda), L(\mu))$.

LEMMA 3.4. (a) *The module* $V(2\lambda_1)$ *is uniserial with composition factors (in descending order)* S_1, $E_{\{0,1\}}$, F.

(b) *We have* $\operatorname{hd}(V(\lambda_1+\lambda_2)) \cong E_{\{0,1\}}$ *and* $JV(\lambda_1+\lambda_2) \cong V(\lambda_1) \oplus F$.

(c) *The module* $V(3\lambda_2)$ *is uniserial with composition factors (in descending order)* E_2, E_1, $E_{\{0,1\}}$, E_1, E.

Proof. As we mentioned earlier there is an algorithm to find the composition factors of Weyl modules from the weight multiplicities of the simple **G**-modules, which we know. The Weyl modules in this lemma have the stated composition factors.

Next, let $\lambda, \mu \in \Lambda_+$ be fixed and let i be a fixed natural number. Then by [5, Theorem 7.1], we have, for sufficiently large (even) values of m, isomorphisms

$$\begin{aligned}\operatorname{Ext}^1_{\mathbf{G}}(L(\lambda), L(\mu)) &\cong \operatorname{Ext}^1_{FG(m)}(L(\lambda), L(\mu)) \\ &\cong \operatorname{Ext}^1_{FG(m)}(L(3^i\lambda), L(3^i\mu)) \cong \operatorname{Ext}^1_{\mathbf{G}}(L(3^i\lambda), L(3^i\mu)),\end{aligned}$$

where the middle isomorphism is induced by the automorphism $\sigma^i|_{G(m)}$ and the end isomorphisms are restriction maps. We may therefore deduce from $S \cong V(2\lambda_2)$ that $\operatorname{Ext}^1_{\mathbf{G}}(S_1, F) = 0$, from which (a) follows. From the structure of $V(\lambda_1)$ we have $\operatorname{Ext}^1_{\mathbf{G}}(E_{\{0,1\}}, F) \cong F$. Also, since $V(\lambda_1+\lambda_2)$ is isomorphic to a submodule of $V(\lambda_1) \otimes V(\lambda_2) \cong V(\lambda_1) \otimes E$, we have

$$\begin{aligned}\dim \operatorname{Hom}_{\mathbf{G}}(E_1, V(\lambda_1+\lambda_2)) &\leq \dim \operatorname{Hom}_{\mathbf{G}}(E_1, V(\lambda_1) \otimes E) \\ &= \dim \operatorname{Hom}_{\mathbf{G}}(E_{\{0,1\}}, V(\lambda_1)) = 0.\end{aligned}$$

Since $\operatorname{Ext}^1_{\mathbf{G}}(E_i, F) = 0$, it follows that $JV(\lambda_1+\lambda_2)$ is the direct sum of F and a non-split extension of E_1 by E. By the uniqueness of such an extension, it is isomorphic to $V(\lambda_1)$, which proves (b). To prove (c), we must show that $JV(3\lambda_2)$ is uniserial with the stated ordering of composition factors. Since $V(3\lambda_2) \subseteq V(\lambda_2) \otimes V(2\lambda_2) \cong E \otimes S$, Lemma 3.2(c) implies that $\operatorname{soc} V(3\lambda_2) \cong E$. Since by (b) we have $\operatorname{Ext}^1_{\mathbf{G}}(E_{\{0,1\}}, E) = 0$ and since $\operatorname{Ext}^1_{\mathbf{G}}(E_1, E) \cong F$, we must have $(\operatorname{soc}^2/\operatorname{soc})V(3\lambda_2) \cong E_1$. Then $(J/\operatorname{soc}^2)V(3\lambda_2)$ has composition factors $E_{\{0,1\}}$ and E_1 and since $\operatorname{Ext}^1_{\mathbf{G}}(E_1, E_2) = 0$, the structure in (c) is forced.

The module $V(4\lambda_2)$ will be considered in § 4.

We now summarize the information on **G**-extensions obtained from the statement and proof of Lemma 3.4.

COROLLARY 3.5. *We have*

(i) $\mathrm{Ext}^1_{\mathbf{G}}(E_{i+1}, E_i) \cong F$,
(ii) $\mathrm{Ext}^1_{\mathbf{G}}(E_i, F) = 0$,
(iii) $\mathrm{Ext}^1_{\mathbf{G}}(S_i, F) = 0$,
(iv) $\mathrm{Ext}^1_{\mathbf{G}}(S_i, E_i) = 0$,
(v) $\mathrm{Ext}^1_{\mathbf{G}}(S_{i+1}, E_{\{i,i+1\}}) \cong F$,
(vi) $\mathrm{Ext}^1_{\mathbf{G}}(E_{\{i,i+1\}}, E_{i+1}) \cong F$,
(vii) $\mathrm{Ext}^1_{\mathbf{G}}(E_{i+2}, E_{\{i,i+1\}}) = 0$,
(viii) $\mathrm{Ext}^1_{\mathbf{G}}(E_{i+2}, E_i) = 0$,
(ix) $\mathrm{Ext}^1_{\mathbf{G}}(S_i, E_{i+1}) = 0$.

The following simple remark will be important in the reduction steps in the proof of the theorem.

LEMMA 3.6. *There is an embedding of* **G***-modules* $\bigwedge^2(E) \hookrightarrow Z$, *and hence an embedding of* $E \otimes E \hookrightarrow F \oplus (E \otimes S)$. *The images are uniquely determined.*

Proof. From the structures of $\bigwedge^2(E)$ and Z described in Lemma 3.2 one sees that there will be an embedding as long as $\dim \mathrm{Hom}_{\mathbf{G}}(\bigwedge^2(E), Z) > 1$. Now $\bigwedge^2(E)$ has a good filtration with factors $H^0(\lambda_2)$ and $H^0(\lambda_1)$. A straightforward calculation then shows that a good filtration of $E \otimes \bigwedge^2(E)$ has factors $H^0(0)$, $H^0(\lambda_2)$, $H^0(\lambda_2)$, $H^0(\lambda_1)$, $H^0(2\lambda_2)$, $H^0(2\lambda_2)$, $H^0(\lambda_1+\lambda_2)$. Thus, by Lemma 3.2(c),

$$\begin{aligned}\dim \mathrm{Hom}_{\mathbf{G}}\big(\textstyle\bigwedge^2(E), Z\big) &= \dim \mathrm{Hom}_{\mathbf{G}}\big(\textstyle\bigwedge^2(E), E \otimes S\big)\\ &= \dim \mathrm{Hom}_{\mathbf{G}}\big(\big(S, E \otimes \textstyle\bigwedge^2(E)\big)\\ &= \dim \mathrm{Hom}_{\mathbf{G}}\big(V(2\lambda_2), E \otimes \textstyle\bigwedge^2(E)\big) = 2.\end{aligned}$$

Thus an embedding exists, and moreover, since $\mathrm{soc}\, Z \cong E \cong \mathrm{hd} \bigwedge^2(E)$, the 2-dimensionality of $\mathrm{Hom}_{\mathbf{G}}(\bigwedge^2(E), Z)$ means that any two embeddings will have the same image. The assertions about $E \otimes E$ are now immediate by Lemma 3.2(a) and (c).

LEMMA 3.7. *Let J and T be subsets of N, not both equal to N. Then $E_J \otimes S_T$ has $2^{|J \cap T|}$ indecomposable direct summands, both as* **G***-modules and as FG-modules. The head and socle of each summand are simple and isomorphic. We have*

(3.7.1)
$$\begin{aligned}\mathrm{soc}(E_J \otimes S_T) &\cong \mathrm{soc}(E_{J\cap T} \otimes S_{J\cap T}) \otimes E_{J\setminus(J\cap T)} \otimes S_{T\setminus(J\cap T)}\\ &\cong \bigotimes_{j\in J\cap T} \mathrm{soc}(E_j \otimes S_j) \otimes E_{J\setminus(J\cap T)} \otimes S_{T\setminus(J\cap T)}\\ &\cong \bigoplus_{(J\setminus(J\cap T))\subseteq K \subseteq J} E_K \otimes S_{(J\cap T)\setminus K}.\end{aligned}$$

Proof. First suppose that $J \subset N$. We know the result when $J \cap T = \varnothing$ and, by Lemma 3.2(c), when $J = T = \{j\}$. It follows that the **G**-module

$$E_J \otimes S_T = E_{J\cap T} \otimes S_{J\cap T} \otimes E_{J\setminus(J\cap T)} \otimes S_{T\setminus(J\cap T)}$$

has at least $2^{|J\cap T|}$ summands and that $E_{J\setminus(J\cap T)}\otimes S_{N\setminus T}$ has at least $2^{|J\setminus(J\cap T)|}$ summands. Therefore the **G**-module

$$E_J\otimes S_N\cong(E_{J\cap T}\otimes S_{J\cap T})\otimes(E_{J\setminus(J\cap T)}\otimes S_{N\setminus T})\otimes S_{T\setminus(J\cap T)}$$

has at least $2^{|J|}$ summands. On the other hand, Corollary 2.4(b) tells us that the restriction of this module to G has exactly $2^{|J|}$ indecomposable direct summands. Thus all inequalities above turn out to be equalities. In particular, $E_J\otimes S_T$ has $2^{|J\cap T|}$ summands. From this we may conclude that each indecomposable FG-summand of the module $E_J\otimes S_T$ is in fact the restriction of a **G**-summand. Corollary 2.4(b) also says that the socle of each summand of the FG-module $E_J\otimes S_N$ is simple and isomorphic to the head of the same summand. In particular, the FG-module $E_J\otimes S_N$ has the same number of indecomposable direct summands as its socle. The same must be true of $E_J\otimes S_N$ regarded as a **G**-module, and by the same counting argument as above, one sees that the same is true of $E_J\otimes S_T$. Thus, each summand of $E_J\otimes S_T$ has a simple socle and head. Now $\operatorname{soc}(E_i\otimes S_i)\cong E_i\oplus S_i$ by Lemma 3.2 and so

$$\bigotimes_{j\in J\cap T}\operatorname{soc}(E_j\otimes S_j)\cong\bigoplus_{L\subseteq J\cap T}E_L\otimes S_{(J\cap T)\setminus L}$$

is a semisimple submodule of $E_{J\cap T}\otimes S_{J\cap T}$ with $2^{|J\cap T|}$ summands, so must be the whole of $\operatorname{soc}(E_{J\cap T}\otimes S_{J\cap T})$. Then since

$$\left(\bigoplus_{L\subseteq J\cap T}E_L\otimes S_{(J\cap T)\setminus L}\right)\otimes E_{J\setminus(J\cap T)}\otimes S_{T\setminus(J\cap T)}\cong\bigoplus_{(J\setminus(J\cap T))\subseteq K\subseteq J}E_K\otimes S_{(J\cup T)\setminus K}$$

is a semisimple submodule of $E_J\otimes S_T$ with $2^{|J\cap T|}$ summands, it is equal to $\operatorname{soc}(E_J\otimes S_T)$. By the self duality of $E_J\otimes S_T$, its head is isomorphic to its socle, but we still have to show that each indecomposable summand has isomorphic head and socle. Our argument above shows that in fact each summand of $E_J\otimes S_T$ is the tensor product of a summand of $E_{J\cap T}\otimes S_{J\cap T}$ with the simple module $E_{J\setminus(J\cap T)}\otimes S_{T\setminus(J\cap T)}$. Thus it will suffice to show that for $I\subset N$, the indecomposable summands of $E_I\otimes S_I$ have isomorphic heads and socles. Suppose for a contradiction that M is a summand of $E_I\otimes S_I$ with $\operatorname{hd}M\not\cong\operatorname{soc}M$. Since the head and socle of each summand is simple and since $\operatorname{hd}(E_I\otimes S_I)\cong\operatorname{soc}(E_I\otimes S_I)$, there must be a different summand M' with $\operatorname{hd}M\cong\operatorname{soc}M'$. Since both $E_I\otimes S_I$ and $E_I\otimes S_N$ have $2^{|I|}$ indecomposable summands, both $M\otimes S_{N\setminus I}$ and $M'\otimes S_{N\setminus I}$ are indecomposable, and since $\operatorname{soc}(E_I\otimes S_N)\cong\operatorname{soc}(E_I\otimes S_I)\otimes S_{N\setminus I}$ and $\operatorname{hd}(E_I\otimes S_N)\cong\operatorname{hd}(E_I\otimes S_I)\otimes S_{N\setminus I}$, we have

$$\operatorname{hd}(M\otimes S_{N\setminus I})\cong(\operatorname{hd}M)\otimes S_{N\setminus I}\cong(\operatorname{soc}M')\otimes S_{N\setminus I}\cong\operatorname{soc}(M\otimes S_{N\setminus I}).$$

But by Corollary 2.4, $E_I\otimes S_N$ is the direct sum of $2^{|I|}$ non-isomorphic projective FG-modules, and so $M\otimes S_{N\setminus I}$ and $M'\otimes S_{N\setminus I}$ are two of these. We have reached a contradiction. This completes the proof of the case where $J\neq N$.

Suppose now that $J=N$, $T\neq N$. We know from the previous case that both as a **G**-module and as an FG-module $E_T\otimes S_T$ is the direct sum of $2^{|T|}$ direct summands, having a simple head of the form $E_{T\setminus K}\otimes S_K$, $K\subseteq T$, and isomorphic socle. Since $(E_{T\setminus K}\otimes S_K)\otimes E_{N\setminus T}\cong E_{N\setminus K}\otimes S_K$ is simple, the lemma will be proved if we show that for any two disjoint subsets A, $B\subseteq N$, we have

$$\text{(3.7.2)}\qquad \operatorname{Hom}_{FG}(E_N\otimes S_T,E_A\otimes S_B)\cong\begin{cases}F & \text{if } A=N\setminus B \text{ and } B\subseteq T,\\ 0 & \text{otherwise.}\end{cases}$$

Suppose first that $A \cup B \neq N$. Pick $r \in N \backslash (A \cup B)$. Then by the case $J \neq N$, we have

$$\mathrm{Hom}_{FG}(E_N \otimes S_T,\, E_A \otimes S_B) \cong \mathrm{Hom}_{FG}(E_{N\backslash\{r\}} \otimes S_T,\, E_{A\cup\{r\}} \otimes S_B) = 0.$$

Therefore we may assume $A = N\backslash B$. By Lemma 3.6,

$$\begin{aligned}
(3.7.3)\qquad & \dim \mathrm{Hom}_{FG}(E_N \otimes S_T,\, E_{N\backslash B} \otimes S_B) \\
&= \dim \mathrm{Hom}_{FG}(E_B \otimes S_T,\, E_{N\backslash B} \otimes E_{N\backslash B} \otimes S_B) \\
&\leqslant \dim \mathrm{Hom}_{FG}\Big(E_B \otimes S_T,\, \bigotimes_{i \in N\backslash B} (F \oplus (E_i \otimes S_i)) \otimes S_B\Big) \\
&= \sum_{K \subseteq N\backslash B} \dim \mathrm{Hom}_{FG}(E_{B\cup K} \otimes S_T,\, S_{B\cup K}).
\end{aligned}$$

For $K = N\backslash B$ we have, by Corollary 2.4,

$$\mathrm{Hom}_{FG}(E_{B\cup K} \otimes S_T,\, S_{B\cup K}) \cong \mathrm{Hom}_{FG}(S_N \otimes E_N,\, S_T) = 0.$$

Assume then that $K \subset N\backslash B$, so $B \cup K \neq N$. Then by the case where $J \neq N$, particularly (3.7.1), we have

$$\mathrm{Hom}_{FG}(E_{B\cup K} \otimes S_T,\, S_{B\cup K}) = \begin{cases} F & \text{if } B \cup K = T, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, all terms of the sum in (3.7.3) are zero unless $B \subseteq T$, in which case there is a single non-zero term, equal to 1, corresponding to $K = T\backslash B$. Since we already know that for $B \subseteq T$, $E_{N\backslash T} \otimes S_T$ is a homomorphic image of $E_N \otimes S_T$, we have proved (3.7.2), and hence the lemma.

Remark. If $J = T = N$, then the FG-module decomposition of $E_J \otimes S_T$ has already been described in Corollary 2.4(a) and we can see that even in this case the simple submodules of $E_N \otimes S_N$ are the same as those obtained by setting $J = T = N$ in the last member of (3.7.1), except that the multiplicity of S_N is $2^m + 1$ instead of 1. Therefore, when applying this lemma in situations where we are not worried about multiplicities, for example in proving that there are no maps from certain modules to $E_J \otimes S_T$, a reference to Lemma 3.7 is intended to include the case $J = T = N$.

One can deduce the **G**-module decomposition of $E_{N(m)} \otimes S_{N(m)}$ from Lemma 3.7; simply choose $m' > m$ so that $N(m) \subset N(m')$ and apply the lemma for $G = G(m')$.

As a corollary of this result we can already exhibit non-trivial extensions for each of (I)–(IV) of the theorem.

Proposition 3.8. *If* (I, J) *and* (A, B) *satisfy one of* (I)–(IV) *of the theorem, then* $\mathrm{Ext}^1_{FG}(E_I \otimes S_J,\, E_A \otimes S_B) \neq 0$.

Proof. By applying a power of the automorphism $\tau|_G$ we may take the index i appearing in (I)–(IV) to be 0. Let I and J be disjoint subsets of N with neither containing 0 or 1. Let Z be the module defined in Lemma 3.2(c) by $S \otimes E \cong$

$S \oplus Z$. It follows from Lemma 3.7 that

$$\mathrm{soc}(Z \otimes E_{I \cup \{1\}} \otimes S_J) \cong E_{I \cup \{0,1\}} \otimes S_J \cong (\mathrm{soc}\, Z) \otimes E_{I \cup \{1\}} \otimes S_J.$$

By Lemma 3.6 and the structure of $\bigwedge^2(E)$ (Lemma 3.2(a)), we have $E_1 \leqslant (\mathrm{soc}^2/\mathrm{soc})(Z)$. Therefore, the module

$$(E_I \otimes S_J) \oplus (E_I \otimes S_{J \cup \{1\}}) \oplus (E_{I \cup \{1\}} \otimes S_J) \cong \mathrm{soc}(E_1 \otimes E_1) \otimes E_I \otimes S_J$$
$$\leqslant E_1 \otimes E_{I \cup \{1\}} \otimes S_J$$

is a semisimple submodule of $(Z \otimes E_{I \cup \{1\}} \otimes S_J)/\mathrm{soc}(Z \otimes E_{I \cup \{1\}} \otimes S_J)$. Thus we have found non-trivial extensions for Cases (I)–(III) of the theorem.

Similarly, Lemma 3.7 implies that

$$\mathrm{soc}(Z \otimes E_I \otimes S_{J \cup \{1\}}) \cong (\mathrm{soc}\, Z) \otimes E_I \otimes S_{J \cup \{1\}} \cong E_{I \cup \{0\}} \otimes S_{J \cup \{1\}}.$$

Since $E_1 \leqslant (\mathrm{soc}^2/\mathrm{soc})Z$ and $S_1 \leqslant \mathrm{soc}(S_1 \otimes E_1)$, we have

$$E_I \otimes S_{J \cup \{1\}} \leqslant E_1 \otimes E_I \otimes S_{J \cup \{1\}} \leqslant (Z \otimes E_I \otimes S_{J \cup \{1\}})/\mathrm{soc}(Z \otimes E_I \otimes S_{J \cup \{1\}}),$$

which shows the non-triviality of the space of extensions in Case (IV). The proposition is proved.

4. *Some technical lemmas*

The proof of the following general lemma is straightforward.

LEMMA 4.1. *Let A and B be simple FG-modules and $d = d(A, B) = \dim \mathrm{Ext}^1_{FG}(A, B)$. Let $X(A, B)$ be an FG-module with $\mathrm{hd}\, X(A, B) \cong A$ and $JX(A, B) \cong B \oplus \ldots \oplus B$ (d copies). Let C be any FG-module and D a simple quotient of $B \otimes C$. Then*

$$\mathrm{Hom}_{FG}(X(A, B) \otimes C, D) = \mathrm{Hom}_{FG}(A \otimes C, D)$$

implies $d \leqslant \dim \mathrm{Ext}^1_{FG}(A \otimes C, D)$.

We shall keep the notations $d(A, B)$ and $X(A, B)$ of this lemma throughout the remainder of the paper.

Whenever we use Lemma 4.1, the module $A \otimes C$ will be simple and we shall be trying for inductive purposes to prove that $d(A, B) \leqslant d(A \otimes C, D)$. Usually, we shall have $A \otimes C \not\cong D$, so that the desired conclusion will follow from $\mathrm{Hom}_{FG}(X(A, B), C^* \otimes D) = 0$. This in turn will be proved by finding a suitable filtration of $C^* \otimes D$ whose factors can be embedded into modules M for which special circumstances will allow us to show that $\mathrm{Hom}_{FG}(X(A, B), M) = 0$. These remarks are intended as motivation for the remaining results of this section.

LEMMA 4.2. *Let $I, J, K \subseteq N$, $J \cap K = \varnothing$. If $|I| > |K| + 2$ then*

$$\mathrm{Ext}^1_{FG}(S_I, E_J \otimes E_K) = 0.$$

Proof. The proof will be by downward induction on $|I|$, starting at the projective module S_N. We may assume $K \subseteq I$, since $\mathrm{Ext}^1_{FG}(S_I, E_J \otimes S_K) \cong \mathrm{Ext}^1_{FG}(S_{I \cup K}, E_J \otimes S_{I \cap K})$. Suppose first that $J \nsubseteq I$. Pick $r \in J \backslash (J \cap I)$. We shall use Lemma 4.1 to show that $d(S_I, E_J \otimes S_K) \leqslant d(S_{I \cup \{r\}}, E_J \otimes S_K)$. Then we shall be reduced to the case where $J \subseteq I$. Since by Lemma 3.2(c), $E_I \otimes S_K$ is a quotient of $S_r \otimes (E_I \otimes S_K)$, the inequality is implied by

$$\mathrm{Hom}_{FG}(X(S_I, E_J \otimes S_K), S_r \otimes (E_J \otimes S_K)) = 0,$$

which is immediate from Lemma 2.2(a), since $|I| > |K| + 2$.

Suppose now that $J \subseteq I$. Pick $r \in N \backslash I$. It will suffice to show that

$$d(S_I, E_J \otimes S_K) \leqslant d(S_{I \cup \{r\}}, E_J \otimes S_{K \cup \{r\}}).$$

Since $|I| > |K| + 2 = |K \cup \{r\}| + 1$, Lemma 2.2(b) yields

$$\mathrm{Hom}_{FG}(X(S_I, E_J \otimes S_K), S_r \otimes (E_I \otimes S_{K \cup \{r\}})) = 0,$$

so the inequality follows from Lemma 4.1. The lemma is proved.

LEMMA 4.3. $\mathrm{Ext}^1(S_1, E_{\{0,2\}}) = 0$.

Proof. We shall show that

$$d(S_1, E_{\{0,2\}}) \leqslant d(S_{\{0,1\}}, E_{\{0,2\}}) \leqslant d(S_{\{0,1,2\}}, E_{\{0,2\}}). \tag{4.3.1}$$

The right-hand end of (4.3.1) is zero by Lemma 4.2. An easy calculation using Lemma 2.1 shows that S_1 is not a composition factor of $S_0 \otimes E_{\{0,2\}}$ and that $S_{\{0,1\}}$ is not a composition factor of $S_2 \otimes E_{\{0,2\}}$. Since $E_{\{0,2\}}$ is a quotient of both $S_0 \otimes E_{\{0,2\}}$ and $S_2 \otimes E_{\{0,2\}}$, the two inequalities follow from Lemma 4.1.

LEMMA 4.4. *The module $V(4\lambda_2)$ has the following Loewy layers:*

$$E_{\{0,2\}},\ E_{\{0,1\}},\ E_1 \oplus S_1,\ E_{\{0,1\}},\ F.$$

The socle layers are the same but in reverse order.

Proof. It is routine to check that the composition factors of $V(4\lambda_2)$ are as stated. By Lemma 3.2(d), we see that $V(4\lambda_2)$ is isomorphic to a submodule of the module X of that lemma. Since $\mathrm{soc}\, X \cong F$, we have $\mathrm{soc}\, V(4\lambda_2) \cong F$. Let us abbreviate $\mathrm{soc}\, V(4\lambda_2)$ by soc, etc. By Corollary 3.5, the only composition factor extending F is $E_{\{0,1\}}$ and $\mathrm{Ext}^1_{\mathbf{G}}(E_{\{0,1\}}, F) \cong F$, so we must have $\mathrm{soc}^2/\mathrm{soc} \cong E_{\{0,1\}}$. Therefore J/soc has a simple socle, so it follows that $\mathrm{soc}^2 \subseteq J^2$. By Lemmas 4.3 and 3.1, we have $\mathrm{Ext}^1_{\mathbf{G}}(E_{\{0,2\}}, S_1) = 0$ and by Corollary 3.5, $\mathrm{Ext}^1_{\mathbf{G}}(E_{\{0,2\}}, E_1) = 0$. Therefore $J/J^2 \cong E_{\{0,1\}}$. Thus, J/soc has a simple head and a simple socle isomorphic to $E_{\{0,1\}}$ and J^2/soc^2 has a composition factors E_1 and S_1. Since $\mathrm{Ext}^1_{\mathbf{G}}(S_1, E_1) = 0$ by Corollary 3.5, the proof is complete.

LEMMA 4.5. *There exists a* **G***-module T, unique up to isomorphism, with the following structure:*

$$\mathrm{soc}\, T \cong E_{\{0,1\}}, \quad T/\mathrm{soc}\, T \cong F \oplus S_1 \oplus \textstyle\bigwedge^2 (E_1) \cong E_1 \otimes E_1.$$

Any **G***-module V with $\mathrm{soc}\, V \cong E_{\{0,1\}}$ and $V/\mathrm{soc}\, V \leqslant T/\mathrm{soc}\, T$ is isomorphic to a submodule of T. The module T is isomorphic to a submodule of $S \otimes E \otimes E_1$.*

Proof. A module T with the structure described can be found as a submodule of $\bigwedge^2 (E) \otimes E_1$, by Lemma 3.2(b). Since by Lemma 3.6 we can embed $\bigwedge^2 (E)$ into $S \otimes E$, we can embed T into $S \otimes E \otimes E_1$. For the uniqueness assertions we have first by Corollary 3.5 that $\mathrm{Ext}^1_{\mathbf{G}}(S_1, E_{\{0,1\}}) \cong F$ and $\mathrm{Ext}^1_{\mathbf{G}}(F, E_{\{0,1\}}) \cong F$, which shows that any submodule of T whose image in $T/\mathrm{soc}\, T$ is isomorphic to a submodule of $F \oplus S_1$ is the unique **G**-module with its Loewy structure. We now consider submodules of $\bigwedge^2(E_1) \leq T/\mathrm{soc}\, T$. Corollary 3.5 gives

$$\mathrm{Ext}^1_{\mathbf{G}}(E_1, E_{\{0,1\}}) \cong F, \quad \mathrm{Ext}^1_{\mathbf{G}}(E, E_{\{0,1\}}) = 0 \quad \text{and} \quad \mathrm{Ext}^1_{\mathbf{G}}(E_1, E) \cong F.$$

By applying the long exact sequence for $\mathrm{Ext}_{\mathbf{G}}$, one sees that any submodule of T whose image in $T/\mathrm{soc}\, T$ is isomorphic to a submodule of $\bigwedge^2(E_1)$ is the unique **G**-module with its Loewy series. The assertions in the statement about uniqueness now follow.

We now look deeper into the structures of the **G**-modules X, Y and Z of Lemma 3.2.

LEMMA 4.6. (a) *$X/\mathrm{soc}\, X$ has a filtration with one factor isomorphic to $E_{\{0,2\}}$ and all other factors embeddable into $S \otimes E$ or $S \otimes E \otimes E_1$.*

(b) *Y and Z have filtrations in which every factor may be embeded into $S \otimes E$ or $S \otimes E \otimes E_1$.*

(c) *$Z/\mathrm{soc}\, Z$ has a filtration in which the bottom factor is isomorphic to E_1 and every other factor can be embedded into $S \otimes E$ or $S \otimes E \otimes E_1$.*

(d) *Let $U \subset Z$ be the image of $\bigwedge^2(E)$ under the embedding of Lemma 3.6. Then $\bar{Z} = Z/U$ has a filtration in which each factor may be embedded into $S \otimes E$ or $S \otimes E \otimes E_1$.*

Proof. (a) By factoring out by the first factor $H^0(0) \cong F$ in a good filtration of X we obtain a good filtration of $\bar{X} = X/\mathrm{soc}\, X$,

$$0 \subset V^1 \subset V^2 \subset \bar{X},$$

with $V^1 \cong H^0(\lambda_1 + \lambda_2)$, $V^2/V^1 \cong H^0(2\lambda_1)$ and $\bar{X}/V^2 \cong H^0(4\lambda_2)$. Let W^2 be the preimage in $\bar{X}$ of $\mathrm{soc}(V^2/V^1)$ under the natural map $\bar{X} \to \bar{X}/V^1$. From the structure of V^1, given in Lemma 3.4(a), and the fact that $\mathrm{Ext}^1_{\mathbf{G}}(S_1, E) = 0$, it follows that $\mathrm{Hom}_{\mathbf{G}}(W^2, E) \cong F$. Let W^1 be the kernel of a non-zero **G**-map from W^2 to E. We have $\mathrm{soc}\, X \cong F$. Since the composition factors of W^1 are F, E_1, S_1 and $E_{\{0,1\}}$, it follows from Corollary 3.5 that we must have $\mathrm{soc}\, W^1 \cong E_{\{0,1\}}$ and $W^1/\mathrm{soc}\, W^1 \cong F \oplus S_1 \oplus E_1$. Therefore by Lemma 4.6, we have an embedding of **G**-modules $W^1 \hookrightarrow S \otimes E \otimes E_1$. By Lemma 3.2(c), $W^2/W^1 \cong E \leq S \otimes E$. By Lemmas 3.4 and 4.5, we see that $V^2/W^2 \cong H^0(2\lambda_1)/\mathrm{soc}(H^0(2\lambda_1))$ also embeds into $S \otimes E \otimes E_1$. Next, the structure of $\bar{X}/V^2 \cong H^0(4\lambda_2)$ has been described in Lemma 4.4. Let W^3 be the preimage in $\bar{X}$ of $\mathrm{soc}(\bar{X}/V^2) \cong E_{\{0,2\}}$. It is immediate from Lemmas 4.4 and 4.5 that $(J^2/\mathrm{soc})H^0(4\lambda_2)$ and $H^0(4\lambda_2)/J^2H^0(4\lambda_2)$ are isomorphic to submodules of $S \otimes E \otimes E_1$. This proves (a).

(b) We start with a Weyl filtration of Y,

$$0 \subset V^1 \subset V^2 \subset V^3 \subset Y,$$

in which $V^1 \cong V(3\lambda_2)$, $V^2/V^1 \cong V(\lambda_1 + \lambda_2)$, $V^3/V^2 \cong V(\lambda_1)$ and $Y/V^3 \cong V(\lambda_2)$, and construct from this a new filtration with the desired properties. Since both $V(\lambda_1)$ and $V(\lambda_2)$ embed into $S \otimes E$, it will suffice to find a suitable filtration of V^2. Let W^2 be the preimage in V^2 of the trivial submodule of V^2/V^1 (see Lemma 3.4) and let $W^1 = \mathrm{soc}^2 V^1$. Then $W^1 \cong V(\lambda_1)$, so it embeds into $S \otimes E$. If F were a submodule of W^2/W^1, then since $\mathrm{Ext}^1_{\mathbf{G}}(F, E_i) = 0$, F would be a submodule of Y, which is not the case. Thus, $\mathrm{soc}(W^2/W^1) \cong E_{\{0,1\}}$. The triviality of $\mathrm{Ext}^1_{\mathbf{G}}(F, E_i)$

also implies that $(W^2/W^1)/\mathrm{soc}(W^2/W^1)$ is isomorphic to a submodule of $F \oplus \bigwedge^2(E_1)$, so, by Lemma 4.5, W^2/W^1 embeds into $S \otimes E \otimes E_1$. Now V^2/W^2 has a submodule isomorphic to $V(\lambda_1)$ with quotient isomorphic to $E_{\{0,1\}}$, by Lemma 3.4(b). The submodule embeds into $S \otimes E$ and the quotient into $S \otimes E \otimes E_1$. Thus, Y has a filtration with the stated properties.

Since this construction used only the Weyl filtration of Y and since Z has a Weyl filtration with exactly the same factors (see Lemma 3.2), the same construction works for Z, so (b) is proved.

(c) We take the filtration on $Z/\mathrm{soc}\, Z$ induced by the filtration on Z of (b). Then the bottom factor is $W^1/\mathrm{soc}\, Z \cong V(\lambda_1)/JV(\lambda_1) \cong E_1$ and all other factors are as in (b).

(d) We start with a good filtration of Z,

$$0 \subset V^1 \subset V^2 \subset V^3 \subset Z,$$

in which $V^1 \cong H^0(\lambda_2)$, $V^2/V^1 \cong H^0(\lambda_1)$, $V^3/V^2 \cong H^0(\lambda_1+\lambda_2)$ and $Z/V^3 \cong H^0(3\lambda_2)$. Since $\mathrm{soc}\, Z \cong E$, it follows easily from Corollary 3.5 that $V^2 \cong \bigwedge^2(E)$, and hence by the uniqueness of U that $V^2 = U$. Thus, $\bar{Z} = Z/U$ has a good filtration

$$0 \subset \overline{V^3} \subset \bar{Z}$$

with $\overline{V^3} \cong H^0(\lambda_1+\lambda_2)$ and $\bar{Z}/\overline{V^3} \cong H^0(3\lambda_2)$. Thus, $\mathrm{soc}\, \bar{Z} \leqslant E_{\{0,1\}} \oplus E_2$. We claim that in fact $\mathrm{soc}\, \bar{Z} \cong E_{\{0,1\}}$. Since by Lemma 3.2(c), Z has a submodule Z' isomorphic to $V(3\lambda_2)$ and, by Lemma 3.4, Z' has Loewy length 5, whereas U has Loewy length 3, it follows that the unique composition factor $E_2 \cong \mathrm{hd}\, V(3\lambda_2)$ of $\bar{Z}$ is not a composition factor of $\mathrm{soc}\, \bar{Z}$, which proves our claim.

We see from Corollary 3.5 that E_1 is the only composition factor of Z which extends E and that $\mathrm{Ext}^1_{\mathbf{G}}(E_1, E) \cong F$. Therefore $(\mathrm{soc}^2/\mathrm{soc})Z \cong E_1$ and so $\mathrm{soc}^2 Z = \mathrm{soc}^2 Z' = \mathrm{soc}^2 U = U \cap Z'$. Let $\overline{Z'}$ be the image of Z' in $\bar{Z}$ and let W be the preimage in $\bar{Z}$ of $\mathrm{soc}^2(\bar{Z}/\overline{V^3})$. Then since E_2 is a composition factor of W, we have $\overline{Z'} \subseteq W$. Let $\hat{W} = \overline{Z'} + \mathrm{soc}^2 W$. Then from the structures of $H^0(\lambda_1+\lambda_2)$ and $\mathrm{soc}^2(\bar{Z}/\overline{V^3})$ we see that $W/\hat{W}$ is an extension of E_1 by E. It is not clear whether this extension splits or not, but we shall show that in either case we can still find a filtration with the required property.

Suppose first that the extension does not split. Then $W/\hat{W} \cong V(\lambda_1)$ can be embedded into $S \otimes E$. It is not hard to see that $\hat{W}/\mathrm{soc}\, \hat{W} \cong F \oplus J \bigwedge^2(E_1)$. Since $\hat{W} \subseteq \bar{Z}$ and $\mathrm{soc}\, \bar{Z} \cong E_{\{0,1\}}$, we have $\mathrm{soc}\, \hat{W} \cong E_{\{0,1\}}$, so by Lemma 4.5, $\hat{W}$ embeds into $S \otimes E \otimes E_1$.

If the extension splits then we have a non-zero $\mathbf{G}$-homomorphism from W to E. Let W'' be the kernel of such a map. Then $W/W'' \cong E$ embeds into $S \otimes E$, while W'' is an extension of $\mathrm{soc}^2 H^0(3\lambda_2)$ by $W'' \cap \overline{V^3}$. Also $\overline{Z'} \subseteq W''$, so $E_2 \leqslant (\mathrm{soc}^3/\mathrm{soc}^2)W''$. It follows that the socle layers of $W''/\mathrm{soc}\, W''$ are (in ascending order) $E_1 \oplus F$, E_2, E_1. Since we know $\mathrm{Ext}^1_{\mathbf{G}}(E_i, F) = 0 = \mathrm{Ext}^1_{\mathbf{G}}(E_1, E_1)$ and $\mathrm{Ext}^1_{\mathbf{G}}(E_2, E_1) \cong F$, it follows that $W''/\mathrm{soc}\, W'' \cong F \oplus \bigwedge^2(E_1)$. Since $\mathrm{soc}\, W'' \subseteq \mathrm{soc}\, \bar{Z} \cong E_{\{0,1\}}$, Lemma 4.5 shows that W'' embeds into $S \otimes E \otimes E_1$. This completes the proof of the lemma.

Recall that

$$S \otimes S \cong S \oplus S \oplus (E_1 \otimes S) \oplus X \oplus Y$$

and that $\mathrm{soc}\, X \cong F$, $S \leqslant S \otimes E$ and $E_1 \otimes S \leqslant S \otimes E \otimes E_1$. Combining these facts with Lemma 4.6 leads to following useful criteria.

COROLLARY 4.7. *Let L and M be FG-modules. Then the following conditions are together sufficient for* $\mathrm{Hom}_{FG}(L, S_i \otimes S_i \otimes M)$ *to be zero. The last three suffice for* $\mathrm{Hom}_{FG}(L, (X/\mathrm{soc}\, X) \otimes M) = 0$.

(i) $\mathrm{Hom}_{FG}(L, M) = 0$;

(ii) $\mathrm{Hom}_{FG}(L, S_i \otimes E_i \otimes M) = 0$;

(iii) $\mathrm{Hom}_{FG}(L, S_i \otimes E_i \otimes E_{i+1} \otimes M) = 0$;

(iv) $\mathrm{Hom}_{FG}(L, E_i \otimes E_{i+2} \otimes M) = 0$.

Proof. Since $\tau|_G$ is an automorphism, we may assume $i = 0$. The result now follows from Lemma 4.6 since **G**-filtrations and **G**-embeddings are certainly filtrations and embeddings of FG-modules.

COROLLARY 4.8. *Let L and M be simple FG-modules such that the following hold:*

(i) $\mathrm{soc}(E_i \otimes S_i \otimes M) \cong \mathrm{soc}(E_i \otimes S_i) \otimes M$;

(ii) $\mathrm{Hom}_{FG}(L, S_i \otimes E_i \otimes M) = 0$;

(iii) $\mathrm{Hom}_{FG}(L, E_{i+1} \otimes M) = 0$;

(iv) $\mathrm{Hom}_{FG}(L, S_i \otimes E_i \otimes E_{i+1} \otimes M) = 0$.

Then L is not a composition factor of $\mathrm{soc}^2(S_i \otimes E_i \otimes M)$.

Proof. We may assume that $i = 0$. By (ii), $L \notin \mathrm{soc}(S_i \otimes E_i \otimes M)$. By (i),

$$(S_i \otimes E_i \otimes M)/\mathrm{soc}(S_i \otimes E_i \otimes M) \cong ((S_i \otimes E_i)/\mathrm{soc}(S_i \otimes E_i)) \otimes M,$$

and by Lemma 3.2(c), this module is isomorphic to $(Z/\mathrm{soc}\, Z) \otimes M$. The result now follows from (ii), (iii), (iv) and Lemma 4.6.

5. *Extensions of simple modules*

LEMMA 5.1. *Let* $I, J, K \subseteq N$, $J \cap K = \varnothing$. *If* $|J| > 2$ *then* $\mathrm{Ext}^1_{FG}(S_I, E_J \otimes S_K) = 0$.

Proof. We may assume that $K \subseteq I$. If $J \subseteq I$ then $|I| > |K| + 2$, so Lemma 4.2 applies. We therefore suppose that $J \nsubseteq I$. We shall prove that, for $r \in J \backslash (J \cap I)$,

$$\mathrm{Hom}_{FG}(X(S_I, E_J \otimes S_K), S_r \otimes E_J \otimes S_K) = 0, \tag{5.1.1}$$

which, since $E_J \otimes S_K$ is a quotient of $S_r \otimes E_J \otimes S_K$, will show by Lemma 4.1 that $d(S_I, E_J \otimes S_K) \leq d(S_{I \cup \{r\}}, E_J \otimes S_K)$ and eventually return us to the case where $J \subseteq I$. Clearly, $\mathrm{Hom}_{FG}(S_I, S_r \otimes E_J \otimes S_K) = 0$, and since by Lemma 3.7,

$$\mathrm{soc}(S_r \otimes E_J \otimes S_K) \cong \mathrm{soc}(S_r \otimes E_r) \otimes E_{J \backslash \{r\}} \otimes S_K,$$

(5.1.1) will follow from

$$\mathrm{Hom}_{FG}(S_I, ((S_r \otimes E_r)/\mathrm{soc}(S_r \otimes E_r)) \otimes E_{J \backslash \{r\}} \otimes S_K) = 0. \tag{5.1.2}$$

A composition series of $(S_r \otimes E_r)/\mathrm{soc}(S_r \otimes E_r)$ induces a filtration on

$$((S_r \otimes E_r)/\mathrm{soc}(S_r \otimes E_r)) \otimes E_{J \backslash \{r\}} \otimes S_K$$

with factors (ignoring multiplicities and order)

$$E_J \otimes S_K,\ E_{J\setminus\{r\}} \otimes S_K,\ E_{r+1} \otimes (E_J \otimes S_K),$$
$$E_{r+1} \otimes (E_{J\setminus\{r\}} \otimes S_K),\ E_{r+2} \otimes (E_{J\setminus\{r\}} \otimes S_K).$$

The first two are simple and not isomorphic to S_I. By Lemma 3.7, for $t \in N$, $E_t \otimes S_I$ is either simple or else its head is isomorphic to $(E_t \otimes S_{I\setminus\{t\}}) \oplus S_I$. Therefore, since $|J| > 2$,

$$\mathrm{Hom}_{FG}(S_I, E_{r+1} \otimes (E_J \otimes S_K)),\quad \mathrm{Hom}_{FG}(S_I, E_{r+1} \otimes (E_{J\setminus\{r\}} \otimes S_K))$$

and

$$\mathrm{Hom}_{FG}(S_I, E_{r+2} \otimes (E_{J\setminus\{r\}} \otimes S_K))$$

are all zero, which proves (5.1.2).

LEMMA 5.2. *Let $I, J, K \subseteq N$, $J \cap K = \varnothing$. If $I \supseteq J \cup K$ then* $\mathrm{Ext}^1_{FG}(S_I, E_J \otimes S_K) = 0$.

Proof. We shall proceed by downward induction on $|I|$, starting with S_N which is projective. Assume $I \subset N$ and pick $r \in N\setminus I$. Our inductive step will be the inequality

$$d(S_I, E_J \otimes S_K) \leq d(S_{I\cup\{r\}}, E_J \otimes S_{K\cup\{r\}}).$$

By applying τ^{-r}, we can assume $r = 0$. Then Lemma 4.1 will yield the inequality if we show that

(5.2.1) $$\mathrm{Hom}_{FG}(X(S_I, E_J \otimes S_K), (S \otimes S) \otimes (E_J \otimes S_K)) = \begin{cases} F & \text{if } J = \varnothing, I = K, \\ 0 & \text{otherwise.} \end{cases}$$

Let us consider the case $E_J \otimes S_K = S_I$ first. By Lemma 3.2, we have

$$(S \otimes S) \otimes S_I \cong S_{I\cup\{0\}} \oplus S_{I\cup\{0\}} \oplus (E_1 \otimes S_{I\cup\{0\}}) \oplus (X \otimes S_I) \oplus (Y \otimes S_I).$$

Clearly, $\mathrm{Hom}_{FG}(X(S_I, S_I), S_{I\cup\{0\}}) = 0$ and by Lemma 3.7,

$$\mathrm{Hom}_{FG}(X(S_I, S_I), E_1 \otimes S_{I\cup\{0\}}) = 0.$$

This second part of Lemma 3.2(d) implies that $\mathrm{soc}(Y \otimes S_I) \cong E \otimes S_I$, so $\mathrm{Hom}_{FG}(X(S_I, S_I), Y \otimes S_I) = 0$ as well. Hence it suffices to prove that $\mathrm{Hom}_{FG}(X(S_I, S_I), X \otimes S_I) \cong F$. The second part of Lemma 3.2(d) shows that $\mathrm{soc}(X \otimes S_I) \cong S_I \cong (\mathrm{soc}\, X) \otimes S_I$, so we are reduced to showing that $\mathrm{Hom}_{FG}(S_I, (X/\mathrm{soc}\, X) \otimes S_I) = 0$. Let us check the last three conditions in Corollary 4.7 with $L = M = S_I$ and $i = 0$:

(ii) $\mathrm{Hom}_{FG}(S_I, S \otimes E \otimes S_I) \cong \mathrm{Hom}_{FG}(S_{I\cup\{0\}}, E \otimes S_I) = 0$;

(iii) $\mathrm{Hom}_{FG}(S_I, S \otimes E \otimes E_1 \otimes S_I) \cong \mathrm{Hom}_{FG}(S_{I\cup\{0\}}, E \otimes E_1 \otimes S_I) = 0$, by Lemma 3.7;

(iv) $\mathrm{Hom}_{FG}(S_I, S \otimes E \otimes E_2 \otimes S_I) \cong \mathrm{Hom}_{FG}(E \otimes S_I, E_2 \otimes S_I) = 0$, by Lemma 3.7.

Thus, this case of (5.2.1) now follows from Corollary 4.7.

Now suppose $E_J \otimes S_K \not\cong S_I$. This case of (5.2.1) will also be proved using Corollary 4.7 if we verify the four conditions of that corollary, with $L = X(S_I, E_J \otimes S_K)$, $M = E_J \otimes S_K$ and $i = 0$.

(i) Obviously, $\mathrm{Hom}_{FG}(X(S_I, E_J \otimes S_K), E_J \otimes S_K) = 0$.

(ii) We have $\mathrm{Hom}_{FG}(X(S_I, E_J \otimes S_K), S \otimes E \otimes E_J \otimes S_K) = 0$, by Lemma 3.7, which computes the socle of the right-hand module.

(iii) We must show that $\mathrm{Hom}_{FG}(X(S_I, E_J \otimes S_K), S \otimes E \otimes E_1 \otimes E_J \otimes S_K) = 0$. If $1 \notin J$, this follows from Lemma 3.7. If $1 \in J$, we must first use Lemma 3.6 to embed $(E_1 \otimes E_1) \otimes S \otimes E \otimes E_{J\setminus\{1\}} \otimes S_K$ into

$$(S_{K \cup \{0\}} \otimes E_{J \cup \{0\} \setminus \{1\}}) \oplus (S_{K \cup \{0,1\}} \otimes E_{J \cup \{0\}})$$

before applying Lemma 3.7.

(iv) We must show that $\mathrm{Hom}_{FG}(X(S_I, E_J \otimes S_K), E \otimes E_2 \otimes E_J \otimes S_K) = 0$. If $2 \notin J$, we can use Lemma 3.7 immediately to compute the socle of $E_{J \cup \{0,2\}} \otimes S_K$, while if $2 \in J$, we can apply Lemma 3.7 after we have used Lemma 3.6 to embed $E_{\{0,2\}} \otimes E_J \otimes S_K \cong (E_2 \otimes E_2) \otimes E_{J \cup \{0\} \setminus \{2\}} \otimes S_K$ into

$$(E_{J \cup \{0\} \setminus \{2\}} \otimes S_K) \oplus (E_{J \cup \{0\}} \otimes S_{K \cup \{2\}}).$$

This completes the proof of (5.2.1), so the induction goes through.

LEMMA 5.3. *Let $J, A, B \subset N$, $A \cap B = \varnothing$, $B \subseteq J$. Then* $\mathrm{Ext}^1_{FG}(S_J, E_A \otimes S_B) = 0$ *unless one of the following holds:*

(i) $A = \{i, i+1\}, J = B$;

(ii) $A = \{i, i+1\}, J = B \cup \{i+1\}$;

(iii) $A = \{i\}, J = B$ *and* $i + 1 \in B$.

(Notice that these are precisely the conditions of the theorem for $I = \varnothing$.)

Proof. By Lemmas 4.2, 5.1 and 5.2 we may assume that $|J \setminus B| \leq 2$, $|A| \leq 2$ and $A \nsubseteq J$. Suppose first that $|J| = |B| + 2$. Then for $r \in A \setminus (J \cap A)$ we have $E_A \otimes S_B \leq \mathrm{hd}(S_r \otimes E_A \otimes S_B)$, by Lemma 3.2(c). A short calculation using Lemma 2.1(b) and Lemma 2.2 shows that S_J is not a composition factor of $(S_r \otimes E_r) \otimes E_{A \setminus \{r\}} \otimes S_B$. Hence, Lemma 4.1 yields

$$d(S_J, E_A \otimes S_B) \leq d(S_{J \cup \{r\}}, E_A \otimes S_B) = 0,$$

by Lemma 4.2. Thus we are left with the cases

(a) $J = B \cup \{t\}, A = \{j\}$,

(b) $J = B \cup \{t\}, A = \{i, j\}$ $(i \neq j)$,

(c) $J = B, A = \{i, j\}$ $(i \neq j)$, and

(d) $J = B, A = \{i\}$.

(a) By Lemma 5.2 we may assume that $j \neq t$. Lemmas 2.1(b) and 2.2 can be used to check that $S_{B \cup \{t\}}$ is not a composition factor of $S_j \otimes (E_j \otimes S_B)$. Since $E_j \otimes S_B \leq \mathrm{hd}(S_j \otimes (E_j \otimes S_B))$, Lemma 4.1 and Lemma 5.2 give

$$d(S_{B \cup \{t\}}, E_j \otimes S_B) \leq d(S_{B \cup \{t,j\}}, E_j \otimes S_B) = 0.$$

(b) Suppose $t \notin \{i, j\}$ or $t = j \neq i + 1$. We claim that

$$d(S_{B \cup \{t\}}, E_{\{i,j\}} \otimes S_B) \leq d(S_{B \cup \{t,i\}}, E_{\{i,j\}} \otimes S_B).$$

Since $E_{\{i,j\}} \otimes S_B \leq \mathrm{hd}(S_i \otimes (E_{\{i,j\}} \otimes S_B))$, this will follow from Lemma 4.1 if we show that

$$\mathrm{Hom}_{FG}(X(S_{B \cup \{t\}}, E_{\{i,j\}} \otimes S_B), S_i \otimes E_i \otimes E_j \otimes S_B) = 0.$$

We shall apply Corollary 4.8 with $L = S_{B \cup \{t\}}$ and $M = E_j \otimes S_B$. Condition (i) of the corollary holds by Lemma 3.7. We now check the other conditions.

(ii) Clearly,

$$\operatorname{Hom}_{FG}(S_{B\cup\{t\}}, S_i \otimes E_i \otimes E_j \otimes S_B) \cong \operatorname{Hom}_{FG}(S_{B\cup\{i,t\}}, E_{\{i,j\}} \otimes S_B) = 0.$$

(iii) We must show that $\operatorname{Hom}_{FG}(S_{B\cup\{t\}}, E_{i+1} \otimes E_j \otimes S_B) = 0$. If $i+1 \neq j$ then either $E_{i+1} \otimes E_j \otimes S_B$ is simple or $i = 1 \in B$ and Lemma 2.2 gives the desired conclusion. If $i+1=j$ then $E_{i+1} \otimes E_j \otimes S_B \cong S_{B\cup\{i+1\}} \oplus S_B \oplus (\bigwedge^2 (E_{i+1}) \otimes S_B)$. Only $S_{B\cup\{i+1\}}$ has as much mass as $S_{B\cup\{t\}}$ but, since $i+1=j$, our hypothesis says that $t \neq i+1$. Therefore (iii) holds.

(iv) We have

$$\operatorname{Hom}_{FG}(S_{B\cup\{t\}}, S_i \otimes E_i \otimes E_{i+1} \otimes E_j \otimes S_B) \cong \operatorname{Hom}_{FG}(S_{B\cup\{i,t\}}, E_{i+1} \otimes E_{\{i,j\}} \otimes S_B) = 0,$$

by considering masses.

Now Corollary 4.8 yields (b).

(c) We may choose notation so that $j \neq i+1$. We shall prove that

$$d(S_B, E_{\{i,j\}} \otimes S_B) \leq d(S_{B\cup\{i\}}, E_{\{i,j\}} \otimes S_B).$$

Then by (b) the right-hand side of the inequality is zero unless $i = j+1$. We shall apply Lemma 4.1 and Corollary 4.8 to prove this inequality. First we note that $E_{\{i,j\}} \otimes S_B \leq \operatorname{hd}(S_i \otimes E_{\{i,j\}} \otimes S_B)$. We now check the conditions of Corollary 4.8 for $L = S_B$ and $M = E_j \otimes S_B$. Condition (i) holds by Lemma 3.7 and (ii) and (iii) are easy to check because $i, i+1 \neq j$. The calculation for (iv) is:

$$\operatorname{Hom}_{FG}(S_B, S_i \otimes E_i \otimes E_{i+1} \otimes E_j \otimes S_B) \cong \operatorname{Hom}_{FG}(S_{B\cup\{i\}}, E_{\{i,i+1,j\}} \otimes S_B) = 0$$

by Lemma 3.7. The inequality now follows from Corollary 4.8 and Lemma 4.1.

(d) We shall show that if $i+1 \notin B$ then $d(S_B, E_i \otimes S_B) \leq d(S_{B\cup\{i\}}, E_i \otimes S_B)$, which is zero by Lemma 5.2. Again, we shall use Lemma 4.1 and Corollary 4.8. Since $E_i \otimes S_B \leq \operatorname{hd}(S_i \otimes E_i \otimes S_B)$, it remains only to check the conditions of Corollary 4.8 with $L = S_B$, $M = S_B$. Lemma 3.7 shows that (i) holds and (ii), (iii) and (iv) are all easy because $i+1 \notin B$.

This completes the proof of the lemma.

LEMMA 5.4. *Let (I, J) and (A, B) be pairs of disjoint subsets of N with $\varnothing \neq I \subseteq A$ and $B \subseteq J$. Then for $i \in I$ we have*

$$d(E_I \otimes S_J, E_A \otimes S_B) \leq d(E_{I\setminus\{i\}} \otimes S_J, E_{A\setminus\{i\}} \otimes S_B) + d(E_{I\setminus\{i\}} \otimes S_{J\cup\{i\}}, E_A \otimes S_B).$$

Proof. We may assume that $i = 0$. By Lemma 3.2(a), we have

$$\begin{aligned}\operatorname{Ext}^1_{FG}(E_I \otimes S_J, E_A \otimes S_B) &\cong \operatorname{Ext}^1_{FG}(E_{I\setminus\{0\}} \otimes S_J, (E \otimes E) \otimes E_{A\setminus\{0\}} \otimes S_B) \\ &\cong \operatorname{Ext}^1_{FG}(E_{I\setminus\{0\}} \otimes S_J, E_{A\setminus\{0\}} \otimes S_B) \\ &\quad \oplus \operatorname{Ext}^1_{FG}\big(E_{I\setminus\{0\}} \otimes S_J, \big(S \oplus \textstyle\bigwedge^2 (E)\big) \otimes E_{A\setminus\{0\}} \otimes S_B\big).\end{aligned}$$

The lemma will follow from the existence of an embedding

$$\operatorname{Ext}^1_{FG}\big(E_{I\setminus\{0\}} \otimes S_J, \big(S \oplus \textstyle\bigwedge^2 (E)\big) \otimes E_{A\setminus\{0\}} \otimes S_B\big) \hookrightarrow \operatorname{Ext}^1_{FG}(E_{I\setminus\{0\}} \otimes S_J, (S \otimes E) \otimes E_{A\setminus\{0\}} \otimes S_B).$$

Lemma 3.6 gives an embedding $S \oplus \bigwedge^2(E) \hookrightarrow S \otimes E$. By Lemma 3.2 and Lemma 3.6, the cokernel is the module $\bar{Z}$ of Lemma 4.6(d). Thus, by the long exact sequence for Ext_{FG}, it will suffice to show that

$$\mathrm{Hom}_{FG}(E_{I\setminus\{0\}} \otimes S_J,\ \bar{Z} \otimes E_{A\setminus\{0\}} \otimes S_B) = 0.$$

We shall apply Lemma 4.6. First,

$$\mathrm{Hom}_{FG}(E_{I\setminus\{0\}} \otimes S_J,\ S \otimes E \otimes E_{A\setminus\{0\}} \otimes S_B) \cong \mathrm{Hom}_{FG}(E_{I\setminus\{0\}} \otimes S_{J\cup\{0\}},\ E_A \otimes S_B) = 0.$$

It therefore remains to show that

$$\mathrm{Hom}_{FG}(E_{I\setminus\{0\}} \otimes S_J,\ S \otimes E \otimes E_1 \otimes E_{A\setminus\{0\}} \otimes S_B) = 0.$$

If $(I, J) = (A, B)$ then the left-hand side of (5.4.1) is isomorphic to $\mathrm{Hom}_{FG}(E_{I\setminus\{0\}} \otimes S_{I\cup\{0\}},\ E_1 \otimes E_I \otimes S_J)$. If $1 \notin I \cup J$, this is obviously zero, and if $1 \in J$ then this is zero by Lemma 3.7. If $1 \in I$, then $E_1 \otimes E_I \otimes S_J$ embeds into $(E_{I\setminus\{1\}} \otimes S_J) \oplus (E_I \otimes S_{J\cup\{1\}})$, and Lemma 3.7 then shows that $E_{I\setminus\{0\}} \otimes S_{J\cup\{0\}}$ is not in the socle of this module.

Now suppose that $(I, J) \neq (A, B)$. Set $K = N\setminus(I \cup J)$, $I' = I \cup (K \cap A)$ and $A' = A\setminus(K \cap A)$. Then (5.4.1) can be transformed to

$$\mathrm{Hom}_{FG}(E_{I'\setminus\{0\}} \otimes S_{I\cup\{0\}},\ E_1 \otimes E_{A'} \otimes S_B) = 0. \tag{5.4.2}$$

Since $A' \subseteq I' \cup J$ and $B \subseteq J$, the assumption $(I, J) \neq (A, B)$ implies that

$$\mathrm{mass}(E_{A'} \otimes S_B) < \mathrm{mass}(E_{I'} \otimes S_J) = \mathrm{mass}(E_{I'\setminus\{0\}} \otimes S_{B\cup\{0\}}) - 1.$$

Therefore by (2.1.1),

$$\mathrm{mass}(E_1 \otimes E_{A'} \otimes S_B) < \mathrm{mass}(E_{I'\setminus\{0\}} \otimes S_{J\cup\{0\}}),$$

from which (5.4.2) is clear. The lemma is proved.

We may now complete the proof of the vanishing in the theorem.

PROPOSITION 5.5. *Let (I, J) and (A, B) be pairs of disjoint subsets of N, with $I \subseteq A$ and $B \subseteq J$, and such that $\mathrm{Ext}^1_{FG}(E_I \otimes S_B, E_A \otimes S_B) \neq 0$. Then these pairs satisfy one of the conditions* (I)–(IV) *of the theorem. Furthermore, according to which of the conditions* (I)–(IV) *is satisfied by (I, J) and (A, B) we have*

(I) $d(E_I \otimes S_J, E_A \otimes S_B) \leq d(S_J, E_{\{i,i+1\}} \otimes S_J)$,

(II) $d(E_I \otimes S_J, E_A \otimes S_B) \leq d(S_{B\cup\{i+1\}}, E_{\{i,i+1\}} \otimes S_B)$,

(III) $d(E_I \otimes S_J, E_A \otimes S_B) \leq d(S_{J\cup\{i+1\}}, E_{\{i,i+1\}} \otimes S_J)$,

(IV) $d(E_I \otimes S_J, E_A \otimes S_B) \leq d(S_J, E_i \otimes S_J)$.

Proof. By iterating 5.4 we obtain

$$d(E_I \otimes S_J, E_A \otimes S_B) \leq \sum_{K \subseteq I} d(S_{J\cup K}, E_{(A\setminus I)\cup K} \otimes S_B). \tag{5.5.1}$$

Lemma 5.3 says that $d(S_{J\cup K}, E_{(A\setminus I)\cup K} \otimes S_B)$ is zero unless one of the following holds:

(i) $(A\setminus I) \cup K = \{i, i+1\}, J \cup K = B$;

(ii) $(A\setminus I) \cup K = \{i, i+1\}, J \cup K = B \cup \{i+1\}$;

(iii) $(A\setminus I) \cup K = \{i\}, J \cup K = B$ and $i+1 \in B$.

If (i) holds then we must have $K=\varnothing$, $B=J$ and $A=I\cup\{i,i+1\}$, since $B\subseteq J$. Thus (I,J) and (A,B) satisfy (I). For these pairs, the terms of (5.5.1) for $K\neq\varnothing$ are zero by Lemma 5.3, which proves the inequality for (I).

Next suppose (ii) holds. Then we may have $K=\varnothing$ or $K=\{i+1\}$ but not $K=\{i,i+1\}$, since $K\cap B=\varnothing$. If $K=\varnothing$ then we have $J=B\cup\{i+1\}$ and $A=I\cup\{i,i+1\}$, which is Condition (II) of the theorem. Moreover, by Lemma 5.3 the terms in (5.5.1) with $K\neq\varnothing$ are zero, which gives the inequality in the statement of this proposition for pairs satisfying (II). If $K=\{i+1\}$ then $B=J$ and $A=I\cup\{i\}$. Since $i+1\in K\subseteq I$, we have Condition (III). The terms of (5.5.1) for $K\neq\{i+1\}$ are zero by Lemma 5.3, so we also obtain the desired inequality in this case.

Finally if (iii) holds then $K=\varnothing$, $B=J$, $A=I\cup\{i\}$ and $i+1\in J$, so (IV) holds. Again the inequality for pairs satisfying (IV) comes from Lemma 5.3, which shows that the terms in (5.5.1) for $K=\varnothing$ are zero. The proposition is proved.

In view of our construction of non-trivial extensions in Proposition 3.8, it remains only to prove the correct upper bound of 1 for the right-hand sides of the inequalities in the last proposition. This will be done by establishing a chain of inequalities and finally computing the end term.

LEMMA 5.6. *Let $J\subset N$ with $i, i+1\in N\backslash J$. Then*

(a) $d(S_J, E_{\{i,i+1\}}\otimes S_J)\leq d(S_{J\cup\{i+1\}}, E_{\{i,i+1\}}\otimes S_J)$;

(b) *if $r\in N\backslash(J\cup\{i,i+1\}$ then*

$$d(S_{J\cup\{i+1\}}, E_{\{i,i+1\}}\otimes S_J)\leq d(S_{J\cup\{r,i+1\}}, E_{\{i,i+1\}}\otimes S_{J\cup\{r\}});$$

(c) *if $r\in N\backslash(J\cup\{i,i+1\})$ then*

$$d(S_{J\cup\{i+1\}}, E_i\otimes S_{J\cup\{i+1\}})\leq d(S_{J\cup\{r,i+1\}}, E_i\otimes S_{J\cup\{r,i+1\}}).$$

Proof. We may assume that $i=0$.

(a) We shall use Lemma 4.1. First we note that

$$E_{\{0,1\}}\otimes S_J\leq \mathrm{hd}(S_1\otimes E_{\{0,1\}}\otimes S_J).$$

Therefore the inequality will follow from

$$\mathrm{Hom}_{FG}(X(S_J, E_{\{0,1\}}\otimes S_J), S_1\otimes E_{\{0,1\}}\otimes S_J)=0. \tag{5.6.1}$$

We check the conditions of Corollary 4.8, with $L=S_J$, $M=E\otimes S_J$ and $i=1$. Condition (i) holds by Lemma 3.7. For the others, we have:

(ii) $\mathrm{Hom}_{FG}(S_J, E_1\otimes S_1\otimes E\otimes S_J)\cong\mathrm{Hom}_{FG}(S_{J\cup\{1\}}, E_{\{0,1\}}\otimes S_J)=0$;

(iii) $\mathrm{Hom}_{FG}(S_J, E_2\otimes E\otimes S_J)=0$, by Lemma 3.7;

(iv) $\mathrm{Hom}_{FG}(S_J, S_1\otimes E_1\otimes E_2\otimes E\otimes S_J)\cong\mathrm{Hom}_{FG}(S_{J\cup\{1\}}, E_{\{0,1,2\}}\otimes S_J)=0$, by Lemma 3.7.

Thus the conditions of Corollary 4.8 are satisfied and (5.6.1) follows via Lemma 4.1.

(b) We can again apply Lemma 4.1 if we first show that

$$\mathrm{Hom}_{FG}(X(S_{J\cup\{1\}}, E_{\{0,1\}}\otimes S_J), S_r\otimes S_r\otimes E_{\{0,1\}}\otimes S_J)=0. \tag{5.6.2}$$

We shall use Corollary 4.7 with $L=X(S_{J\cup\{1\}}, E_{\{0,1\}}\otimes S_J)$, $M=E_{\{0,1\}}\otimes S_J$ and $i=r$. Condition (i) is clearly satisfied.

(ii) We have $\mathrm{Hom}_{FG}(X(S_{J\cup\{1\}}, E_{\{0,1\}}\otimes S_J), S_r\otimes E_r\otimes E_{\{0,1\}}\otimes S_J)=0$, since by Lemma 3.7, neither $S_{J\cup\{1\}}$ nor $E_{\{0,1\}}\otimes S_J$ occurs in $\mathrm{soc}(E_{\{0,1,r\}}\otimes S_{J\cup\{r\}})$.

(iii) We must show that

$$\mathrm{Hom}_{FG}(X(S_{J\cup\{1\}}, E_{\{0,1\}}\otimes S_J), S_r\otimes E_r\otimes E_{\{0,1\}}\otimes S_J)=0.$$

If $r+1\neq 0$, then we can see this immediately by applying Lemma 3.7, as in the verification of (i). If $r+1=0$, then we must first use Lemma 3.6 to embed $S_r\otimes E_r\otimes(E\otimes E)\otimes E_1\otimes S_J$ into $(E_{\{1,r\}}\otimes S_{J\cup\{r\}})\oplus(E_{\{0,1,r\}}\otimes S_{J\cup\{0,r\}})$ before applying Lemma 3.7 in order to see that neither $S_{J\cup\{1\}}$ nor $E_{\{0,1\}}\otimes S_J$ is a submodule.

(iv) We must show that

$$\mathrm{Hom}_{FG}(X(S_{J\cup\{1\}}, E_{\{0,1\}}\otimes S_J), E_r\otimes E_{r+2}\otimes E_{\{0,1\}}\otimes S_J)=0.$$

If $r+2\notin\{0,1\}$ then Lemma 3.7 shows that no composition factor of the left module is a submodule of the right module. If $r+2=0$ or $r+2=1$, we may reach the same conclusion by first using Lemma 3.6 as in the verification of (iii) to embed the right module into one to which we can apply Lemma 3.7.

Having checked the conditions of Corollary 4.7 we have proved (5.6.2), and hence (b).

(c) The proof is similar to that of (b). We must check the conditions of Corollary 4.7 with $L=X(S_{J\cup\{1\}}, E\otimes S_{J\cup\{1\}})$, $M=E\otimes S_{J\cup\{1\}}$ and $i=r$. It is clear that (i) holds, and (ii) holds by Lemma 3.7. For (iii), if $r+1\neq 0$ then Lemma 3.7 applies, while if $r+1=0$, we must first embed $S_r\otimes E_r\otimes(E\otimes E)\otimes S_{J\cup\{1\}}$ into $(E_r\otimes S_{J\cup\{1,r\}})\oplus(E_{\{0,1\}}\otimes S_{J\cup\{0,1,r\}})$ using Lemma 3.6, and then Lemma 3.7 yields the desired conclusion. Condition (iv) is checked similarly, using Lemma 3.7, immediately if $r+2\neq 0$, and after embedding $E_r\otimes(E\otimes E)\otimes S_{J\cup\{1\}}$ into $(E_r\otimes S_{J\cup\{1\}})\oplus(E_{\{0,r\}}\otimes S_{J\cup\{0,1\}})$ if $r+2=0$.

The lemma is proved.

The proof of the theorem will be completed by the following result.

LEMMA 5.7. *We have*

(a) $\mathrm{Ext}^1_{FG}(S_{N\setminus\{i\}}, E_{\{i,i+1\}}\otimes S_{N\setminus\{i,i+1\}})\cong F$;

(b) $\mathrm{Ext}^1_{FG}(S_{N\setminus\{i\}}, E_i\otimes S_{N\setminus\{i\}})\cong F$.

Proof. Since we know by Proposition 3.8 that these groups are not trivial, it will be enough to show that their dimensions are no greater than 1. Also, we may assume $i=0$. By Lemma 3.2(c) we have

$$P(E\otimes S_{N\setminus\{0\}})\cong Z\otimes S_{N\setminus\{0\}}.$$

Since moreover, $\mathrm{soc}(Z\otimes S_{N\setminus\{0\}})\cong(\mathrm{soc}\,Z)\otimes S_{N\setminus\{0\}}$, the two parts of the lemma will follow from

$$\dim\mathrm{Hom}_{FG}(S_{N\setminus\{0,1\}}\otimes E_1, (Z/\mathrm{soc}\,Z)\otimes S_{N\setminus\{0\}})\leqslant 1 \tag{5.7.1}$$

and

$$\dim\mathrm{Hom}_{FG}(S_{N\setminus\{0\}}, (Z/\mathrm{soc}\,Z)\otimes S_{N\setminus\{0\}})\leqslant 1. \tag{5.7.2}$$

By Lemma 4.6(c), $(Z/\mathrm{soc}\,Z)$ has a filtration with one factor isomorphic to E_1 and all other factors embeddable into $S\otimes E$ or $S\otimes E\otimes E_1$. We have

$$\mathrm{Hom}_{FG}(S_{N\setminus\{0,1\}}\otimes E_1, E_1\otimes S_{N\setminus\{0\}})\cong F,$$

by Lemma 3.7. Also

$$\mathrm{Hom}_{FG}(S_{N\setminus\{0,1\}}\otimes E_1,\ S\otimes E\otimes S_{N\setminus\{0\}})\cong \mathrm{Hom}_{FG}(S_{N\setminus\{0,1\}}\otimes E_{\{0,1\}},\ S_N)=0$$

and

$$\mathrm{Hom}_{FG}(S_{N\setminus\{0,1\}}\otimes E_1,\ S\otimes E\otimes E_1\otimes S_{N\setminus\{0\}})$$
$$\cong \mathrm{Hom}_{FG}(S_{N\setminus\{0,1\}}\otimes E_{\{0,1\}},\ S_N\otimes E_1)=0,$$

by Corollary 2.4. This proves (5.7.1), and hence (a). Similarly, (5.7.2) follows from the following calculations:

$$\mathrm{Hom}_{FG}(S_{N\setminus\{0\}},\ E_1\otimes S_{N\setminus\{0\}})\cong F,$$

by Lemma 3.7,

$$\mathrm{Hom}_{FG}(S_{N\setminus\{0\}},\ S\otimes E\otimes S_{N\setminus\{0\}})\cong \mathrm{Hom}_{FG}(S_N,\ E\otimes S_{N\setminus\{0\}})=0$$

and

$$\mathrm{Hom}_{FG}(S_{N\setminus\{0\}},\ S\otimes E\otimes E_1\otimes S_{N\setminus\{0\}})\cong \mathrm{Hom}_{FG}(S_N\otimes E_1,\ E\otimes S_{N\setminus\{0\}})=0,$$

by Corollary 2.4.

The proof is complete.

Appendix: $m\leqslant 2$

We shall describe here the extensions of simple $FG(m)$-modules for $m=1$ and $m=2$. These do not follow the general pattern and we use some special arguments to compute them.

Case 1: $m=1$. The simple $FG(1)$-modules are F, E and S. The module S is a projective $FG(1)$-module, so the following describes all extensions of simple modules.

Proposition A1. *We have*

(1) $\mathrm{Ext}^1_{FG}(F,F)\cong F$;

(2) $\mathrm{Ext}^1_{FG}(F,E)\cong F$;

(3) $\mathrm{Ext}^1_{FG}(E,E)\cong F^2$.

Proof. It is well known that for $G={}^2G_2(3)$ we have an exact sequence

$$1\to L\to G\to \mathbb{Z}/3\mathbb{Z}\to 1$$

with $L\cong \mathrm{SL}_2(8)$. Therefore (1) is clear. We shall prove (2) and (3) by means of the inflation-restriction sequence

$$0\to H^1(G/L,M^L)\to H^1(G,M)\xrightarrow{\mathrm{res}} H^1(L,M) \tag{A.1.1}$$

for appropriate choices of the FG-module M. The restriction to L of E remains simple, since, for example, the index of L and the dimension of E are coprime. If we think of L as the group of 2×2 matrices over $\mathbb{F}_8$ of determinant 1, and let B be the subgroup of lower triangular matrices in L then the permutation FL-module on the right cosets of B is a 9-dimensional projective FL-module, which is easily seen to be the projective cover $P_L(F)$ of the trivial module. In all there are five simple FL-modules. They are F, E and three non-isomorphic 9-dimensional projective simple modules which are the FL-summands of S. Thus we must have $\mathrm{Ext}^1_{FL}(F, E) \neq 0$ and then it follows from dimensions that $\mathrm{Ext}^1_{FL}(F, E) \cong F$ and that $P_L(F)$ is uniserial with series F, E, F. Next, a straightforward matrix calculation shows that $\mathrm{Hom}_{FL}(E \otimes E, E) \cong F$. From what we know about the **G**-module structure of $E \otimes E$ (Lemma 3.2), it follows that as an FL-module, $\bigwedge^2(E)$ is uniserial with series E, E, E. Thus, using Lemma 3.2(a), we have

$$\mathrm{Ext}^1_{FL}(E, E) \cong H^1(L, E \otimes E) \cong H^1(L, F \oplus S \oplus \textstyle\bigwedge^2(E))$$
$$= H^1(L, \textstyle\bigwedge^2(E)) \cong \mathrm{Hom}_{FL}(JP_L(F), \textstyle\bigwedge^2(E)) \cong F.$$

Let $M = E \otimes E$ in (A.1.1). Since $\bigwedge^2(E)$ remains uniserial as an FL-module, the restriction map in (A.1.1) is not the zero map, and (3) is proved. Since FG has only the simple modules F, E and S, it is obvious that $\mathrm{Ext}^1_{FG}(F, E) \neq 0$. The correct upper bound for (2) is obtained by setting $M = E$ in (A.1.1).

Case 2: $m = 2$.

PROPOSITION A2. *In the table below the dimension of the space of extensions between two simple modules is the entry of the row labelled by the first module and the column labelled by the second.*

Ext^1_{FG}	F	E	E_1	S	S_1	$E_{\{0,1\}}$	$E \otimes S_1$	$E_1 \otimes S$	$S_{\{0,1\}}$
F	0	0	0	0	0	2	0	0	0
E	0	0	2	0	0	1	0	1	0
E_1	0	2	0	0	0	1	1	0	0
S	0	0	0	0	0	1	0	1	0
S_1	0	0	0	0	0	1	1	0	0
$E_{\{0,1\}}$	2	1	1	1	1	0	0	0	0
$E \otimes S_1$	0	0	1	0	1	0	0	0	0
$E_1 \otimes S$	0	1	0	1	0	0	0	0	0
$S_{\{0,1\}}$	0	0	0	0	0	0	0	0	0

Proof. We begin by showing that the spaces of extensions have dimensions greater than or equal to the entries in our table. Since $\mathrm{Hom}_{FG}(E \otimes E, E_1) = 0$, it follows, as in Lemma 3.2(a), that $\bigwedge^2(E)$ is uniserial with series E, E_1, E and hence that $\bigwedge^2(E_1)$ is uniserial with series E_1, $E_2 \cong E$, E_1. We shall show that $J\bigwedge^2(E) \not\cong (\bigwedge^2(E_1))/(\mathrm{soc}\bigwedge^2(E_1))$, thereby exhibiting two non-isomorphic non-split extensions of E_1 by E. We have

$$\mathrm{Hom}_{FG}(E_1 \otimes E_1, E \otimes E) \cong \mathrm{Hom}_{FG}(E_{\{0,1\}}, E_{\{0,1\}}) \cong F.$$

Since F is a direct summand of both $E \otimes E$ and $E_1 \otimes E_1$, it follows that $\mathrm{Hom}_{FG}(\bigwedge^2(E_1), \bigwedge^2(E)) = 0$, which is what we wanted. Therefore the entries '2' are certainly lower bounds.

The above calculation also shows that $\mathrm{Hom}_{FG}(\bigwedge^2(E)\otimes E_1, E_1)\cong \mathrm{Hom}_{FG}(\bigwedge^2(E), E_1\otimes E_1)=0$, so from the filtration of $\bigwedge^2(E)\otimes E_1$ having factors $E_{\{0,1\}}$, $E_1\otimes E_1$, $E_{\{0,1\}}$, we see that $\mathrm{Ext}^1_{FG}(E_{\{0,1\}}, E_1)\neq 0$, and by conjugating by τ we obtain $\mathrm{Ext}^1_{FG}(E_{\{0,1\}}, E)\neq 0$. Since we may twist by τ and apply standard isomorphisms, the existence of non-split extensions for all other non-zero entries in the table will follow if we show that $\mathrm{Ext}^1_{FG}(E\otimes S_1, S_1)\neq 0$ and $\mathrm{Ext}^1_{FG}(E\otimes S_1, E_1)\neq 0$. It is easily checked that, just as in Lemma 3.2(c), we have $E\otimes S_{\{0,1\}}\cong S_{\{0,1\}}\oplus P(E\otimes S_1)$, so that $P(E\otimes S_1)\cong Z\otimes S_1$ and $\mathrm{soc}(Z\otimes S_1)=(\mathrm{soc}\, Z)\otimes S_1$ as before. By Lemma 4.6(c), $Z/\mathrm{soc}\, Z$ has a filtration with bottom factor isomorphic to E_1. The required non-vanishing therefore follows from $\mathrm{Hom}_{FG}(S_1, E_1\otimes S_1)\neq 0$ and $\mathrm{Hom}_{FG}(E_1, E_1\otimes S_1)\neq 0$, which both follow from Lemma 3.2(c).

This completes the proof of the existence of the non-trivial extensions in the table. It remains to show that these lower bounds are sharp.

We use the filtration of $Z/\mathrm{soc}\, Z$ in Lemma 4.6(c) in which the bottom factor is E_1 and the other factors embed into either $S\otimes E$ or $S\otimes E\otimes E_1$. We make the following computations.

(a)

$$\mathrm{Hom}_{FG}(E_1, E_1\otimes S_1)\cong F,$$
$$\mathrm{Hom}_{FG}(E_1, S\otimes E\otimes S_1)\cong \mathrm{Hom}_{FG}(S_{\{0,1\}}, E_{\{0,1\}})=0,$$
$$\mathrm{Hom}_{FG}(E_1, S\otimes E\otimes E_1\otimes S_1)\cong \mathrm{Hom}_{FG}(S_{\{0,1\}}, E_1\otimes E_{\{0,1\}})=0,$$

and thus $\mathrm{Ext}^1_{FG}(E\otimes S_1, E_1)\cong F$;

(b)

$$\mathrm{Hom}_{FG}(S_1, E_1\otimes S_1)\cong F,$$
$$\mathrm{Hom}_{FG}(S_1, S\otimes E\otimes S_1)\cong \mathrm{Hom}_{FG}(S_{\{0,1\}}, E\otimes S_1)=0,$$
$$\mathrm{Hom}_{FG}(S_1, S\otimes E\otimes E_1\otimes S_1)\cong \mathrm{Hom}_{FG}(S_{\{0,1\}}, E\otimes E_1\otimes S_1)=0,$$

and thus $\mathrm{Ext}^1_{FG}(E\otimes S_1, S_1)\cong F$;

(c)

$$\mathrm{Hom}_{FG}(F, E_1\otimes S_1)=0,$$
$$\mathrm{Hom}_{FG}(F, S\otimes E\otimes S_1)=0,$$
$$\mathrm{Hom}_{FG}(F, S\otimes E\otimes E_1\otimes S_1)=0,$$

and thus $\mathrm{Ext}^1_{FG}(E\otimes S_1, F)=0$;

(d)

$$\mathrm{Hom}_{FG}(E, E_1\otimes S_1)=0,$$
$$\mathrm{Hom}_{FG}(E, S\otimes E\otimes S_1)\cong \mathrm{Hom}_{FG}(S_{\{0,1\}}, E\otimes E)=0,$$
$$\mathrm{Hom}_{FG}(E, S\otimes E\otimes E_1\otimes S_1)\cong \mathrm{Hom}_{FG}(S_{\{0,1\}}, E\otimes E\otimes E_1)=0,$$

and thus $\mathrm{Ext}^1_{FG}(E\otimes S_1, E)=0$.

Since $S_{\{0,1\}}$ is projective, we have

$$\mathrm{Ext}^1_{FG}(E\otimes S_1, S)=0 \quad\text{and}\quad \mathrm{Ext}^1_{FG}(E\otimes S_1, E_1\otimes S)=0.$$

We now do the same thing with $P(S_1)$. Let X be the $\mathbf{G}$-module defined in Lemma 3.2. An easy calculation shows that $X\otimes S_1\cong P(S_1)\otimes S_{\{0,1\}}$ as FG-modules and that $(\mathrm{soc}\, X)\otimes S_1\cong \mathrm{soc}\, P(S_1)$. According to Lemma 4.6, $X/\mathrm{soc}\, X$

has a filtration in which each factor embeds into either $S \otimes E$ or $S \otimes E \otimes E_1$ or is isomorphic to $E \otimes E_2 \cong E \otimes E$. We make some more computations:

(e)

$$\operatorname{Hom}_{FG}(E_1, E \otimes E \otimes S_1) = 0,$$
$$\operatorname{Hom}_{FG}(E_1, S \otimes E \otimes S_1) = 0,$$
$$\operatorname{Hom}_{FG}(E_1, S \otimes E \otimes E_1 \otimes S_1) \cong \operatorname{Hom}_{FG}(S_{\{1,0\}}, E \otimes E_1 \otimes E_1) = 0,$$

and thus $\operatorname{Ext}^1_{FG}(S_1, E_1) = 0$;

(f)

$$\operatorname{Hom}_{FG}(F, E \otimes E \otimes S_1) = 0,$$
$$\operatorname{Hom}_{FG}(F, S \otimes E \otimes S_1) = 0,$$
$$\operatorname{Hom}_{FG}(F, S \otimes E \otimes E_1 \otimes S_1) = 0,$$

and thus $\operatorname{Ext}^1_{FG}(S_1, F) = 0$.

Combining (a)–(f) with standard isomorphisms, τ-conjugation and the fact that $H^1(G, F) = 0$ because G has no homomormorphic image of order 3, we will have computed all of the entries in the table once we obtain the correct upper bounds for

(i) $\operatorname{Ext}^1_{FG}(E, F)$, (ii) $\operatorname{Ext}^1_{FG}(F, E)$, (iii) $\operatorname{Ext}^1_{FG}(E, E_{\{0,1\}})$,
(iv) $\operatorname{Ext}^1_{FG}(S, S)$, (v) $\operatorname{Ext}^1_{FG}(E_{\{0,1\}}, E_{\{0,1\}})$, (vi) $\operatorname{Ext}^1_{FG}(E_{\{0,1\}}, S)$,
(vii) $\operatorname{Ext}^1_{FG}(E \otimes S_1, E \otimes S_1)$.

We first consider (ii). In the notation of Lemma 4.1 and by Lemma 3.7, we have

$$\dim \operatorname{Hom}_{FG}(X(F, E), S \otimes E)$$
$$\leqslant \dim \operatorname{Hom}_{FG}(F, E_1) + \dim \operatorname{Hom}_{FG}(F, S \otimes E) + \dim \operatorname{Hom}_{FG}(F, S \otimes E \otimes E_1) = 0.$$

Therefore, since E is a homomorphic image of $S \otimes E$, we may apply Lemma 4.1 to conclude that $d(F, E) \leqslant d(S, E) = 0$, by (e). For (i), we have $\operatorname{Ext}^1_{FG}(E, E) \cong \operatorname{Ext}^1_{FG}(F, E \otimes E) = 0$, because the composition factors of $E \otimes E$ are E, E_1, F and S.

In the next stage of our calculation we consider the embedding of **G**-modules $E \otimes E \hookrightarrow F \oplus (S \otimes E)$ of Lemma 3.6. The cokernel is isomorphic to the module $\bar{Z}$ of Lemma 4.6, which is filtered by modules which embed into either $S \otimes E$ or $S \otimes E \otimes E_1$.

We apply this first to (iii), and then to (vi), (v) and (vii).

(iii) We have $\operatorname{Hom}_{FG}(E_1, S \otimes E) = 0$ and $\operatorname{Hom}_{FG}(E_1, S \otimes E \otimes E_1) = 0$, so we have an injection

$$\operatorname{Ext}^1_{FG}(E_1, E \otimes E) \hookrightarrow \operatorname{Ext}^1_{FG}(E_1, F \oplus S \otimes E) \cong F,$$

by (a) and (ii). This is the correct upper bound for (iii).

(vi) Since $\operatorname{Hom}_{FG}(E_1, S \otimes E \otimes S_1) = 0$ and $\operatorname{Hom}_{FG}(E_1, S \otimes E \otimes E_1 \otimes S_1) = 0$, we obtain an injection

$$\operatorname{Ext}^1_{FG}(E_{\{0,1\}}, E \otimes S_1) \cong \operatorname{Ext}^1_{FG}(E_1, (E \otimes E) \otimes S_1)$$
$$\hookrightarrow \operatorname{Ext}^1_{FG}(E_1, S_1 \oplus (E \otimes S_{\{1,0\}})) = 0.$$

by (e).

(v) The equations

$$\mathrm{Hom}_{FG}(E_1, S\otimes E\otimes E_1)=0 \quad \text{and} \quad \mathrm{Hom}_{FG}(E_1, S\otimes E\otimes E_1\otimes E_1)=0$$

yield

$$\begin{aligned}\mathrm{Ext}^1_{FG}(E_{\{0,1\}}, E_{\{0,1\}}) &\cong \mathrm{Ext}^1_{FG}(E_1, (E\otimes E)\otimes E_1)\\ &\hookrightarrow \mathrm{Ext}^1_{FG}(E_1, E_1\oplus(S\otimes E\otimes E_1))\\ &\cong \mathrm{Ext}^1_{FG}(E_1, E_1)\oplus \mathrm{Ext}^1_{FG}(E_{\{0,1\}}, S\otimes E_1)=0,\end{aligned}$$

by (i) and (vi).

(vii) Since $\mathrm{Hom}_{FG}(S_1, S\otimes E\otimes S_1)=0$ and $\mathrm{Hom}_{FG}(S_1, S\otimes E\otimes E_1\otimes S_1)=0$, we have

$$\begin{aligned}\mathrm{Ext}^1_{FG}(E\otimes S_1, E\otimes S_1) &\cong \mathrm{Ext}^1_{FG}(S_1, (E\otimes E)\otimes S_1)\\ &\hookrightarrow \mathrm{Ext}^1_{FG}(S_1, S_1)\oplus \mathrm{Ext}^1_{FG}(S_1, S\otimes E\otimes S_1)=\mathrm{Ext}^1_{FG}(S_1, S_1),\end{aligned}$$

as $S_{\{0,1\}}$ is projective.

Thus (vii) reduces to (iv) and all statements will be proved once we show that $\mathrm{Ext}^1_{FG}(S, S)=0$. This will occupy the rest of this appendix.

Bearing in mind our earlier discussion of $P(S_1)$, it will suffice to show that $\mathrm{Hom}_{FG}(S_1\otimes S_1, X/\mathrm{soc}\,X)=0$. In Lemma 4.6, we constructed a filtration of $\bar{X}=X/\mathrm{soc}\,X$ by **G**-modules

$$0\subset V^1\subset W^2\subset V^2\subset W^3\subset \bar{X},$$

in which $\bar{X}/V^2\cong H^0(4\lambda_2)$ and $W^3/V^2\cong E_{\{0,2\}}$. Furthermore, it was shown there that $\bar{X}/W^3$ and V^2 have filtrations whose factors embed into either $S\otimes E$ or $S\otimes E\otimes E_1$. Since $\mathrm{Hom}_{FG}(S_1\otimes S_1, (S\otimes E)\oplus(S\otimes E\otimes E_1))=0$, we are reduced to proving that there are no FG-maps from $S_1\otimes S_1$ into W^3 such that the composition with the natural map $W^3\to W^3/V^2$ is non-zero. We note that $(S_1\otimes S_1)/\mathrm{soc}(S_1\otimes S_1)$ has no composition factor S_1 and that $\mathrm{Hom}_{FG}(S_1, E\otimes E)=0$, $\mathrm{Hom}_{FG}(S_1, S\otimes E)=0$ and $\mathrm{Hom}_{FG}(S_1, S\otimes E\otimes E_1)=0$. The last three equations imply (using the filtration of $\bar{X}$) that $\mathrm{Hom}_{FG}(S_1, \bar{X})=0$. It follows that the image of any FG-map $S_1\otimes S_1\to \bar{X}$ does not have S_1 as a composition factor. We claim, on the other hand, that any FG-submodule of W^3 having non-zero image in W^3/V^2 has S_1 as a composition factor. The desired result obviously follows from this claim, which in turn is a consequence of the following two statements:

(a) if M is an FG-submodule of W^3 such that W^3/M has no composition factor $E_{\{0,1\}}$, then M has S_1 as a composition factor;

(b) if M is an FG-submodule of W^3 with non-zero image in W^3/V^2, then W^3/M has no composition factor $E_{\{0,1\}}$.

(a) Since the **G**-module W^3 has a composition factor $E_{\{0,2\}}$, it is not hard to see from the Weyl filtration of X that W^3 has a submodule $W\cong V(4\lambda_2)/\mathrm{soc}\,V(4\lambda_2)$. The **G**-module JW will play an important role in what follows. Since its composition factors are $E_{\{0,1\}}$, E_1 and S_1, Lemma 3.1 shows that it has the same Loewy structure when considered as an FG-module. In particular, its head as an FG-module is isomorphic to $E_{\{0,1\}}$. Thus, if W^3/M does not have a composition factor $E_{\{0,1\}}$, we must have $JW\subseteq M$, which proves (a).

(b) Consider the **G**-module W^3/W^2. It has a submodule isomorphic to $H^0(2\lambda_1)/\mathrm{soc}\,H^0(2\lambda_1)$ with quotient isomorphic to $E_{\{0,2\}}$. If $E_{\{0,2\}}$ were a submodule of W^3/W^2 then its preimage in W^3 would be a submodule of W^3 having $E_{\{0,2\}}$ as a composition factor and only one composition factor $E_{\{0,1\}}$, contrary to the Loewy structure of $V(4\lambda_2)$. Thus the socle of the $\mathbf{G}$-module W^3/W^2 is $E_{\{0,1\}}$ and since $\mathrm{Ext}^1_{\mathbf{G}}(E_{\{0,2\}}, F) = 0$, its (ascending) socle layers are $E_{\{0,1\}}$, $F \oplus E_{\{0,2\}}$. By Lemma 3.1, the restriction of the extension of F by $E_{\{0,1\}}$ to G does not split. Since $\mathrm{Ext}^1_{\mathbf{G}}(E_{\{0,2\}}, E_{\{0,1\}}) \cong F$, it is not difficult to show that the extension of $E_{\{0,2\}}$ by $E_{\{0,1\}}$ is isomorphic to $E \otimes J \bigwedge^2(E_1)$. As an FG-module, the latter has a submodule isomorphic to $E_{\{0,1\}}$ with quotient isomorphic to $E \otimes E_2 \cong E \otimes E \cong F \oplus S \oplus \bigwedge^2(E)$. Now, $E \otimes J \bigwedge^2(E_1)$ embeds into $E \otimes E_1 \otimes S_1$ and $\mathrm{Hom}_{FG}(F \oplus S \oplus E, E \otimes E_1 \otimes S_1) = 0$, so we may conclude that as an FG-module, W^3/W^2 has socle $E_{\{0,1\}}$. By Lemma 3.1, the restriction to G of $V^1 \cong H^0(\lambda_1 + \lambda_2)$ has a simple socle $E_{\{0,1\}}$ and $V^2/V^1 \cong H^0(2\lambda_1)$ has a simple socle $W^2/V^1 \cong S_1$. Since $W^3/W^2 \cong (W^3/V^1)/(W^2/V^1)$ also has a simple socle as an FG-module, it follows that W^3/V^1 has a simple socle S_1 as an FG-module. Since we have seen that $\mathrm{Hom}_{FG}(S_1, \bar{X}) = 0$, we have proved that the socle of the FG-module W^3 is isomorphic to $E_{\{0,1\}}$. Suppose M is an FG-submodule of W^3 with non-zero image in W^3/V^2. Then since $\mathrm{soc}(W^3/W^2) \cong E_{\{0,1\}}$, we see that the image of M in W^3/W^2 must have $E_{\{0,1\}}$ as a composition factor. Then, since $\mathrm{soc}\, W^3 \cong E_{\{0,1\}} \subseteq M \cap W^2$, it follows that M has two composition factors isomorphic to $E_{\{0,1\}}$, so W^3/M has none. This completes the proof of (b) and of the proposition.

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