EXTENSIONS OF SIMPLE MODULES FOR $SL_3(2^n)$ AND $SU_3(2^n)$

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ABSTRACT

The group of extensions between any two irreducible 2-modular representations of the groups $SL_3(2^m)$ and $SU_3(2^m)$ is determined.

1. *Introduction*

Let F be a fixed algebraic closure of $\mathbb{F}_2$. We shall regard finite extensions of $\mathbb{F}_2$ as subfields of F. Let m be a natural number. We denote by $G = G_m$ either the group $SL_3(2^m)$ or the group $SU_3(2^m)$. The latter is defined to be the subgroup of $SL_3(2^{2m})$ preserving the hermitian form on $\tilde{V} = \mathbb{F}_{2^{2m}}^3$ which is represented in the standard basis by the identity matrix. Thus, we have $G_m \subset SL_3(2^{2m}) \subset SL_3(F)$.

For any finite-dimensional vector space E over F, we shall denote its dual by E^*. Let σ: $\lambda \mapsto \lambda^2$ be the Frobenius automorphism of F and for $i \in \mathbb{N}$ let E_i be the F-vector space whose underlying group is the same as E but on which $\lambda \in F$ operates as $\sigma^{-i}(\lambda)$ operates on E. If E is an FG-module then so is E_i, since the actions of G and F commute. For any $FSL_3(2^m)$-module E we have $E_m \cong E$. Let $V = F^3 \cong \tilde{V} \otimes_{\mathbb{F}_{2^{2m}}} F$. Extending our notation to $\mathbb{F}_{2^{2m}}$-modules in the obvious way, we have $\tilde{V}_m \cong \tilde{V}^*$ as $FSU_3(2^m)$-modules, by definition of $SU_3(2^m)$, and hence $V_m \cong V^*$. In view of these remarks, we see that indices for Frobenius twists should be read modulo m when considering modules for $SL_3(2^m)$, but they should be read modulo $2m$ for $FSU_3(2^m)$-modules.

We shall now describe the simple FG-modules. The 'restricted' modules are V, V^* and the 8-dimensional space W of traceless 3×3 matrices over F on which $SL_3(F)$ acts by conjugation. These are of course related by the formula

(1.1) $$V \otimes V^* \cong F \oplus W,$$

where F denotes the trivial module and the symbol '$\otimes$' stands for tensor product over F (we shall keep these conventions throughout).

Let $N = \{0, 1, \ldots, m-1\} \subseteq \mathbb{N}$ and for each subset $I \subseteq \mathbb{N}$, we define

$$V_I = \bigotimes_{i \in I} V_i, \quad V_I^* = \bigotimes_{i \in I} V_i^* \quad \text{and} \quad W_I = \bigotimes_{i \in I} W_i.$$

Then by Steinberg's tensor product theorem, the 2^{2m} modules $V_I \otimes V_J^* \otimes W_K$, where I, J and K are disjoint subsets of N, form a complete set of non-isomorphic simple FG-modules (by convention, the empty tensor product is F). They are also the '2^m-restricted' modules for $SL_3(F)$. The module W_N is the Steinberg module. For convenience, we shall refer to the ordered triple (I, J, K) of disjoint subsets of N merely as a *triple* from now on.

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Galois action. The powers σ^r of the Frobenius map permute the (isomorphism classes of) simple FG-modules, but in different ways for $SL_3(2^m)$ and $SU_3(2^m)$. For $SL_3(2^m)$, the effect of σ^r is simply to add r and take the remainder modulo m to all indices in a triple. For $SU_3(2^m)$, we must first add r and take remainders modulo $2m$ and then 'dualize' those resulting indices which lie between m and $2m-1$; for example, for $SU_3(2^3)$, we have

$$(V_0 \otimes W_1)^{\sigma^5} = V_5 \otimes W_6 = V_5 \otimes W_0 = V_2^* \otimes W_0.$$

Note that σ^m simply dualizes each simple module for $SU_3(2^m)$.

It is clear that in both cases, the permutations induced on the set of triples preserves the total size $|I \cup J \cup K|$ of a triple (I, J, K) and also the size of the third component.

The automorphism τ. The group $SL_3(F)$ has an outer automorphism τ sending an element to the transpose of its inverse, which clearly maps G back to itself. It interchanges the FG-module structures of V and V^* and hence, by the description of the simple modules given above, it also interchanges the FG-module structures of every simple module and its dual. However, it is not true for a general FG-module M that M^* is isomorphic to the τ-twisted module M^τ, so the simple modules possess an extra symmetry. An example of this is the following: if (I, J, K) and (A, B, C) are triples, then applying τ and duality, we obtain

$$\begin{aligned}\mathrm{Ext}_{FG}(V_I \otimes V_J^* \otimes W_K, V_A \otimes V_B^* \otimes W_C) &\cong \mathrm{Ext}_{FG}(V_J \otimes V_I^* \otimes W_K, V_B \otimes V_A^* \otimes W_C)\\ &\cong \mathrm{Ext}_{FG}(V_A \otimes V_B^* \otimes W_C, V_I \otimes V_J^* \otimes W_K)\\ &\cong \mathrm{Ext}_{FG}(V_B \otimes V_A^* \otimes W_C, V_J \otimes V_I^* \otimes W_K).\end{aligned}$$

Regarding these formulae as statements about pairs of triples, we see that they are obtained from the pair $((I, J, K), (A, B, C))$ by applying the permutations id, $(IJ)(AB)$, $(IA)(JB)(KC)$ and $(IB)(JA)(KC)$. In the preceding paragraph, we saw that the group of field automorphisms of G acts on the set of triples, and hence also on the set of pairs of triples. We shall call two statements about pairs of triples which can be obtained from each other by a combination of field automorphisms of G and the four permutations above *variants* of each other. Of course, the set of variants of a given pair of triples will depend on which group we are considering, so when necessary, we shall use the terms 'L-variant' and 'U-variant' according to whether G is the linear group or the unitary group.

With all of these conventions we may now state our result.

THEOREM. *Suppose*† $m > 2$. *Then for triples* (I, J, K) *and* (A, B, C) *we have*

$$\mathrm{Ext}^1_{FG}(V_I \otimes V_J^* \otimes W_K, V_A \otimes V_B^* \otimes W_C) = 0$$

unless a variant of one of the following holds, in which case the space of extensions is one-dimensional:

(a)(i) $K = C, I = A \cup \{0, 1\}$, *and* $B = J$ $(0, 1 \notin A)$;

(ii) $K = C, I = A \cup \{0\}$, *and* $B = J \cup \{1\}$ $(0 \notin A, 1 \notin J)$;

† The cases where $m \leqslant 2$ do not all follow the same pattern and are described in the appendix.

(b)(i) $K = C \cup \{1\}, I = A$ *and* $B = J \cup \{0, 1\}$ $(0, 1 \notin J)$;

(ii) $K = C \cup \{1\}, I = A \cup \{0\}$ *and* $B = J \cup \{1\}$ $(0 \notin A, 1 \notin J)$;

(c) $K = C, I = (I \cap A) \cup \{0\}, J = B \cup \{1\}$ *and* $A = (I \cap A) \cup \{1\}$ $(0 \notin A, 1 \notin B)$.

REMARKS. (1) The parts (i) and (ii) in (a) and (b) reflect the natural isomorphism $\mathrm{Ext}(A, B^* \otimes C) \cong \mathrm{Ext}(A \otimes B, C)$.

(2) The variants of the statements (a)–(c) may be described easily. For example, to obtain the L-variants corresponding to field automorphisms we replace 0 and 1 by i and $i+1$ for $0 \leq i \leq n-2$ or by n and 0. The U-variants are slightly less obvious. Let us consider just the U-variants obtained by applying σ^{m-1}. For instance, the U-variant of (a)(i) would be

$$K = C, \quad J = B \cup \{0\}, \quad I = A \cup \{m-1\}.$$

(3) It is known from a result of Cline, Parshall, Scott and van der Kallen [**7**, Theorem 7.2D] (see also [**2**, Proposition 2.7]) that the restriction to $SL_3(2^m)$ (or $SU_3(2^m)$) of a non-split extension between two simple (rational) $SL_3(F)$-modules having '2^m-restricted' highest weights does not split. Since it will be clear that the non-trivial extensions of our theorem are in fact restrictions of $SL_3(F)$-module extensions, the theorem also describes the extensions between simple modules for the algebraic group $SL_3(F)$. This also explains why it is possible to treat the twisted and untwisted groups simultaneously, and why the answer is in some sense the same for the two cases. These comments require clarification. In particular, we point out that not all non-split extensions of simple modules for the finite groups will be restrictions of non-split extensions of 2^m-*restricted* simple modules for the algebraic group, as the following example illustrates.

Let $\tilde{G} = SL_3(2^{2m})$, so that according to the theorem there is a non-split extension of V_m^* by V_{m-1} as $F\tilde{G}$-modules. By the result mentioned above, the restriction to $SL_3(2^m)$ yields a non-split extension of V_0^* by V_{m-1} and the restriction to $SU_3(2^m)$ gives a non-split extension of V_0 by V_{m-1}. However, there are no non-split extensions of V_0 or of V_0^* by V_{n-1} as rational $SL_3(F)$-modules. These 'wrap-around' effects are due to the identifications of modules for the finite groups with some of their Frobenius twists.

Tensor factors. Given an FG-module we shall often refer to its tensor factors, by which we shall mean those modules which may be tensored with some other to give back the module in question. Clearly the tensor factors of the simple module $V_I \otimes V_J^* \otimes W_K$ for a triple (I, J, K) are precisely those simple modules indexed by triples (I', J', K') for $I' \subseteq I$, $J' \subseteq J$ and $K' \subseteq K$.

Except in the appendix, we shall assume throughout that $m > 2$.

2. *A reduction*

LEMMA 2.1. *Let (I, J, K) and (A, B, C) be triples. Then*

$$\mathrm{Ext}^1_{FG}(V_I \otimes V_J^* \otimes W_K, V_A \otimes V_B^* \otimes W_C)$$

$$\cong \bigoplus_{((I',J',K'),(A',B',C'))} \mathrm{Ext}^1_{FG}(V_{I'} \otimes V_{J'}^* \otimes W_{K'}, V_{A'} \otimes V_{B'}^* \otimes W_{C'}),$$

where the summation runs over certain pairs $((I', J', K'), (A', B', C'))$ *of triples satisfying:*

(1) $A' \subseteq J' \cup K'$, $B' \subseteq I' \cup K'$, $C' \subseteq K'$;

(2) *if* $((I, J, K), (A, B, C))$ *satisfies one of the conditions of the theorem (that is, a variant of* (a), (b) *or* (c)) *then there is a unique pair* $((I', J', K'), (A', B', C'))$ *which satisfies a condition of the theorem (though not necessarily the same one);*

(3) *if any pair* $((I', J', K'), (A', B', C'))$ *satisfies a condition of the theorem, then so must the original pair* $((I, J, K), (A, B, C))$.

Proof. We set $L = N\backslash(I \cup J \cup K)$ and $D = N\backslash(A \cup B \cup C)$. Subdividing the sets in the obvious fashion and applying (1.1), we obtain

$$\begin{aligned}
&\mathrm{Ext}^1_{FG}(V_I \otimes V_J^* \otimes W_K, V_A \otimes V_B^* \otimes W_C)\\
&= \mathrm{Ext}^1_{FG}(V_{(I\cap A)\cup(I\cap B)\cup(I\cap C)\cup(I\cap D)} \otimes V^*_{(J\cap A)\cup(J\cap B)\cup(J\cap C)\cup(J\cap D)} \otimes W_K,\\
&\qquad V_{(I\cap A)\cup(J\cap A)\cup(K\cap A)\cup(L\cap A)} \otimes V^*_{(I\cap B)\cup(J\cap B)\cup(K\cap B)\cup(L\cap B)} \otimes W_C)\\
&\cong \bigoplus_{T\subseteq(I\cap A)\cup(J\cap B)} \mathrm{Ext}^1_{FG}(V_{(I\cap B)\cup(I\cap D)\cup(L\cap B)} \otimes V^*_{(J\cap A)\cup(J\cap D)\cup(L\cap A)} \otimes W_{K\cup C\cup T},\\
&\qquad V_{(J\cap A)\cup(K\cap A)\cup(J\cap C)} \otimes V^*_{(I\cap B)\cup(K\cap B)\cup(I\cap C)} \otimes W_{K\cap C}).
\end{aligned}$$

Setting

$$I' = (I\cap B)\cup(I\cap D)\cup(L\cap B), \quad J' = (J\cap A)\cup(J\cap D)\cup(L\cap A),$$
$$K' = K\cup C\cup T, \quad A' = (J\cap A)\cup(K\cap A)\cup(J\cap C),$$
$$B' = (I\cap B)\cup(K\cap B)\cup(I\cap C), \quad C' = K\cap C,$$

we see that (1) holds for the pairs $((I', J', K'), (A', B', C'))$, and we shall check that (2) and (3) also hold. It is not hard to see that variants of the original pair $((I, J, K), (A, B, C))$ correspond to variants of the primed pairs $((I', J', K'), (A', B', C'))$, so in checking (2) and (3) we may assume that Conditions (a), (b) or (c) of the theorem are satisfied rather than a variant. First we check (2) for each of the Conditions (a), (b) and (c).

(a)(i) We easily obtain $I' = \{0, 1\}$, $J' = \emptyset$, $K' = K \cup T$, $A' = \emptyset$, $B' = \emptyset$ and $C' = K$. Thus, for $T = \emptyset$ we obtain a pair $((I', J', K'), (A', B', C'))$ satisfying (a)(i). We claim that none of the pairs for $T \neq \emptyset$ satisfies a variant of (a), (b) or (c). To see this, we may consider, for example, the quantity

$$\big||I\cup J| + 2|K| - |A\cup B| - 2|C|\big|$$

for each pair of triples $((I, J, K), (A, B, C))$, which is clearly preserved upon taking variants. If $T \neq \emptyset$, this number would be at least 4 for the primed pairs of triples but it is at most 2 for the pairs in (a), (b) and (c).

(a)(ii) We obtain $I' = \{0, 1\}$, $J' = \emptyset$, $K' = K \cup T$, $A' = \emptyset$, $B' = \emptyset$ and $C' = K$, which is the same as in Case (a)(i).

(b)(i) We obtain $I' = \{0\}$, $J' = \emptyset$, $K' = C \cup \{1\} \cup T$, $A' = \emptyset$, $B' = \{1\}$ and $C' = C$. As in Case (a)(i), we see that the only pair satisfying a condition of the theorem is the one for $T = \emptyset$, which satisfies (b)(ii).

(b)(ii) We again obtain $I' = \{0\}$, $J' = \emptyset$, $K' = C \cup \{1\} \cup T$, $A' = \emptyset$, $B' = \{1\}$ and $C' = C$, the same pairs as in Case (b)(i).

(c) We obtain $I' = \{0\}$, $J' = \{1\}$, $K' = K \cup T$, $A' = \{1\}$, $B' = \emptyset$ and $C' = K$, and again $T = \emptyset$ gives the unique pair which satisfies a condition of the theorem, in this instance Condition (c). This proves (2).

Now we shall prove (3), so we suppose that some pair $((I', J', K'), (A', B', C'))$ satisfies (a), (b) or (c) of the theorem. We consider them in turn.

(a)(i) We have $K \cup C \cup T = K \cap C$, so $T = \emptyset$ and $K = C$. Also we have

$$(I \cap B) \cup (I \cap D) \cup (L \cap B) = (J \cap A) \cup (K \cap A) \cup (J \cap C) \cup \{0, 1\},$$

where $0, 1 \notin (J \cap A) \cup (K \cap A) \cup (J \cap C)$, and

$$(J \cap A) \cup (J \cap D) \cup (L \cap A) = (I \cap B) \cup (K \cap B) \cup (I \cap C).$$

From these equations and the fact that the sets in a triple are disjoint, it is easy to deduce that $I = A \cup (I \cap D)$, $B = J \cup (L \cap B)$, $K = C$ and $(I \cap D) \cup (L \cap B) = \{0, 1\}$. The four possibilities for the latter equation all imply that the pair $((I, J, K), (A, B, C))$ satisfies a variant of Condition (a) of the theorem, for example, if $I \cap D = \{0\}$ and $L \cap B = \{1\}$, then the pair $((I, J, K), (A, B, C))$ satisfies (a)(i).

(a)(ii) Since (1) is preserved under τ and taking Galois conjugates, we see that no variant of the pair of triples in (a)(ii) of the theorem satisfies (1), so there is nothing to check in this case. The same applies to (b)(i).

(b)(ii) We have

$$K \cup C \cup T = (K \cap C) \cup \{0, 1\},$$
$$(I \cap B) \cup (I \cap D) \cup (L \cap B) = (J \cap A) \cup (K \cap A) \cup (J \cap C) \cup \{0\},$$

and

$$(J \cap A) \cup (J \cap D) \cup (L \cap A) \cup \{1\} = (I \cap B) \cup (K \cap B) \cup (I \cap C),$$

where $0 \notin (J \cap A) \cup (K \cap A) \cup (J \cap C)$ and $1 \notin (J \cap A) \cup (J \cap D) \cup (L \cap A)$. Thus, $I \cap B = \emptyset$ and $J \subseteq B$, so $1 \in (K \cap B) \cup (I \cap C) \subseteq K \cup C$, whence $T = \emptyset$. Suppose $1 \in K \cap B$. Then $K = C \cup \{1\}$ and $I \cap C = \emptyset$. Then if $I \cap D = \{0\}$, we obtain $I = A \cup \{0\}$, $B = J \cup \{1\}$, $K = C \cup \{1\}$, which is (b)(ii), and if $L \cap B = \{0\}$, we obtain $I = A$, $B = J \cup \{0, 1\}$, $K = C \cup \{1\}$, which is (b)(i). The case where $1 \in I \cap C$ is the variant of the case where $1 \in K \cap B$ under the permutation $(IB)(JA)(KC)$, so in this case the original pair $((I, J, K), (A, B, C))$ must satisfy a variant of (b) as well.

(c) We have $K \cup C \cup T = K \cap C$, so $T = \emptyset$ and $K = C$. Also we have

$$(I \cap B) \cup (I \cap D) \cup (L \cap B) = \{0\},$$
$$(J \cap A) \cup (K \cap A) \cup (J \cap C) = \{1\}$$

and

$$(J \cap A) \cup (J \cap D) \cup (L \cap A) = (I \cap B) \cup (K \cap B) \cup (I \cap C) \cup \{1\},$$

where $1 \notin (I \cap B) \cup (K \cap B) \cup (I \cap C)$. The second of these equations reduces to $J \cap A = \{1\}$ and then it is easy to infer that

$$J = (J \cap B) \cup \{1\}, \quad I = (I \cap A) \cup (I \cap D), \quad A = (I \cap A) \cup \{1\},$$
$$B = (J \cap B) \cup (L \cap B) \quad \text{and} \quad (I \cap D) \cup (L \cap B) = \{0\}.$$

If $0 \in I \cap D$, we obtain $J = B \cup \{1\}$, $I = (I \cap A) \cup \{0\}$ and $A = (I \cap A) \cup \{1\}$, which is (c). The case where $0 \in L \cap B$ is the $(IB)(JA)(KC)$-variant of this.

The lemma is proved.

3. *Lemmas on characters*

The results of this section may be stated and proved in terms of Brauer characters, but for ease of reference later we shall present them in module-theoretic language, using the notation of § 1. Most of the results in this section are simply variations of results proved in [**6**] and independently in [**4**].

LEMMA 3.1 (cf. [**6**, Lemma 3.2]).

(a) $V_i \otimes V_i$ *has composition factors* V_i^* (*twice*) *and* V_{i+1}.

(b) $V_i \otimes W_i$ *has composition factors* V_i (*three times*), V_{i+1}^* (*twice*) *and* $V_i^* \otimes V_{i+1}$.

(c) $V_i \otimes V_i \otimes V_i$ *has composition factors* W_i (*twice*), F (*twice*) *and* $V_{\{i,i+1\}}$.

(d) $W_i \otimes W_i$ *has composition factors* F (*four times*), W_i (*twice*), W_{i+1}, $V_{\{i,i+1\}}$ (*twice*) *and* $V^*_{\{i,i+1\}}$ (*twice*).

(e) $V_i \otimes V_i \otimes V_{i+1}^*$ *has composition factors* $V^*_{\{i,i+1\}}$ (*twice*), F *and* W_{i+1}.

Proof. For any finite-dimensional vector space E over F, $E \otimes E$ has a natural filtration with factors $\Lambda^2(E)$, E_1, $\Lambda^2(E)$, so (a) is immediate from this and the fact that since V is 3-dimensional, there is a natural isomorphism $\Lambda^2(V) \cong V^* \otimes \Lambda^3(V)$. The other parts follow easily from (a) and (1.1).

Later we shall return to the modules of Lemma 3.1 and study their submodule structures. We shall denote the projective cover of the simple module $V_I \otimes V_J^* \otimes W_K$ by $P(V_I \otimes V_J^* \otimes W_K)$.

LEMMA 3.2.

(a)
$$W_N \otimes V_N \cong \begin{cases} P(V_N) \oplus W_N, & \text{if } G = \mathrm{SL}_3(2^m), \\ P(V_N), & \text{if } G = \mathrm{SU}_3(2^m). \end{cases}$$

(b) *If I and J are disjoint, proper subsets of N, then*

$$W_N \otimes (V_I \otimes V_J^*) \cong P(V_I \otimes V_J^* \otimes W_{N \setminus (I \cup J)}).$$

(c)
$$W_i \otimes W_N \cong P(W_{N \setminus \{i\}}) \oplus W_N \oplus W_N.$$

Proof. These are all consequences of [**6**, Lemma 6.1]; for (c) it is necessary that $m > 1$.

The above lemma, though character-theoretic in nature, has powerful implications about modules. For example, from (b) we may deduce that any tensor factor of $W_N \otimes (V_I \otimes V_J^*)$, such as $V_I \otimes W_K$, must have a simple head and a simple socle (the socle of a module is the maximal semisimple submodule and the head is the maximal semisimple quotient, namely the quotient by the radical). Lemma 3.2 will often be applied in this way.

We define the *mass* of the simple module $V_I \otimes V_J^* \otimes W_K$ to be $|I| + |J| + 2\,|K|$ and the mass of an arbitrary FG-module to be the maximum of the masses of its composition factors. We observe that mass is preserved when taking duals. Galois conjugates and τ-conjugates. The following statements follow from Lemma 3.1 by an easy induction.

LEMMA 3.3. *Let (I, J, K) and (A, B, C) be triples.*

(a) *If $i \in I \cup K$ then*

$$\operatorname{mass}(V_i \otimes (V_I \otimes V_J^* \otimes W_K)) \leqslant |I| + |J| + 2|K|.$$

(b) $$\operatorname{mass}(W_i \otimes (V_I \otimes V_J^* \otimes W_K)) \leqslant \begin{cases} |I| + |J| + 2|K| + 1, & \text{if } i \in I \cup J, \\ |I| + |J| + 2|K|, & \text{if } i \in K. \end{cases}$$

(c) $$\operatorname{mass}((V_I \otimes V_J^* \otimes W_K) \otimes (V_A \otimes V_B^* \otimes W_C)) \leqslant |I| + |J| + 2|K| + |A| + |B| + 2|C|,$$

with equality if and only if $(I \cup K) \cap (A \cup C) = \varnothing = (J \cup K) \cap (B \cup C)$.

In the following lemma and also later, when we are interested only in the isomorphism classes of the composition factors (or, more generally, filtration factors) of a module, we shall usually ignore both the multiplicities and the order of the factors.

LEMMA 3.4. *Let (I, J, K) be a triple and $i \in N$.*

(a) *$V_i \otimes (V_I \otimes V_J^* \otimes W_K)$ has no composition factor of the form W_T with $|T| > |K| + 1$, and none with $|T| > |K|$ if $i \in K$.*

(b) *$W_i \otimes (V_I \otimes V_J^* \otimes W_K)$ has no composition factor of the form W_T with $|T| > |K| + 1$, and none with $|T| > |K|$ if $i \in K$.*

Proof. (a) We use induction on $|I \cup J \cup K|$ and, given $|I \cup J \cup K|$, on $|K|$. We may assume by taking Galois conjugates that $i = 0$. All statements are obvious if $0 \notin I \cup J \cup K$, so we shall consider the cases

$$\text{(i) } 0 \in J, \quad \text{(ii) } 0 \in I \quad \text{and} \quad \text{(iii) } 0 \in K.$$

(i) By (1.1), we have

$$V_0 \otimes (V_I \otimes V_J^* \otimes W_K) \cong (V_I \otimes V_{J \setminus \{0\}}^* \oplus W_K) \otimes (V_I \otimes V_{J \setminus \{0\}} \otimes W_{K \cup \{0\}}),$$

and neither of these simple modules is W_T for $|T| > |K| + 1$.

(ii) By Lemma 3.1(a), the composition factors of

$$V_0 \otimes (V_I \otimes V_J^* \otimes W_K) = (V_0 \otimes V_0) \otimes (V_{I \setminus \{0\}} \otimes V_J^* \otimes W_K)$$

are $V_{I \setminus \{0\}} \otimes V_{J \cup \{0\}}^* \otimes W_K$ and those of $V_1 \otimes (V_{I \setminus \{0\}} \otimes V_J^* \otimes W_K)$, so induction on $|I \cup J \cup K|$ applies (after conjugation by σ^{-1}).

(iii) By Lemma 3.1(b), the composition factors of $W_0 \otimes (V_I \otimes V_J^* \otimes W_K) = (V_0 \otimes W_0) \otimes (V_I \otimes V_J^* \otimes W_{K \setminus \{0\}})$ are $V_{I \cup \{0\}} \otimes V_J^* \otimes W_{K \setminus \{0\}}$, those of

$$V_1^* \otimes (V_I \otimes V_J^* \otimes W_{K \setminus \{0\}})$$

and those of $V_1 \otimes (V_I \otimes V_{J \cup \{0\}}^* \otimes W_{K \setminus \{0\}})$. We can apply induction on $|I \cup J \cup K|$ to the second last of these (suitably dualized) and induction on $|K|$ to the last module.

(b) If $0 \in I \cup J$, say $0 \in I$, then

$$W_0 \otimes (V_I \otimes V_J^* \otimes W_K) \cong V_0 \otimes (V_{I \setminus \{0\}} \otimes V_J^* \otimes W_{K \cup \{0\}})$$

and we have finished, by (a). If $0 \in K$, we use induction on $|K|$. For $K = \{0\}$, Lemma 3.1(d) gives the composition factors of $(W_0 \otimes W_0) \otimes (V_I \otimes V_J^*)$ as $W_0 \otimes V_I \otimes V_J^*$, $V_I \otimes V_J^*$, those of $V_1 \otimes (V_{I \cup \{0\}} \otimes V_J^*)$, those of $V_1^* \otimes (V_I \otimes V_{J \cup \{0\}})$ and those of $W_1 \otimes (V_I \otimes V_J^*)$, all of which are covered by (a).

If $|K|>1$, the composition factors of $(W_0 \otimes W_0) \otimes (V_I \otimes V_J^* \otimes W_{K \setminus \{0\}})$ are $V_I \otimes V_J^* \otimes W_{K \setminus \{0\}}$, $V_I \otimes V_J^* \otimes W_K$, those of $V_1 \otimes (V_I \otimes V_{J \cup \{0\}} \otimes W_{K \setminus \{0\}})$, those of $V_1^* \otimes (V_I \otimes V_{J \cup \{0\}} \otimes W_{K \setminus \{0\}})$ and those of $W_1 \otimes (V_I \otimes V_J^* \otimes W_{K \setminus \{0\}})$. The result is true for the last module by (a) and for the other two non-simple modules by induction. This completes the proof of the lemma.

We end this section with a special case of Theorem 4.5 of [**6**].

LEMMA 3.5. *Let (I, J, K) be a triple.*

(a) *If I and J are proper subsets of N with $I \cup J \neq \varnothing$, then $V_I \otimes V_J^*$ is not a composition factor of $(V_I \otimes V_J^*) \otimes (V_I \otimes V_J^*)$.*

(b) *$V_N \otimes V_N$ has a unique composition factor isomorphic to V_N if $G = \mathrm{SL}_3(2^m)$ and no such composition factor if $G = \mathrm{SU}_3(2^m)$.*

4. *Lemmas on modules*

Let J denote the Jacobson radical of FG. By the *Loewy layers* of an FG-module M we shall mean the sequence $\mathrm{hd}(M) = M/JM$, $JM/J^2M, \dots$, and by the *socle layers* the sequence $\mathrm{soc}(M)$, $\mathrm{soc}^2(M)/\mathrm{soc}(M), \dots$, where $\mathrm{soc}^i(M)$ is defined recursively as the preimage in M of $\mathrm{soc}(M/\mathrm{soc}^{i-1}(M))$. Thus, for a uniserial module the Loewy layers are the socle layers in reverse order; we shall call the Loewy layers of such a module its *series*.

We begin with a simple observation of a very general kind which will provide the basic mechanism for the inductive parts of the proof in § 5 of the theorem. It is a more general form of Alperin's induction step used in [**1**] and [**8**].

LEMMA 4.1. *Let H be a finite group, k an algebraically closed field, L and M simple kH-modules and $d = \dim_k \mathrm{Ext}^1_{kH}(L, M)$. Let X be a kH-module with head isomorphic to L and maximal submodule isomorphic to the direct sum of d copies of M. Suppose S is a kH-module such that $L \otimes_k S$ is semisimple, and let E be a simple quotient of $M \otimes_k S$. Then*

$$\mathrm{Hom}_{kH}(X \otimes_k S, E) = \mathrm{Hom}_{kH}(L \otimes_k S, E)$$

implies that $d \leqslant \dim_k \mathrm{Ext}^1_{kH}(L \otimes_k S, E)$.

Since the Loewy length of X is at most 2, the condition of the lemma is equivalent to

$$(4.1') \qquad \mathrm{Hom}_{kH}(X, \mathrm{soc}^2(S^* \otimes_k E)) = \mathrm{Hom}_{kH}(L \otimes_k S, E),$$

which is how the lemma will sometimes be applied. Most frequently, we shall have $\mathrm{Hom}_{kH}(L \otimes_k S, E) = 0$, and we shall actually check the stronger condition

$$(4.1'') \qquad \mathrm{Hom}_{kH}(X, S^* \otimes_k E) = 0,$$

by checking, for example, that L is not a composition factor of $S^* \otimes_k E$.

The situations we have in mind are when $L = V_I \otimes V_J^* \otimes W_K$ and S is chosen to be either $V_t \otimes V_t^*$ or W_t for $t \notin I \cup J \cup K$.

It will often be necessary to consider modules like X in Lemma 4.1, so for any two simple FG-modules L and M we introduce the notation $d(L, M) = \dim_F \mathrm{Ext}^1_{FG}(L, M)$ and $X(L, M)$ for the (unique up to isomorphism) FG-module with head L and maximal submodule a direct sum of $d(L, M)$ copies of M.

REMARK 4.2. At this point we wish to make a further observation about the automorphism τ. We consider FG-modules M such that $M^* \cong M^\tau$. As we have already pointed out, these include the simple modules, and hence also their projective covers and direct sums and tensor products of any of these. Given such a module M, suppose it has a quotient Q with the same property. Then by duality, Q^* is a submodule of M^* and by τ-conjugacy, Q is isomorphic to a submodule of M. This additional symmetry implies, for example, that the Loewy layers and socle layers of such a module are the same, that is, $M/JM \cong \mathrm{soc}(M)$, $JM/J^2M \cong \mathrm{soc}^2(M)/\mathrm{soc}(M)$, etc. We shall make use of these properties many times, starting with the next lemma.

LEMMA 4.3. *We have the following statements:*

(a) $V_i \otimes V_i$ *is uniserial with series* V_i^*, V_{i+1}, V_i^*;

(b) $V_i \otimes W_i$ *has head and socle isomorphic to* V_i. *It has a quotient and a submodule isomorphic to* $V_i^* \otimes V_i^*$;

(c) $V_i \otimes V_i \otimes V_i \cong W_i \oplus W_i \oplus L_i$, *where* L_i *is uniserial with series* F, $V_{\{i,i+1\}}$, F;

(d) $V_i \otimes V_i \otimes V_{i+1}^*$ *has Loewy layers* $V_{\{i,i+1\}}^*$, $F \oplus W_{i+1}$, $V_{\{i,i+1\}}^*$.

Proof. The composition factors are given in Lemma 3.1. Part (a) follows from Remark 4.2 and the calculation

$$\mathrm{Hom}_{FG}(V_i \otimes V_i, V_{i+1}) = \mathrm{Hom}_{FG}(V_i, V_i^* \otimes V_{i+1}) = 0.$$

(b) That $V_i \otimes W_i$ has a simple head and socle follows from the fact that it is a tensor factor of $V_i \otimes W_N$ and Lemma 3.2(b). Since $V_i \otimes V_i \otimes V_i^* \cong V_i \oplus (V_i \otimes W_i)$, by (1.1), and since, by (a), $(V_i \otimes V_i) \otimes V_i^*$ has a quotient $V_i^* \otimes V_i^*$ which, again by (a), is uniserial of length 3, it follows that $V_i^* \otimes V_i^*$ is a homomorphic image of $V_i \otimes W_i$. In view of Remark 4.2, (b) is proved.

(c) We have

$$\mathrm{Hom}_{FG}(V_i \otimes V_i \otimes V_i, W_i) \cong \mathrm{Hom}_{FG}(V_i \otimes V_i, V_i^* \otimes W_i),$$

which, by (a) and (b), has dimension not less than 2. Since W_i occurs twice as a composition factor of $V_i \otimes V_i \otimes V_i$, it follows from Remark 4.2 that $W_i \oplus W_i$ is a direct summand. The structure of a complementary summand now follows from the composition factors and the calculation

$$\mathrm{Hom}_{FG}(V_i \otimes V_i \otimes V_i, V_{\{i,i+1\}}) \cong \mathrm{Hom}_{FG}(V_i \otimes V_i, V_{i+1} \oplus (V_{i+1} \otimes W_i)) = 0,$$

using (1.1) and (a).

(d) By (a), we know that $(V_i \otimes V_i) \otimes V_{i+1}^*$ has a filtration with factors (in order)

$$V_{\{i,i+1\}}^*, \quad F \otimes W_{i+1}, \quad V_{\{i,i+1\}}^*.$$

The result therefore follows from the calculation

$$\mathrm{Hom}_{FG}(V_i \otimes V_i \otimes V_{i+1}^*, F \oplus W_{i+1}) \approx \mathrm{Hom}_{FG}(V_i \otimes V_{i+1}^*, V_i^* \oplus (V_i^* \otimes W_{i+1})) = 0$$

and Remark 4.2.

LEMMA 4.4. *We have* $W_i \otimes W_i \cong W_i \oplus W_i \oplus D_i$, *where* D_i *is an indecomposable module with the following properties:*

(a) $\mathrm{hd}(D_i) \cong F \cong \mathrm{soc}(D_i)$;

(b) $J(D_i)$ *has two filtrations:*

$$0 \subset Q_i \subset R_i \subset J(D_i)$$

with

$$Q_i \cong L_i, \quad R_i/Q_i \cong V_i \otimes V_i \otimes V_{i+1}^*, \quad J(D_i)/R_i \cong V_{\{i,i+1\}},$$

and

$$0 \subset Q_i^\tau \subset R_i^\tau \subset J(D_i)$$

with

$$Q_i^\tau \cong L_i^\tau, \quad R_i^\tau/Q_i^\tau \cong V_i^* \otimes V_i^* \otimes V_{i+1}, \quad J(D_i)/R_i^\tau \cong V_{\{i,i+1\}}^*;$$

(c) *we have* $Q_i \subseteq R_i^\tau$, $Q_i^\tau \subseteq R_i$, $Q_i \cap Q_i^\tau = \mathrm{soc}(D_i)$ *and* $R_i + R_i^\tau = J(D_i)$.

Proof. We may assume $i = 0$. First, we have $\mathrm{Hom}_{FG}(W_0 \otimes W_0, F) \cong \mathrm{Hom}_{FG}(W_0, W_0) \cong F$. Next, we compute $\mathrm{Hom}_{FG}(W_0 \otimes W_0, W_0)$. Since

$$(V_0 \otimes V_0^*) \otimes (V_0 \otimes V_0^*) \cong F \oplus W_0 \oplus W_0 \oplus (W_0 \otimes W_0)$$

and

$$\begin{aligned}\mathrm{Hom}_{FG}(V_0 \otimes V_0^* \otimes V_0 \otimes V_0^*, F \oplus W_0) &\cong \mathrm{Hom}_{FG}(V_0 \otimes V_0^* \otimes V_0 \otimes V_0^*, V_0 \otimes V_0^*)\\ &\cong \mathrm{Hom}_{FG}(V_0 \otimes V_0 \otimes V_0, V_0 \otimes V_0 \otimes V_0),\end{aligned}$$

which, by Lemma 4.3(c), is 6-dimensional, it follows easily that

$$\mathrm{Hom}_{FG}(W_0 \otimes W_0, W_0)$$

is 2-dimensional. Since W_0 occurs twice as a composition factor of $W_0 \otimes W_0$, Remark 4.2 implies that $W_0 \otimes W_0$ has a summand isomorphic to $W_0 \oplus W_0$. Let D_0 be a complementary summand. By Lemma 3.2(c), we have

$$\begin{aligned}W_N \oplus W_N \oplus P(W_{N\setminus\{0\}}) &\cong W_0 \otimes W_N\\ &\cong (W_0 \otimes W_0) \otimes W_{N\setminus\{0\}}\\ &\cong (W_0 \oplus W_0 \oplus D_0) \otimes W_{N\setminus\{0\}},\end{aligned}$$

from which we see that

$$D_0 \otimes W_{N\setminus\{0\}} \cong P(W_{N\setminus\{0\}}). \tag{4.5}$$

This shows that D_0 has simple head and socle, which we have already seen must be F, proving (a). Let π be the projection of $(V_0 \otimes V_0^*) \otimes (V_0 \otimes V_0^*)$ onto D_0. The socle series of $V_0^* \otimes V_0^*$ induces a filtration

$$0 \subset B \subset C \subset (V_0 \otimes V_0) \otimes (V_0^* \otimes V_0^*)$$

of $(V_0 \otimes V_0^*) \otimes (V_0 \otimes V_0^*)$ with $B \cong V_0 \otimes V_0 \otimes V_0$ and $C/B \cong V_0 \otimes V_0 \otimes V_1^*$. We define $R_0 = \pi(C)$ and $Q_0 = \pi(B)$. By Lemma 4.3(c), $B = Z \oplus L$, where $Z \cong W_0 \oplus W_0$ and $L \cong L_0$. Clearly, $Z \subseteq \mathrm{Ker}\,\pi$. We claim that $\pi|_L$ is injective. Since $V_{\{0,1\}}$ occurs twice as a composition factor of both D_0 and of

$$(V_0 \otimes V_0^*) \otimes (V_0 \otimes V_0^*),$$

we see from the structure of L_0 that $J(L) \not\subseteq \operatorname{Ker} \pi$. Then since $\operatorname{soc}(D_0) \cong F$, we cannot have $L \cap \operatorname{Ker} \pi = \operatorname{soc}(L)$ either, so our claim is true. Thus $Q_0 \cong L_0$. The induced map $\bar{\pi}$: $C/B \to D_0/Q_0$ is also injective, since $\operatorname{soc}(V_0 \otimes V_0 \otimes V_1^*) \cong V^*_{\{0,1\}}$, by Lemma 4.3(d), and by comparing the composition factors of D_0/Q_0 and $(V_0 \otimes V_0) \otimes (V_0^* \otimes V_0^*)$, we see that $V^*_{\{0,1\}}$ is not a composition factor of $\ker \bar{\pi}$. Thus, $R_0/Q_0 \cong V_0 \otimes V_0 \otimes V_1^*$. Now by considering the composition factors of R_0, we see that it is a maximal submodule of $J(D_0)$ with $J(D_0)/R_0 \cong V_{\{0,1\}}$. The existence of the second filtration follows from the fact that D_0 is the unique non-simple indecomposable summand of $W_0 \otimes W_0 \cong (W_0 \otimes W_0)^\tau$, so that $D_0 \cong D_0^\tau$. Thus, (b) is proved. Since R_0 and R_0^τ are distinct maximal submodules of $J(D_0)$, their sum equals $J(D_0)$, and since F is the only module isomorphic to both a submodule of L_0 and a submodule of L_0^τ, we have $Q_0 \cap Q_0^\tau = \operatorname{soc}(D_0)$. Finally, since $Q_0^\tau \cong L_0^\tau$ has no quotient $V_{\{0,1\}} \cong J(D_0)/R_0$, we have $Q_0^\tau \subseteq R_0$ and similarly $Q_0 \subseteq R_0^\tau$, proving (c).

It is often helpful to keep the following pictures of modules in mind (although we shall not attempt to use such pictures in our proofs):

$$V_i \otimes V_i:\ \begin{array}{c} V_i^* \\ V_{i+1} \\ V_i^* \end{array} \qquad L_i:\ \begin{array}{c} F \\ V_{\{i,i+1\}} \\ F \end{array} \qquad V_i \otimes V_i \otimes V_{i+1}^*:\ \begin{array}{ccc} & V^*_{\{i,i+1\}} & \\ / & & \backslash \\ F & & W_{i+1} \\ \backslash & & / \\ & V^*_{\{i,i+1\}} & \end{array}$$

$$D_i:\ \begin{array}{ccccc} & & F & & \\ & / & & \backslash & \\ & V^*_{\{i,i+1\}} & & V^*_{\{i,i+1\}} & \\ / & & \backslash \quad / & & \backslash \\ F & & W_{i+1} & & F \\ \backslash & & / \quad \backslash & & / \\ & V_{\{i,i+1\}} & & V^*_{\{i,i+1\}} & \\ & \backslash & & / & \\ & & F & & \end{array}$$

LEMMA 4.6. *Let $J \subseteq \{i, i+1, \ldots, i+m-1\} \subseteq \mathbb{N}$. Then $V_i \otimes V_J$ is uniserial. Moreover, if $J = \{i, i+1, \ldots, i+k\}$, with $0 \leq k \leq m-1$, then the series of $V_i \otimes V_J$ is*

$$V_i^* \otimes V_{\{i+1,\ldots,i+k\}},\ V_{i+1}^* \otimes V_{\{i+2,\ldots,i+k\}},\ \ldots,\ V_{i+k}^*,\ V_{i+k+1},\ V_{i+k}^*,\ \ldots,\ V_{i+1}^* \otimes V_{\{i+2,\ldots,i+k\}},\ \ldots,\ V_i^* \otimes V_{\{i+1,\ldots,i+k\}}.$$

Proof. We may assume $i = 0$. It is enough to prove the second statement since $V_0 \otimes V_J$ is a tensor factor of $V_0 \otimes V_N$. We proceed by induction on k, the result being true for $k = 0$ by Lemma 4.3(a). We assume $k \geq 1$.

Now $V_0 \otimes V_{\{0,\ldots,k\}} = (V_0 \otimes V_0) \otimes V_{\{1,\ldots,k\}}$ has a filtration (induced by the socle series of $V_0 \otimes V_0$), with factors (in order)

$$V_0^* \otimes V_{\{1,\ldots,k\}},\ V_1 \otimes V_{\{1,\ldots,k\}},\ V_0^* \otimes V_{\{1,\ldots,k\}}.$$

The top and bottom factors are simple and by the inductive hypothesis, the middle factor is uniserial with series

$$V_1^* \otimes V_{\{2,\ldots,k\}},\ \ldots,\ V_k^*,\ V_{k+1},\ V_k^*,\ \ldots,\ V_1^* \otimes V_{\{2,\ldots,k\}},$$

so the lemma follows from the calculation

$$\mathrm{Hom}_{FG}(V_0 \otimes V_{\{0,\dots,k\}}, V_1^* \otimes V_{\{2,\dots,k\}}) \cong \mathrm{Hom}_{FG}(V_{\{0,\dots,k\}}, V_{\{0,1\}}^* \otimes V_{\{2,\dots,k\}}) = 0.$$

LEMMA 4.7. *V_N is not a composition factor of* $\mathrm{soc}^2(V_N \otimes V_N)$.

Proof. By Lemma 3.5(b), we may assume $G = \mathrm{SL}_3(2^m)$ and in that case we know that $V_N \otimes V_N$ has a unique composition factor isomorphic to V_N. By Lemma 4.6, the uniserial module $V_1 \otimes V_N = (V_1 \otimes V_{N\setminus\{0\}}) \otimes V_0$ has a unique filtration

$$0 \subset X \subset Y \subset V_1 \otimes V_N$$

such that $Y/X \cong V_0 \otimes V_0$. Now V_1 is a composition factor of $V_0 \otimes V_0$. Therefore, in the induced filtration

$$0 \subset X \otimes V_{N\setminus\{1\}} \subset Y \otimes V_{N\setminus\{1\}} \subset V_N \otimes V_N$$

of $V_N \otimes V_N$, the unique composition factor V_N occurs in the factor

$$(Y/X) \otimes V_{N\setminus\{1\}} \cong (V_0 \otimes V_0) \otimes V_{N\setminus\{1\}}.$$

We claim that the lemma will be proved as soon as we show that

(1) $\mathrm{Hom}_{FG}(V_N, (V_0 \otimes V_0) \otimes V_{N\setminus\{1\}}) = 0$ and

(2) $\mathrm{Hom}_{FG}(\mathrm{soc}(V_0^* \otimes V_{N\setminus\{1\}}), V_N \otimes V_N) = 0$.

Indeed, (1) implies that V_N is a composition factor of

$$((Y/X) \otimes V_{N\setminus\{1\}})/\mathrm{soc}((Y/X) \otimes V_{N\setminus\{1\}}),$$

which in turn implies (by Lemma 3.1(a)) that any simple submodule of $(Y/X) \otimes V_{N\setminus\{1\}}$ is isomorphic to one in $\mathrm{soc}(V_0^* \otimes V_{N\setminus\{1\}})$. Then (2) implies that $\mathrm{soc}((Y/X) \otimes V_{N\setminus\{1\}})$ maps injectively into $(V_N \otimes V_N)/(X + \mathrm{soc}(V_N \otimes V_N))$, which shows that V_N is not a composition factor of $\mathrm{soc}^2(V_N \otimes V_N)$.

It therefore remains to check (1) and (2).

(1) We have

$$\begin{aligned}\mathrm{Hom}_{FG}(V_N, V_0 \otimes V_0 \otimes V_{N\setminus\{1\}}) &\cong \mathrm{Hom}_{FG}((V_0^* \otimes V_0) \otimes V_{N\setminus\{0\}}, V_0 \otimes V_{N\setminus\{1\}}) \\ &\cong \mathrm{Hom}_{FG}(V_{N\setminus\{0\}} \oplus (V_{N\setminus\{0\}} \otimes W_0), V_0 \otimes V_{N\setminus\{1\}}) \\ &\cong \mathrm{Hom}_{FG}(V_{N\setminus\{0\}} \otimes V_0^*, V_{N\setminus\{1\}}) \\ &\quad \oplus \mathrm{Hom}_{FG}(V_{N\setminus\{0\}} \otimes W_0, V_0 \otimes V_{N\setminus\{1\}}).\end{aligned}$$

The first term in the last line is obviously zero, and the second is zero because by Lemma 4.6, $V_0 \otimes V_{N\setminus\{1\}}$ is uniserial and, using Lemma 4.3(a), one sees that its socle is $V_0^* \otimes V_{\{2,\dots,m-1\}}$.

(2) We have $V_0^* \otimes V_{N\setminus\{1\}} \cong V_{N\setminus\{0,1\}} \oplus (V_{N\setminus\{0,1\}} \otimes W_0)$. By Lemma 3.2(b) and the remark following it, both of these direct summands have simple socles, which we can determine using Lemma 4.3(b), yielding

$$\mathrm{soc}(V_0^* \otimes V_{N\setminus\{1\}}) \cong V_{N\setminus\{0,1\}} \oplus V_{N\setminus\{1\}}.$$

By (1.1), we obtain

$$\mathrm{Hom}_{FG}(V_{N\setminus\{0,1\}}, V_N \otimes V_N) \cong \bigoplus_{T \subseteq N\setminus\{0,1\}} \mathrm{Hom}_{FG}(W_T \otimes V_{\{0,1\}}^*, V_N) = 0$$

and

$$\mathrm{Hom}_{FG}(V_{N\setminus\{1\}}, V_N \otimes V_N) \cong \bigoplus_{T \subseteq N\setminus\{1\}} \mathrm{Hom}_{FG}(W_T \otimes V_{\{1\}}^*, V_N) = 0.$$

Thus (2) holds and the lemma is proved.

5. *Proof of the theorem*

As in [1] and [8], our plan will be to determine Ext^1_{FG} for 'maximal' simple modules, in this case $W_{N\setminus\{i\}}$ and $V_i \otimes W_{N\setminus\{i\}}$, for $i \in N$, and then to reduce the general case to these with the aid of Lemma 4.1.

We keep the notation of Lemmas 4.3 and 4.4.

LEMMA 5.1. *Let (I, J, K) be a triple. Then*

$$\mathrm{Ext}^1_{FG}(W_{N\setminus\{0\}}, V_I \otimes V_J^* \otimes W_K) \cong F \quad \textit{if} \quad K = N\setminus\{0, 1\} \quad \textit{and} \quad \{I, J\} = \{\varnothing, \{0, 1\}\},$$

and is zero otherwise.

Proof. By (4.5), we have $D_0 \otimes W_{N\setminus\{0\}} \cong P(W_{N\setminus\{0\}})$. Since $\mathrm{hd}(D_0 \otimes W_{N\setminus\{0\}}) = \mathrm{hd}(D_0) \otimes W_{N\setminus\{0\}}$, we have $JP(W_{N\setminus\{0\}}) \cong J(D_0) \otimes W_{N\setminus\{0\}}$, so the lemma is equivalent to

$$\mathrm{hd}(J(D_0) \otimes W_{N\setminus\{0\}}) \cong (V_{\{0,1\}} \otimes W_{N\setminus\{0,1\}}) \oplus (V^*_{\{0,1\}} \otimes W_{N\setminus\{0,1\}}). \tag{5.2}$$

By Lemma 4.4(c), $J(D_0)$ has homomorphic images $V_{\{0,1\}}$ and $V^*_{\{0,1\}}$, so by Lemma 4.3(b), the right-hand side of (5.2) is certainly a homomorphic image of $J(D_0) \otimes W_{N\setminus\{0\}}$. Since $J(D_0) = R_0 + R_0^\tau$, the lemma will be proved once we establish the following statements:

(1) $(R_0/Q_0) \otimes W_{N\setminus\{0\}}$ has a simple head;

(2) $Q_0 \otimes W_{N\setminus\{0\}} \subseteq J^2(D_0 \otimes W_{N\setminus\{0\}})$.

Indeed, it is immediate from (2) that $(Q_0 + Q_0^\tau) \otimes W_{N\setminus\{0\}} \subseteq J^2(D_0 \otimes W_{N\setminus\{0\}})$, and from (1) that the head of $((R_0 + R_0^\tau)/(Q_0 + Q_0^\tau)) \otimes W_{N\setminus\{0\}}$ is a direct sum of two simple modules. It remains to prove (1) and (2).

(1) By Lemma 4.4(b), we have $R_0/Q_0 \cong (V_0 \otimes V_0) \otimes V_1^*$, which by Lemma 4.3(b) is isomorphic to a quotient of $V_0^* \otimes W_0 \otimes V_1^*$. Thus, $(R_0/Q_0) \otimes W_{N\setminus\{0\}}$ is isomorphic to a quotient of $(V_0^* \otimes V^*_{\{1\}}) \otimes W_N$, which by Lemma 3.2(b) has a simple head for $m > 2$.

(2) Since by Lemmas 4.4 and 4.3, Q_0 and R_0^τ/Q_0^τ have no non-zero homomorphic images in common, we have $(Q_0 + Q_0^\tau)/Q_0^\tau \subseteq J(R_0^\tau/Q_0^\tau)$. Then by (1) applied to R_0^τ/Q_0^τ and by (4.5), we have

$$((Q_0 + Q_0^\tau)/Q_0^\tau) \otimes W_{N\setminus\{0\}} \subseteq J((R_0^\tau/Q_0^\tau) \otimes W_{N\setminus\{0\}}) \subseteq J^2(D_0) \otimes W_{N\setminus\{0\}}.$$

Also, by Lemma 4.4 and (4.5),

$$(Q_0 \cap Q_0^\tau) \otimes W_{N\setminus\{0\}} = \mathrm{soc}(D_0) \otimes W_{N\setminus\{0\}} = \mathrm{soc}(D_0 \otimes W_{N\setminus\{0\}}) \subseteq J^2(D_0 \otimes W_{N\setminus\{0\}}).$$

Together, these imply (2), so the lemma is proved.

COROLLARY 5.3. *For $0 \in T \subseteq N$ we have*

(a) $$\mathrm{soc}(W_0 \otimes W_T) \cong W_T \oplus W_T \oplus W_{T\setminus\{0\}},$$

(b) $$\mathrm{soc}^2(W_0 \otimes W_T)/\mathrm{soc}(W_0 \otimes W_T) \cong \mathrm{soc}((V_{\{0,1\}} \otimes W_{T\setminus\{0\}}) \oplus (V^*_{\{0,1\}} \otimes W_{T\setminus\{0\}})),$$

with

$$\mathrm{soc}(V_{\{0,1\}} \otimes W_{T\setminus\{0\}}) \cong \begin{cases} V_{\{0,1\}} \otimes W_{T\setminus\{0\}} & \textit{if } 1 \notin T, \\ V_{\{0,1\}} \otimes W_{T\setminus\{0,1\}} & \textit{if } 1 \in T. \end{cases}$$

Proof. Part (a) follows from Lemma 4.4(a) and (4.5). It is clear from Lemma 4.4 and the self-duality of D_0 that

$$\mathrm{soc}^2(W_0 \otimes W_0)/\mathrm{soc}(W_0 \otimes W_0) \cong \mathrm{soc}^2(D_0)/\mathrm{soc}(D_0)$$

has at least two simple summands. By Lemma 5.1, $\mathrm{soc}^2(W_0 \otimes W_N)/\mathrm{soc}(W_0 \otimes W_N)$ has two simple summands. Now since $W_0 \otimes W_N = (W_0 \otimes W_T) \otimes W_{N \setminus T}$, $W_0 \otimes W_T = (W_0 \otimes W_0) \otimes W_{T \setminus \{0\}}$, $\mathrm{soc}(W_0 \otimes W_N) = \mathrm{soc}(W_0 \otimes W_T) \otimes W_{N \setminus T}$ and $\mathrm{soc}(W_0 \otimes W_T) = \mathrm{soc}(W_0 \otimes W_0) \otimes W_{T \setminus \{0\}}$, it follows that the second socle layer of $(W_0 \otimes W_T)$ has two simple factors, which are then easily determined from Lemma 4.4 and Lemma 4.3(b).

LEMMA 5.4. *Suppose* $0, 1 \notin K \subseteq N$. *Then*

$$\mathrm{Ext}^1_{FG}(V_{\{0,1\}} \otimes W_K, W_K) \cong F \cong \mathrm{Ext}^1_{FG}(V_{\{0,1\}} \otimes W_K, W_{K \cup \{1\}}).$$

Proof. From Lemma 4.3(a), we see that $\mathrm{Ext}^1_{FG}(V_{\{0,1\}}, F) \neq 0$. The lemma will therefore follow from Lemma 5.1 and the inequalities

$$\text{(5.5)}\quad \begin{aligned} &d(V_{\{0,1\}} \otimes W_K, W_K) \leq d(V_{\{0,1\}} \otimes W_K, W_{K \cup \{1\}}) \quad \text{and} \\ &d(V_{\{0,1\}} \otimes W_K, W_{K \cup \{1\}}) \leq d(V_{\{0,1\}} \otimes W_{K \cup \{t\}}, W_{K \cup \{t\}}), \quad \text{for } t \notin K \cup \{0, 1\}. \end{aligned}$$

We shall apply Lemma 4.1. In order to prove the second inequality of (5.5), we must check that

$$\mathrm{Hom}_{FG}(X(V_{\{0,1\}} \otimes W_K, W_{K \cup \{1\}}), \mathrm{soc}^2(W_t \otimes W_{K \cup \{t\}})) = 0.$$

By Corollary 5.3, $V_{\{0,1\}} \otimes W_K$ is not even a composition factor of

$$\mathrm{soc}^2(W_t \otimes W_{K \cup \{t\}}).$$

The first inequality in (5.5) is equivalent by duality to

$$d(W_K, V^*_{\{0,1\}} \otimes W_K) \leq d(W_{K \cup \{1\}}, V^*_{\{0,1\}} \otimes W_K).$$

We shall apply Lemma 4.1 again. We note that $V^*_{\{0,1\}} \otimes W_K$ occurs in

$$\mathrm{hd}(W_1 \otimes (V^*_{\{0,1\}} \otimes W_K)),$$

by Lemma 4.3(b). Thus, by Lemma 4.1, a sufficient condition for the last inequality to hold is

$$\mathrm{Hom}_{FG}(X(W_K, V^*_{\{0,1\}} \otimes W_K), W_1 \otimes (V^*_{\{0,1\}} \otimes W_K)) = 0.$$

Now $W_1 \otimes (V^*_{\{0,1\}} \otimes W_K)$ has a filtration (induced by a composition series of $V^*_1 \otimes W_1$) with factors isomorphic to $V^*_{\{0,1\}} \otimes W_K$, $V_2 \otimes (V^*_0 \otimes W_K)$ and

$$V^*_2 \otimes (V_1 \otimes V^*_0 \otimes W_K).$$

There are obviously no (non-zero) homomorphisms from $X(W_K, V^*_{\{0,1\}} \otimes W_K)$ to the first of these three modules. The second and third modules are simple if $2 \notin K$, since $m > 2$, so there are no (non-zero) homomorphisms in this case either. If $2 \in K$, then there are no (non-zero) homomorphisms from $X(W_K, V^*_{\{0,1\}} \otimes W_K)$, because by Lemma 3.2(b) and Lemma 4.3(b), their socles are isomorphic to $V_2 \otimes V^*_0 \otimes V_{K \setminus \{2\}}$ and $V^*_{\{0,2\}} \otimes V_1 \otimes W_{K \setminus \{2\}}$ respectively. This completes the proof of the lemma.

LEMMA 5.6. (a) *If* (I, J, K) *is a triple and* $I \cup J \cup K \subseteq T \subseteq N$, *then*

$$\mathrm{Ext}^1_{FG}(V_I \otimes V_J^* \otimes W_K, W_T) = 0.$$

(b) *For any* $S, K \subseteq N$, *we have* $\mathrm{Ext}^1_{FG}(W_S, W_K) = 0$.

Proof. Part (b) follows from (a) since $\mathrm{Ext}^1_{FG}(W_S, W_K) \cong \mathrm{Ext}^1_{FG}(W_{S\cap K}, W_{S\cup K})$. In order to prove (a) we may assume by taking Galois conjugates that $0 \notin T$. It will suffice to show that

$$d(V_I \otimes V_J^* \otimes W_K, W_T) \leq d(V_I \otimes V_J^* \otimes W_{K\cup\{0\}}, W_{T\cup\{0\}}),$$

because W_N is projective. By Lemma 4.1, this inequality will be proved if we verify that

$$\mathrm{Hom}_{FG}(X(V_I \otimes V_J^* \otimes W_K), \mathrm{soc}^2(W_0 \otimes W_{T\cup\{0\}})) \cong \begin{cases} F & \text{if } I = J = \varnothing \text{ and } K = T, \\ 0 & \text{otherwise.} \end{cases}$$

This is immediate from Corollary 5.3, so the lemma is proved.

COROLLARY 5.7. *For any triple* (I, J, K) *we have*

$$\mathrm{Ext}^1_{FG}(V_I \otimes V_J^* \otimes W_K, V_I \otimes V_J^* \otimes W_K) = 0.$$

Proof. We have

$$\mathrm{Ext}^1_{FG}(V_I \otimes V_J^* \otimes W_K, V_I \otimes V_J^* \otimes W_K) \cong \bigoplus_{K \subseteq T \subseteq I\cup J\cup K} \mathrm{Ext}^1_{FG}(W_K, W_T).$$

LEMMA 5.8. *Let* $I, J \subseteq N$ *with* $I \cup J = N$ *and* $I \cap J = \varnothing$. *Then*

$$\mathrm{Ext}^1_{FG}(W_T, V_I \otimes V_J^*) = 0$$

for $T \subseteq N$.

Proof. We shall argue by downward induction on $|T|$, the result being true for $T = N$. We are free to replace W_T and $V_I \otimes V_J^*$ by their (simultaneous) conjugates by τ or field automorphisms, as the hypotheses and conclusion are not affected and neither is the inductive hypothesis. It will suffice to find some $i \in N\backslash T$ such that

$$d(W_T, V_I \otimes V_J^*) \leq d(W_{T\cup\{i\}}, V_I \otimes V_J^*) = 0.$$

Since by Lemma 4.3(b), $V_I \otimes V_J^*$ occurs in the head of $W_i \otimes (V_I \otimes V_J^*)$ for all $i \in N$, this inequality will follow from Lemma 4.1 if we can find $i \in N\backslash T$ such that

$$\mathrm{Hom}_{FG}(X(W_T, V_I \otimes V_J^*), W_i \otimes (V_I \otimes V_J^*)) = 0.$$

First suppose $|T| > 1$. By taking appropriate conjugates, we may assume $0 \notin T$, and then by Lemma 3.4, W_T is not a composition factor of the right-hand side, so we have finished in this case. Therefore, we may assume from now on that $|T| \leq 1$.

Next we consider the case in which some conjugate of $V_I \otimes V_J^*$ is V_N. Let us take such a conjugate so that $I = N$. If $0 \notin T$, we choose $i = 0$. Then $W_0 \otimes V_N = (W_0 \otimes V_0) \otimes V_{N\backslash\{0\}}$ has a filtration with factors of the form $V_0 \otimes V_{N\backslash\{0\}}$, $V_1^* \otimes V_{N\backslash\{0\}}$ and $(V_0^* \otimes V_1) \otimes V_{N\backslash\{0\}}$. Since $m > 2$, the first two modules have no composition factor W_T. It is easy to see that the last module has no composition factor V_N, so that any homomorphism from $X(W_T, V_N)$ into it must factor through W_T. But we have

$$\mathrm{Hom}_{FG}(W_T, V_0^* \otimes V_1 \otimes V_{\{1,\ldots,m-1\}}) \cong \mathrm{Hom}_{FG}(W_T \otimes V_0, V_1 \otimes V_{\{1,\ldots,m-1\}}) = 0$$

by Lemma 4.6. Thus, if $0 \notin T$, we have finished. If $G = \mathrm{SL}_3(2^m)$, then V_N is invariant under field automorphisms, and so, since $|T| \leqslant 1$, we can always choose conjugates with $I = N$ and $0 \notin T$. If $G = \mathrm{SU}_3(2^m)$ and $T = \{0\}$, we choose $i = 1$. It is straightforward to check using Lemmas 3.1 and 4.6 that $W_1 \otimes V_N$ does not have W_0 as a composition factor.

We may therefore assume from now on that no conjugate of $V_I \otimes V_J^*$ by τ or a field automorphism is V_N and that $|T| \leqslant 1$.

We claim that one of the following holds:

(a) there is some conjugate $V_{I'} \otimes V_{J'}^*$ of $V_I \otimes V_J^*$ with $0 \in I'$, $1 \in I'$ and $2 \in J'$;

(b) for every conjugate $V_{I'} \otimes V_{J'}^*$ of $V_I \otimes V_J^*$ such that $0 \in I'$, we have $1 \in J'$ and $2 \in I'$.

To see this, consider the set of conjugates $V_{I''} \otimes V_{J''}^*$ of $V_I \otimes V_J^*$ with $0 \in I''$. Suppose there is one with $1 \in I''$. Then since $I'' \neq N$, we may choose the smallest r with $r \in J''$. Conjugating by $\sigma^{-(r-2)}$ yields a conjugate satisfying the conditions in (a). Suppose then that for every conjugate $V_{I''} \otimes V_{J''}^*$ such that $0 \in I''$ we have $1 \in J''$. Fix such a conjugate. Then conjugating by $\tau\sigma^{-1}$ gives another conjugate $V_{I'} \otimes V_{J'}^*$ with $0 \in I'$. Thus, $1 \in J'$, which is to say that $2 \in I''$, so (b) holds. This establishes the claim.

For $V_{I'} \otimes V_{J'}^*$ as in (a), the composition factors of $W_0 \otimes (V_{I'} \otimes V_{J'}^*)$ are readily calculated to be $V_{I'} \otimes V_{J'}^*$, $V_{I' \setminus \{0,1\}} \otimes V_{J'}^*$, $V_{I' \setminus \{0,1\}} \otimes V_{J'}^* \otimes W_1$, $V_{I' \setminus \{0,1\}} \otimes V_{J' \cup \{0,1\}}^*$, $V_{I' \setminus \{0\}} \otimes V_{J' \cup \{0\} \setminus \{2\}}^*$ and $V_{I' \setminus \{0\}} \otimes V_{J' \cup \{0\} \setminus \{2\}}^* \otimes W_2$. Since $m > 2$, none of these is F or W_t for $t \in N$.

For $V_{I'} \otimes V_{J'}^*$ as in (b), the composition factors of $W_0 \otimes (V_{I'} \otimes V_{J'}^*)$ are $V_{I'} \otimes V_{J'}^*$, $V_{I' \setminus \{0\} \cup \{1\}} \otimes V_{J' \setminus \{1\}}^*$, $V_{I' \setminus \{0,2\}} \otimes V_{J' \setminus \{1\}}^*$, $V_{I' \setminus \{0,2\}} \otimes V_{J' \setminus \{1\}}^* \otimes W_2$, $V_{I' \setminus \{0\}} \otimes V_{J' \cup \{0\} \setminus \{1\}}^*$ and $V_{I' \setminus \{0\}} \otimes V_{J' \cup \{0\} \setminus \{1\}}^* \otimes W_1$. If $m \geqslant 4$, none of these is F or W_t for $t \in N$.

We consider first the generic case $m \geqslant 4$. Suppose (b) holds. Let $V_{I'} \otimes V_{J'}^*$ be a conjugate of $V_I \otimes V_J^*$ with $0 \in I'$ and let $W_{T'}$ be the corresponding conjugate of W_T. Since $|T| \leqslant 1$, these can be chosen with $0 \notin T'$. We choose $i = 0$ and then the above calculations show that $W_{T'}$ is not a composition factor of $W_0 \otimes V_{I'} \otimes V_{J'}^*$.

Suppose (a) holds. We choose a conjugate $V_{I'} \otimes V_{J'}^*$ as in (a), and denote by $W_{T'}$ the corresponding conjugate of W_T. If $T' \neq \{0\}$, then we may choose $i = 0$ by the above calculations. Suppose then that $T' = \{0\}$. If $3 \in I'$, then we choose $i = 1$, and may use the result of our calculation in Case (b) above (after conjugating by σ^{-1}) to conclude that no composition factor is W_0. If $3 \in J'$, then we have $2 \in J'$ and $3 \in J'$, and it is not difficult to see that there will be another conjugate $V_{I''} \otimes V_{J''}^*$ satisfying (a) but with $T'' \neq \{0\}$ (for $G = \mathrm{SU}_3(2^m)$, we must remember that $V_I \otimes V_J^*$ is not conjugate to V_N). Thus, choosing this conjugate places us back in the case where $T \neq \{0\}$. This completes the argument for $m \geqslant 4$.

Finally, suppose $m = 3$. We can assume that T is either $\{0\}$ or empty. If $G = \mathrm{SL}_3(2^3)$, we can conjugate by τ if necessary so that $V_I \otimes V_J^*$ is $V_0 \otimes V_1^* \otimes V_2^*$ or $V_0 \otimes V_1 \otimes V_2^*$ or $V_0 \otimes V_1^* \otimes V_2^*$. We choose $i = 2$ in the first two cases and $i = 1$ in the last. Then it is easy to check that no composition factor of $W_i \otimes V_I \otimes V_J^*$ is

F or W_0. If $G = SU_3(2^3)$, we may assume $V_I \otimes V_J^* = V_0 \otimes V_1^* \otimes V_2$. We choose $i = 2$. The module $W_2 \otimes V_0 \otimes V_1^* \otimes V_2 = (W_2 \otimes V_2) \otimes V_0 \otimes V_1^*$ has a filtration with factors of the form $V_2 \otimes V_0 \otimes V_1^*$, $V_3^* \otimes V_0 \otimes V_1^*$ and $(V_2^* \otimes V_3) \otimes V_0 \otimes V_1^*$. Since $V_3 \cong V_0^*$, we see using Lemma 4.3(d) that there are no non-zero homomorphisms from $X(W_T, V_I \otimes V_J^*)$ into any of these factors. This completes the proof of the lemma.

PROPOSITION 5.9. *Let (I, J, K) be a triple and $T \subseteq N$. Then*

$$\mathrm{Ext}^1_{FG}(W_T, V_I \otimes V_J^* \otimes W_K) \cong F$$

if $((\varnothing, \varnothing, T), (I, J, K))$ is conjugate (by field automorphisms and τ) to $((\varnothing, \varnothing, K), (\{0, 1\}, \varnothing, K))$ or $((\varnothing, \varnothing, K \cup \{1\}), (\{0, 1\}, \varnothing, K))$, where $0, 1 \notin K$, and is zero otherwise.

Proof. We may assume that $K \subseteq T$. If $I \cup J \cup K \subseteq T$, then we have finished by Lemma 5.6. Suppose first that $|T \backslash K| > 1$. There is no harm in replacing (I, J, K) and T by (simultaneous) conjugates, so we may assume without loss of generality that $0 \in I \backslash T$. Using Lemma 4.1, we shall prove the inequality

$$d(W_T, V_I \otimes V_J^* \otimes W_K) \leq d(W_{T \cup \{0\}}, V_I \otimes V_J^* \otimes W_K).$$

Clearly, this will give a reduction to the case where $I \cup J \cup K \subseteq T$. By Lemma 4.3(b), $W_0 \otimes (V_I \otimes V_J^* \otimes W_K)$ has $V_I \otimes V_J^* \otimes W_K$ in its head. Since $|T| > |K| + 1$, Lemma 3.4 implies that

$$\mathrm{Hom}_{FG}(X(W_T, V_I \otimes V_J^* \otimes W_K), W_0 \otimes (V_I \otimes V_J^* \otimes W_K)) = 0,$$

so the inequality follows from Lemma 4.1.

The cases in which $T = K$ and $|T| = |K| + 1$ require closer analysis. We have seen in Lemma 5.4 that if the conditions of the proposition hold then the cohomology groups are isomorphic to F as claimed, so it remains to prove vanishing in all other cases. We therefore assume that the conditions of the proposition do not hold. By Lemma 5.8 we may also assume that $I \cup J \subset N$ (where $\subset$ indicates strict inclusion). We first show how to reduce the case where $K = T$ to the case in which $|T| = |K| + 1$. Again, by taking Galois conjugates if necessary, we may assume that $0 \in I \cup J$ and $1 \notin I \cup J$. Applying τ if necessary, we can also assume $0 \in I$. We shall show that

$$d(W_K, V_I \otimes V_J^* \otimes W_K) \leq d(K_{T \cup \{0\}}, V_I \otimes V_J^* \otimes W_K).$$

Since $W_0 \otimes (V_I \otimes V_J^* \otimes W_K)$ has $V_I \otimes V_J^* \otimes W_K$ in its head, the inequality will follow from the equation

$$\mathrm{Hom}_{FG}(X(W_K, V_I \otimes V_J^* \otimes W_K), W_0 \otimes (V_I \otimes V_J^* \otimes W_K)) = 0.$$

Now $W_0 \otimes (V_I \otimes V_J^* \otimes W_K)$ has a filtration with factors isomorphic to

$$V_I \otimes V_J^* \otimes W_K, \; V_1^* \otimes (V_{I \backslash \{0\}} \otimes V_J^* \otimes W_K)$$

and $V_1 \otimes (V_{I \backslash \{0\}} \otimes V_{J \cup \{0\}}^* \otimes W_K)$. If $1 \notin I \cup J \cup K$ then all three of these modules are simple and not isomorphic to W_K. If $1 \in K$, then the first module is unchanged, and by Lemma 3.2(b) and Lemma 4.3(a), the socles of the second and third modules are isomorphic to $V_{I \backslash \{0\}} \otimes V_{J \cup \{1\}}^* \otimes W_{K \backslash \{1\}}$ and $V_{I \backslash \{0\} \cup \{1\}} \otimes V_{J \cup \{0\}}^* \otimes W_{K \backslash \{1\}}$ respectively. Thus in all cases there are no (non-zero) homomorphisms from $X(W_K, V_I \otimes V_J^* \otimes W_K)$.

We are reduced to considering the case where $T = K \cup \{t\}$, for some $t \notin K$. This time it will be convenient to take Galois conjugates so that $t = m-1$. Our assumption is that $\{I, J\} \neq \{\emptyset, \{m-2, m-1\}\}$, and we must prove that $\mathrm{Ext}^1_{FG}(W_{K\cup\{m-1\}}, V_I \otimes V_J^* \otimes W_K) = 0$. We shall show that we can find $i \in (I \cup J)\setminus\{m-1\}$ such that

$$d(W_{K\cup\{m-1\}}, V_I \otimes V_J^* \otimes W_K) \leqslant d(W_{K\cup\{m-1,i\}}, V_I \otimes V_J^* \otimes W_K),$$

which will put us back into the case where $|T| > |K| + 1$.

We claim that one of the following must hold:

(a) there exists $i \in (I \cup J)\setminus\{m-1\}$ with $i+1 \notin I \cup J \cup K$;

(b) there exists $i \in (I \cup J)\setminus\{m-1\}$ with $i+1 \in K$;

(c) $I \cup J = \{m-1-s, m-1-s+1, \ldots, m-2, m-1\}$, for some s with $1 \leqslant s \leqslant m-2$.

This is easy to see; let S be the set of elements x of $I \cup J\setminus\{m-1\}$ such that $x+1 \notin I \cup J\setminus\{m-1\}$. Then $S \neq \emptyset$ since $I \cup J \neq N$. If S contains an element different from $m-2$ then (a) or (b) will hold. If $S = \{m-2\}$ and neither (a) nor (b) holds, then $m-1 \in I \cup J$ and (c) must hold.

We shall prove the inequality using Lemma 4.1 by showing that for a suitable choice of $i \in (I \cup J)\setminus T$,

$$\mathrm{Hom}_{FG}(X(W_{K\cup\{m-1\}}, V_I \otimes V_J^* \otimes W_K), W_i \otimes (V_I \otimes V_J^* \otimes W_K)) = 0.$$

Suppose first that it is possible to choose i as in (a) or (b). We may assume that $i \in I$ by conjugating everything by τ if necessary.

Suppose (a) holds. Then the composition factors of $W_i \otimes (V_I \otimes V_J^* \otimes W_K)$ are $V_I \otimes V_J^* \otimes W_K$, $V_{I\setminus\{i\}} \otimes V_{J\cup\{i+1\}}^* \otimes W_K$ and $V_{I\cup\{i+1\}} \otimes V_{J\cup\{i\}}^* \otimes W_K$, none of which is $W_{K\cup\{m-1\}}$, so we have finished in this case.

If (b) holds, then $W_i \otimes (V_I \otimes V_J^* \otimes W_K)$ has a filtration with factors isomorphic to $V_I \otimes V_J^* \otimes W_K$, $(V_{i+1}^* \otimes W_{i+1}) \otimes (V_{I\setminus\{i\}} \otimes V_J^* \otimes W_{K\setminus\{i+1\}})$ and

$$(V_{i+1} \otimes W_{i+1}) \otimes (V_{I\setminus\{i\}} \otimes V_{J\cup\{i\}}^* \otimes W_{K\setminus\{i+1\}}).$$

The first module is simple and by Lemmas 3.2(b) and 4.3(b), the socles of the second and third are isomorphic to $V_{I\setminus\{i\}} \otimes V_{J\cup\{i+1\}}^* \otimes W_{K\setminus\{i+1\}}$ and $V_{I\setminus\{i\}\cup\{i+1\}} \otimes V_{J\cup\{i\}}^* \otimes W_{K\setminus\{i+1\}}$ respectively. Therefore there are no (non-zero) homomorphisms from $X(W_{K\cup\{m-1\}}, V_I \otimes V_J^* \otimes W_K)$ into any of the three filtration factors.

Finally, if it is not possible to choose i as in (a) or (b), then (c) must hold. We choose $i = m-2$. We may assume that $m-2 \in I$. Since by assumption we do not have $I = \{m-2, m-1\}$ and $J = \emptyset$, we must have either $m-1 \in J$ or $|I \cup J| = s > 1$.

If $m-1 \in J$, then there is a filtration of $W_{m-2} \otimes (V_I \otimes V_J^* \otimes W_K)$ with factors isomorphic to $V_I \otimes V_J^* \otimes W_K$,

$$V_{m-1}^* \otimes (V_{I\setminus\{m-2\}} \otimes V_J^* \otimes W_K) = (V_{m-1}^* \otimes V_{m-1}^*) \otimes (V_{I\setminus\{m-2\}} \otimes V_{J\setminus\{m-1\}}^* \otimes W_K)$$

and

$$V_{m-1} \otimes (V_{I\setminus\{m-2\}} \otimes V_{J\cup\{m-2\}}^* \otimes W_K) \cong (V_{I\setminus\{m-2\}} \otimes V_{J\cup\{m-2\}\setminus\{m-1\}}^* \otimes W_K) \oplus (V_{I\setminus\{m-2\}} \otimes V_{J\cup\{m-2\}\setminus\{m-1\}}^* \otimes W_{K\cup\{m-1\}}).$$

The first of these factors is simple and the third is semisimple and neither has a factor $W_{K\cup\{m-1\}}$. By Lemma 4.3(b), the second filtration factor is isomorphic to a submodule of

$$(V_{m-1}\otimes W_{m-1})\otimes(V_{I\setminus\{m-2\}}\otimes V^*_{J\setminus\{m-1\}}\otimes W_K)$$
$$= V_{I\setminus\{m-2\}\cup\{m-1\}}\otimes V^*_{J\setminus\{m-1\}}\otimes W_{K\cup\{m-1\}},$$

which, by Lemma 3.2(b) and Lemma 4.3(b) has socle

$$V_{I\setminus\{m-2\}\cup\{m-1\}}\otimes V^*_{J\setminus\{m-1\}}\otimes W_K.$$

Therefore, there are no (non-zero) homomorphisms of

$$X(W_{K\cup\{m-1\}}, V_I\otimes V_J^*\otimes W_K)$$

into any of the three filtration factors.

We may therefore assume that $m-1\in I$, and hence that $|I\cup J|>1$. Then we have a filtration of $W_{m-2}\otimes(V_I\otimes V_J^*\otimes W_K)$ with factors of the form

$$V_I\otimes V_J^*\otimes W_K,$$
$$(V_{I\setminus\{m-2,m-1\}}\otimes V_J^*\otimes W_K)\oplus(V_{I\setminus\{m-2,m-1\}}\otimes V_J^*\otimes W_{K\cup\{m-1\}}),$$

and

$$(V_{m-1}\otimes V_{m-1})\otimes V_{I\setminus\{m-2,m-1\}}\otimes V^*_{J\cup\{m-2\}}\otimes W_K.$$

The first two factors are semisimple and have no factor $W_{K\cup\{m-1\}}$, as $|I\cup J|>1$, and the third filtration factor has simple socle $V_{I\setminus\{m-2,m-1\}}\otimes V^*_{J\cup\{m-2,m-1\}}\otimes W_K$, by Lemma 4.3(b) and Lemma 3.2(b). Therefore, there are no (non-zero) homomorphisms of $X(W_{K\cup\{m-1\}}, V_I\otimes V_J^*\otimes W_K)$ into any of the three filtration factors.

The proposition is proved.

LEMMA 5.10. *$V_i\otimes V_i\otimes W_i$ has a direct summand isomorphic to $V_{i+1}\otimes W_i$.*

Proof. By Galois conjugation we can assume that $i=0$. By Remark 4.2, it suffices to show that $V_1\otimes W_0$ is a homomorphic image, since $V_0\otimes V_0\otimes W_0$ has a unique composition factor $V_1\otimes W_0$. By Lemma 4.3(d), $V_0\otimes V_0\otimes V_1^*$ has a uniserial quotient with series $V^*_{\{0,1\}}$, F. By Lemma 4.4, $W_0\otimes W_0$ has a uniserial submodule with series $V^*_{\{0,1\}}$, F. Since $\mathrm{Ext}^1_{FG}(V^*_{\{0,1\}}, F)\cong F$ by Lemma 5.4, these uniserial modules are in fact isomorphic. Thus,

$$\mathrm{Hom}_{FG}(V_0\otimes V_0\otimes W_0, V_1\otimes W_0)\cong\mathrm{Hom}_{FG}(V_0\otimes V_0\otimes V_1^*, W_0\otimes W_0)\neq 0.$$

LEMMA 5.11. $\mathrm{Ext}^1_{FG}(V_i, V_i^*\otimes V_{i+1})=0.$

Proof. We may assume that $i=0$. By Lemma 5.4, $\mathrm{Ext}^1_{FG}(V_0^*, V_1)\cong F$. By Lemma 4.3(a), $V_0\otimes V_0$ is uniserial with series, V_0^*, V_1, V_0^*. Thus, if $\mathrm{Ext}^1_{FG}(V_0, V_0^*\otimes V_1)\cong\mathrm{Ext}^1_{FG}(V_0\otimes V_0, V_1)$ were not zero, there would exist a uniserial module U such that $U/\mathrm{soc}(U)\cong J(V_0\otimes V_0)$ and

$$JU\cong(V_0\otimes V_0)/\mathrm{soc}(V_0\otimes V_0).$$

Therefore, by Lemma 5.10, we have

$$JU\otimes W_0\cong(V_1\otimes W_0)\oplus(V_0^*\otimes W_0)\cong U/\mathrm{soc}(U).$$

By Corollary 5.7, we have $\mathrm{Ext}^1_{FG}(V_1 \otimes W_0, V_1 \otimes W_0) = 0$, so we deduce that

$$U \otimes W_0 \cong (V_1 \otimes W_0) \oplus (V_1 \otimes W_0) \oplus (V_0^* \otimes W_0).$$

Thus, $\mathrm{Hom}_{FG}(U \otimes W_0, V_1 \otimes W_0)$ is 2-dimensional.

On the other hand, from the structure of $U/\mathrm{soc}(U)$ and JU given above and the structure of $V_0 \otimes V_0 \otimes V_1^*$ given in Lemma 4.3(d), we see that the Loewy layers of $U \otimes V_1^*$ are

$$F \oplus W_1, \quad V^*_{\{0,1\}}, \quad F \oplus W_1.$$

Thus, F is the only homomorphic image of $U \otimes V_1^*$ with no composition factor W_1. From the filtration $0 \subset Q_0^\tau \subset R_0^\tau \subset J(D_0)$ of Lemma 4.4(b), we see that the unique composition factor W_1 of $W_0 \otimes W_0$ is a composition factor of $R_0^\tau/Q_0^\tau \cong V_0^* \otimes V_0^* \otimes V_1$. Since $\mathrm{soc}(V_0^* \otimes V_0^* \otimes V_1) \cong V_{\{0,1\}}$, by Lemma 4.3(d), it follows that any submodule of $W_0 \otimes W_0$ which has W_1 as a composition factor also has $V_{\{0,1\}}$ as a composition factor. Since the latter is not a composition factor of $U \otimes V_1^*$, we conclude that the only (non-zero) homomorphisms of $U \otimes V_1^*$ into $W_0 \otimes W_0$ have image isomorphic to F. Thus, from the structure of $W_0 \otimes W_0$ given in Lemma 4.4, we have

$$\mathrm{Hom}_{FG}(U \otimes V_1^*, W_0 \otimes W_0) \cong F,$$

which is contrary to our previous calculation. Therefore, U does not exist and the lemma is proved.

LEMMA 5.12. *$V_i \otimes W_i$ has Loewy layers*

$$V_i, \quad V^*_{i+1}, \quad V_i \oplus (V_i^* \otimes V_{i+1}), \quad V^*_{i+1}, \quad V_i.$$

Proof. We may assume that $i = 0$. By Lemma 4.3(b), $V_0 \otimes W_0$ has a submodule S and a quotient Q, both isomorphic to $V_0^* \otimes V_0^*$. Since $S \subseteq J(V_0 \otimes W_0)$, by Lemma 4.3(b), and since V_0 occurs three times as a composition factor of $V_0 \otimes W_0$, it must be the case that the image of $\mathrm{hd}(S)$ in Q is $\mathrm{soc}(Q)$. Thus, the Loewy length of $V_0 \otimes W_0$ is at least 5. It is then easily seen from Lemma 5.11 that the Loewy layers are as claimed.

The structure of $V \otimes W$ as a module for $\mathrm{SL}_3(F)$ has been given previously in [3].

In the following two lemmas we need to assume that $G \neq \mathrm{SU}_3(8)$. Lemma 5.13 is false for this group, but the statement of Lemma 5.14 is true and will be proved in the supplementary Lemma 5.14′ below.

LEMMA 5.13. *Assume $G \neq \mathrm{SU}_3(8)$. Suppose $0 \notin K \subseteq N$. Then*

$$\mathrm{Ext}^1_{FG}(W_K, V_2 \otimes (V_0^* \otimes W_K)) = 0.$$

Proof. If $2 \notin K$ then the result is a special case of Proposition 5.9. We assume $2 \in K$. Then $V_2 \otimes (V_i^* \otimes W_K)$ has a filtration (induced by a composition series of $V_2 \otimes W_2$) with factors (ignoring multiplicities and order) isomorphic to

$$V_2 \otimes V_0^* \otimes W_{K \setminus \{2\}}, \quad V_3^* \otimes (V_0^* \otimes W_{K \setminus \{2\}}) \text{ and } V_3 \otimes (V^*_{\{0,2\}} \otimes W_{K \setminus \{2\}}).$$

It will suffice to show that $\mathrm{Ext}^1_{FG}(W_K, -)$ vanishes on each of these three modules. For $V_2 \otimes V_0^* \otimes W_{K\setminus\{2\}}$ this is just Proposition 5.9 again. Next we note that by Proposition 5.9, $\mathrm{Ext}^1_{FG}(W_K, -)$ vanishes on all modules of mass less than $2|K|$. By Lemma 3.3(c), the second module can have as large a mass as $2|K|$ only if $3 \notin K \cup \{0\}$, and if this is so then Proposition 5.9 gives the required vanishing. For the third module, we may again apply Proposition 5.9 immediately if $3 \notin K \cup \{0\}$. If $V_3 \cong V_0$, then the module is $V_2^* \otimes W_{K\setminus\{2\}} \oplus V_2^* \otimes W_{K\setminus\{2\}\cup\{0\}}$ and Proposition 5.9 applies. Finally, if $3 \in K$, we get a filtration on $V_3 \otimes V^*_{\{0,2\}} \otimes W_{K\setminus\{2\}}$ (induced by a composition series of $V_3 \otimes W_3$) with factors of the form

$$V_3 \otimes V^*_{\{0,2\}} \otimes W_{K\setminus\{2,3\}}, \quad V_4^* \otimes (V^*_{\{0,2\}} \otimes W_{K\setminus\{2,3\}}) \text{ and } V_4 \otimes (V^*_{\{0,2,3\}} \otimes W_{K\setminus\{2,3\}}).$$

By Lemma 3.3(c), all of these factors have mass less than $2|K|$ except perhaps the last one, which will have mass $2|K|$ if W_4 is not a tensor factor of W_K and V_4 is not isomorphic to V_0^*. But then either $V_4 \cong V_0$ and the module is

$$(V^*_{\{2,3\}} \otimes W_{K\setminus\{2,3\}}) \oplus (V^*_{\{2,3\}} \otimes W_{K\setminus\{2,3\}\cup\{0\}})$$

or else the module is simple. In both cases $\mathrm{Ext}^1_{FG}(W_K, -)$ vanishes on it, by Proposition 5.9.

LEMMA 5.14. *Assume $G \neq \mathrm{SU}_3(8)$. For $0, 1 \notin K \subseteq N$ we have*

$$\mathrm{Ext}^1_{FG}(V_1^* \otimes W_K, V_0^* \otimes V_1 \otimes W_K) \cong F.$$

Proof. We have

$$\mathrm{Ext}^1_{FG}(V_1^* \otimes W_K, V_0^* \otimes V_1 \otimes W_K) \cong \mathrm{Ext}^1_{FG}(V_0 \otimes W_K, V_1 \otimes V_1 \otimes W_K).$$

Now $(V_1 \otimes V_1) \otimes W_K$ has a filtration with factors (in order)

$$V_1^* \otimes W_K, \quad V_2 \otimes W_K, \quad V_1^* \otimes W_K.$$

Also, $(V_1 \otimes V_1) \otimes W_K$, being a quotient of $(V_1^* \otimes W_1) \otimes W_K = V_1^* \otimes W_{K\cup\{1\}}$, has simple head and socle isomorphic to $V_1^* \otimes W_K$. Thus,

$$\mathrm{soc}^2((V_1 \otimes V_1) \otimes W_K)/\mathrm{soc}((V_1 \otimes V_1) \otimes W_K) \cong \mathrm{soc}(V_2 \otimes W_K) \cong \begin{cases} V_2 \otimes W_K & \text{if } 2 \notin K, \\ V_2 \otimes W_{K\setminus\{2\}} & \text{if } 2 \in K. \end{cases}$$

Therefore, $\mathrm{Hom}_{FG}(V_0 \otimes W_K, \mathrm{soc}^2((V_1 \otimes V_1) \otimes W_K)/\mathrm{soc}((V_1 \otimes V_1) \otimes W_K)) = 0$, so we have an exact sequence

$$0 \to \mathrm{Ext}^1_{FG}(V_0 \otimes W_K, V_1^* \otimes W_K) \to \mathrm{Ext}^1_{FG}(V_0 \otimes W_K, (V_1 \otimes V_1) \otimes W_K),$$

which by Lemma 5.4 proves that $\mathrm{Ext}^1_{FG}(V_0 \otimes W_K, (V_1 \otimes V_1) \otimes W_K) \neq 0$.

Suppose for a contradiction that $d(V_0 \otimes W_K, (V_1 \otimes V_1) \otimes W_K) > 1$. Then from the structure of $(V_1 \otimes V_1) \otimes W_K$ described above, and by Lemma 5.13 and Lemma 5.4, there would exist a module M with $\mathrm{hd}(M) \cong V_0 \otimes W_K$ and $J(M) \cong (V_1 \otimes V_1) \otimes W_K$.

We would then have, on the one hand,

$$\mathrm{Hom}_{FG}(M \otimes V_1, W_{K\cup\{1\}}) \cong \mathrm{Hom}_{FG}(M, V_1^* \otimes W_{K\cup\{1\}}).$$

The composition factors of $V_1^* \otimes W_{K\cup\{1\}}$ are $V_1^* \otimes W_K$, those of $V_2 \otimes W_K$ and those of $V_2^* \otimes (V_1 \otimes W_K)$. Using Lemma 3.3(c), we can easily see that the first two modules have no composition factor $V_0 \otimes W_K$ (since $m > 2$). This is also clear for the last module if $2 \notin K$. If $2 \in K$ then its composition factors are $V_1 \otimes V_2^* \otimes W_{K\setminus\{2\}}$, those of $V_3 \otimes (V_1 \otimes W_{K\setminus\{2\}})$ and those of $V_3^* \otimes (V_{\{1,2\}} \otimes W_{K\setminus\{2\}})$, and now we can use Lemma 3.3(c) to see that none of these composition factors is $V_0 \otimes W_K$. Thus, $\mathrm{Hom}_{FG}(M \otimes V_1, W_{K\cup\{1\}}) = 0$.

On the other hand, $(M/JM) \otimes V_1 \cong V_{0,1} \otimes W_K$ and

$$JM \otimes V_1 \cong (V_1 \otimes V_1 \otimes V_1 \otimes W_K) \cong W_{K\cup\{1\}} \oplus W_{K\cup\{1\}} \oplus (L_1 \otimes W_K),$$

by Lemma 4.3(c). By Lemma 5.4, $\mathrm{Ext}^1_{FG}(V_{0,1} \otimes W_K, W_{K\cup\{1\}})$ is only one-dimensional, so $M \otimes V_1$ must have $W_{K\cup\{1\}}$ as a homomorphic image. This contradiction proves the lemma.

LEMMA 5.14′. *Let* $G = \mathrm{SU}_3(8)$. *Then*

(a) $\mathrm{Ext}^1_{FG}(V_0^* \otimes V_1, V_1^*) \cong F$,

(b) $\mathrm{Ext}^1_{FG}(V_0^* \otimes V_1 \otimes W_2, V_1^* \otimes W_2) \cong F$.

Proof. We know by Lemma 4.6 that the group in (a) is not trivial. We shall show first that $d(V_0^* \otimes V_1, V_1^*) \leq d(V_0^* \otimes V_1 \otimes W_2, V_1^* \otimes W_2)$. By Lemma 4.1, this follows from the fact that $W_2 \otimes W_2 \otimes V_1^*$ does not have $V_0^* \otimes V_1$ as a composition factor, which is easy to check using Lemma 3.1.

Now we shall show that $d(V_0^* \otimes V_1 \otimes W_2, V_1^* \otimes W_2) \leq 1$, that is, that

$$\dim_F \mathrm{Hom}_{FG}(JP(V_0^* \otimes V_1 \otimes W_2), V_1^* \otimes W_2) \leq 1.$$

By Lemma 3.2(b) we have

$$P(V_0^* \otimes V_1 \otimes W_2) \cong V_0^* \otimes V_1 \otimes W_N \cong (V_1 \otimes W_1) \otimes V_0^* \otimes W_0 \otimes W_2.$$

Now by Lemmas 4.3 and 3.1, we know that $V_1 \otimes W_1$ has a filtration

$$0 \subset S_1 \subset J(V_1 \otimes W_1) \subset V_1 \otimes W_1,$$

where $S_1 \cong V_1^* \otimes V_1^*$ and the head of $V_1 \otimes W_1$ is isomorphic to V_1. It will therefore suffice to check the following three statements:

(1) $\mathrm{Hom}_{FG}(J(V_1 \otimes V_0^* \otimes W_0 \otimes W_2), V_1^* \otimes W_2) \cong F$;

(2) $\mathrm{Hom}_{FG}((J(V_1 \otimes W_1)/S_1) \otimes V_0^* \otimes W_0 \otimes W_2, V_1^* \otimes W_2) = 0$;

(3) $\mathrm{Hom}_{FG}(V_1^* \otimes V_1^* \otimes V_0^* \otimes W_0 \otimes W_2, V_1^* \otimes W_2) = 0$.

Part (3) is immediate since $V_1^* \otimes V_1^* \otimes V_0^* \otimes W_0 \otimes W_2$ is a quotient of $P(V_1^* \otimes V_0^* \otimes W_2)$.

In Part (2) we know by Lemmas 4.3 and 3.1 that the composition factors of $J(V_1 \otimes W_1)/S_1$ are V_2^* and $V_1^* \otimes V_2$, so (2) follows because by Lemma 3.2(b), both $V_2^* \otimes V_0^* \otimes W_0 \otimes W_2$ and $(V_1^* \otimes V_2) \otimes V_0^* \otimes W_0 \otimes W_2$ have simple heads isomorphic to $V_2^* \otimes V_0^*$ and $V_1^* \otimes V_2 \otimes V_0^*$ respectively.

To prove (1) we consider the filtration

$$0 \subset S_0^* \subset J(V_0^* \otimes W_0) \subset V_0^* \otimes W_0$$

similar to the one above. Now

$$\mathrm{hd}(V_0^* \otimes W_0 \otimes V_1 \otimes W_2) = \mathrm{hd}(V_0^* \otimes W_0) \otimes V_1 \otimes W_2,$$

so $J(V_1 \otimes V_0^* \otimes W_0 \otimes W_2) = J(V_0^* \otimes W_0) \otimes V_1 \otimes W_2$. Therefore, by considering the subquotients in the above filtration, we see that (1) follows from the three statements

$$\mathrm{Hom}_{FG}(V_1 \otimes (V_1 \otimes W_2),\ V_1^* \otimes W_2) \cong F,$$

$$\mathrm{Hom}_{FG}((V_0 \otimes V_1^*) \otimes (V_1 \otimes W_2),\ V_1^* \otimes W_2) = 0$$

and

$$\mathrm{Hom}_{FG}((V_0 \otimes V_0) \otimes (V_1 \otimes W_2),\ V_1^* \otimes W_2) = 0,$$

which are all easily verified.

LEMMA 5.15. $\mathrm{Ext}^1_{FG}(V_0 \otimes W_{\mathcal{N}\setminus\{0\}},\ V_0^* \otimes V_1 \otimes W_{\mathcal{N}\setminus\{0,1\}}) = 0$.

Proof. By Lemma 3.2(b),

$$P(V_0 \otimes W_{\mathcal{N}\setminus\{0\}}) \cong V_0 \otimes W_{\mathcal{N}} \cong (V_0 \otimes W_0) \otimes W_{\mathcal{N}\setminus\{0\}}.$$

Since $\mathrm{hd}(V_0 \otimes W_{\mathcal{N}}) = \mathrm{hd}(V_0 \otimes W_0) \otimes W_{\mathcal{N}\setminus\{0,1\}}$, we have

$$JP(V_0 \otimes W_{\mathcal{N}\setminus\{0\}}) \cong J(V_0 \otimes W_0) \otimes W_{\mathcal{N}\setminus\{0\}}.$$

Now by Lemma 4.3(b), $J(V_0 \otimes W_0)$ has a submodule $S \cong V_0^* \otimes V_0^*$, and by Lemma 5.12, we see that $T = J(V_0 \otimes W_0)/S$ is uniserial with series V_1^*, $V_0^* \otimes V_1$. Now $S \otimes W_{\mathcal{N}\setminus\{0\}}$ is a quotient of $V_0 \otimes W_{\mathcal{N}} \cong P(V_0 \otimes W_{\mathcal{N}\setminus\{0\}})$, by Lemma 4.3(b) and Lemma 3.2(b), so it has no quotient isomorphic to $V_0^* \otimes V_1 \otimes W_{\mathcal{N}\setminus\{0,1\}}$. It remains to show that $\mathrm{Hom}_{FG}(T \otimes W_{\mathcal{N}\setminus\{0\}},\ V_0^* \otimes V_1 \otimes W_{\mathcal{N}\setminus\{0,1\}}) = 0$. We claim that T is a homomorphic image of $V_0^* \otimes V_0^* \otimes V_0^* \otimes V_1^*$. From the structure of L_0^* given in Lemma 4.3(c) and from Lemma 4.3(a), $(V_0^* \otimes V_0^* \otimes V_0^*) \otimes V_1^*$ certainly has a quotient with composition factors (in descending order) V_1^* and $V_0^* \otimes V_1$, so the claim will follow from Lemma 5.13 if we show that this quotient is not semisimple. This in turn is immediate from the calculation

$$\begin{aligned}&\mathrm{Hom}_{FG}(V_0^* \otimes V_0^* \otimes V_0^* \otimes V_1^*,\ V_0^* \otimes V_1)\\ &\qquad \cong \mathrm{Hom}_{FG}(V_0^* \otimes V_1^*,\ V_0 \otimes V_1) \oplus \mathrm{Hom}_{FG}(V_0^* \otimes V_0^* \otimes V_1^*,\ W_0 \otimes V_1) = 0,\end{aligned}$$

by Lemma 3.1(e).

Thus, in order to prove that $\mathrm{Hom}_{FG}(T \otimes W_{\mathcal{N}\setminus\{0\}},\ V_0^* \otimes V_1 \otimes W_{\mathcal{N}\setminus\{0,1\}}) = 0$, it will suffice to show that

$$\mathrm{Hom}_{FG}((V_0^* \otimes V_0^* \otimes V_0^* \otimes V_1^*) \otimes W_{\mathcal{N}\setminus\{0\}},\ V_0^* \otimes V_1 \otimes W_{\mathcal{N}\setminus\{0,1\}}) = 0.$$

We have

$$\begin{aligned}&\mathrm{Hom}_{FG}(V_0^* \otimes V_0^* \otimes V_0^* \otimes V_1^* \otimes W_{\mathcal{N}\setminus\{0\}},\ V_0^* \otimes V_1 \otimes W_{\mathcal{N}\setminus\{0,1\}})\\ &\qquad \cong \mathrm{Hom}_{FG}(V_0^* \otimes V_0^* \otimes V_1^* \otimes W_{\mathcal{N}\setminus\{0\}},\ (V_1 \otimes W_{\mathcal{N}\setminus\{0,1\}}) \oplus (V_1 \otimes W_{\mathcal{N}\setminus\{1\}})).\end{aligned}$$

Now $V_0^* \otimes V_0^*$ is a quotient of $V_0 \otimes W_0$, by Lemma 4.3(a), so

$$(V_0^* \otimes V_0^*) \otimes V_1^* \otimes W_{\mathcal{N}\setminus\{0\}}$$

is a quotient of $V_0 \otimes V_1^* \otimes W_{\mathcal{N}} \cong P(V_0 \otimes V_1^* \otimes W_{\mathcal{N}\setminus\{0,1\}})$, by Lemma 3.2(b). Thus the above space of homomorphisms is zero and the lemma is proved.

LEMMA 5.16. *Let* $0, 1 \notin K \subseteq N$. *Then*

(a) $\mathrm{Ext}^1_{FG}(V_{\{0,1\}} \otimes W_K,\ V_{\{0,1\}}^* \otimes W_K) = 0$,

(b) $\mathrm{Ext}^1_{FG}(V_0 \otimes W_K,\ V_0^* \otimes V_1 \otimes W_K) = 0$,

(c) $\mathrm{Ext}^1_{FG}(V_0 \otimes W_{K\cup\{1\}},\ V_0^* \otimes V_1 \otimes W_K) = 0$.

Proof. We shall first reduce (a) and (b) to (c). By Lemma 4.3(b),

$$(V^*_{\{0,1\}} \otimes W_K) \otimes V_1^*$$

has $V_0^* \otimes V_1 \otimes W_K$ in its head. Also

$$\mathrm{Hom}_{FG}(X(V_{\{0,1\}} \otimes W_K, V^*_{\{0,1\}} \otimes W_K), V_1 \otimes (V_0^* \otimes V_1 \otimes W_K)) = 0,$$

because by Lemma 3.1(a) and Lemma 3.3, $(V_1 \otimes V_1) \otimes V_0^* \otimes W_K$ has no composition factor $V_{\{0,1\}} \otimes W_K$. Therefore, by Lemma 4.1, we have

$$d(V_{\{0,1\}} \otimes W_K, V^*_{\{0,1\}} \otimes W_K) \leqslant d(V_0 \otimes W_K, V_0^* \otimes V_1 \otimes W_K) + d(V_0 \otimes W_{K\cup\{1\}}, V_0^* \otimes V_1 \otimes W_K)$$

and (a) is reduced to (b) and (c).

Next, $W_1 \otimes (V_0^* \otimes V_1 \otimes W_K)$ has $V_0^* \otimes V_1 \otimes W_K$ in its head, by Lemma 4.3(b). Also it has a filtration (induced by a composition series of $V_1 \otimes W_1$) with factors isomorphic to $V_0^* \otimes V_1 \otimes W_K$, $V_2^* \otimes (V_0^* \otimes W_K)$ and $V_2 \otimes (V^*_{\{0,1\}} \otimes W_K)$. If $2 \notin K$, then these three modules are simple, because $m > 2$, and not isomorphic to $V_0 \otimes W_K$. If $2 \in K$, then the first of the three is unchanged and since $m > 2$, Lemma 4.3(b) and Lemma 3.2(b) show that the second and third have socles isomorphic to $V^*_{\{0,2\}} \otimes W_{K\setminus\{2\}}$, and $V_2 \otimes V^*_{\{0,1\}} \otimes W_{K\setminus\{2\}}$ respectively. Therefore,

$$\mathrm{Hom}_{FG}(X(V_0 \otimes W_K, V_0^* \otimes V_1 \otimes W_K), W_1 \otimes (V_0^* \otimes V_1 \otimes W_K)) = 0,$$

so by Lemma 4.1 we have

$$d(V_0 \otimes W_K, V_0^* \otimes V_1 \otimes W_K) \leqslant d(V_0 \otimes W_{K\cup\{1\}}, V_0^* \otimes V_1 \otimes W_K).$$

Thus, we are reduced to proving (c).

We shall show that for $r \in N\setminus(K \cup \{0, 1\})$, we have

$$d(V_0 \otimes W_{K\cup\{1\}}, V_0^* \otimes V_1 \otimes W_K) \leqslant d(V_0 \otimes W_{K\cup\{1,r\}}, V_0^* \otimes V_1 \otimes W_{K\cup\{r\}}).$$

Then (c) will follow from Lemma 5.15. By Lemma 4.1, this inequality will be proved if we prove that

$$\mathrm{Hom}_{FG}(X(V_0 \otimes W_{K\cup\{1\}}, V_0^* \otimes V_1 \otimes W_K), W_r \otimes (V_0^* \otimes V_1 \otimes W_{K\cup\{r\}})) = 0.$$

The module $W_r \otimes (V_0^* \otimes V_1 \otimes W_{K\cup\{r\}})$ has a filtration (induced by a composition series of $W_r \otimes W_r$) with factors of the form $V_0^* \otimes V_1 \otimes W_{K\cup\{r\}}$, $V_0^* \otimes V_1 \otimes W_K$,

(α) $V_{r+1} \otimes (V_0^* \otimes V_{\{1,r\}} \otimes W_K)$,

(β) $V^*_{r+1} \otimes (V^*_{\{0,r\}} \otimes V_1 \otimes W_K)$, and

(γ) $W_{r+1} \otimes (V_0^* \otimes V_1 \otimes W_K)$,

so it suffices to show that there are no (non-zero) homomorphisms of

$$X(V_0 \otimes W_{K\cup\{1\}}, V_0^* \otimes V_1 \otimes W_K)$$

into any of these modules. Clearly, we need only concern ourselves with (α), (β) and (γ).

(α) If $r+1 \notin K \cup \{0\}$ and $r \neq m-1$, the result is clear. If $r+1 \in K$, then by Lemma 3.2(b) and Lemma 4.3(b), the module has simple socle

$$V_{\{1,r+1\}} \otimes V_0^* \otimes W_{K\setminus\{r+1\}},$$

so there are no (non-zero) homomorphisms from $X(V_0 \otimes W_{K\cup\{1\}}, V_0^* \otimes V_1 \otimes W_K)$. If $r = m-1 \notin K$, we must consider the cases $G = \mathrm{SL}_3(2^m)$ and $G = \mathrm{SU}_3(2^m)$

separately. For $SL_3(2^m)$, the module is $(V_{\{1,m-1\}} \otimes W_K) \oplus (V_{\{1,m-1\}} \otimes W_{K\cup\{0\}})$, so there are no (non-zero) homomorphisms. For $SU_3(2^m)$, the module is isomorphic to a submodule of

$$(V_0^* \otimes W_0) \otimes V_{\{1,m-1\}} \otimes W_K = V_{\{1,m-1\}} \otimes V_0^* \otimes W_{K\cup\{0\}},$$

so by Lemma 3.2(b) and Lemma 4.3(b) it has a simple socle isomorphic to $V_{\{1,m-1\}} \otimes V_0^* \otimes W_K$, so again there are no maps.

(β) If $r+1 \notin K \cup \{0\}$ and $r \neq m-1$, the result is clear. If $r+1 \in K$, then by Lemma 3.2(b) and Lemma 4.3(b), the module has simple socle

$$V_{\{1,r,r+1\}} \otimes V_0^* \otimes W_{K\setminus\{r+1\}},$$

so there are no (non-zero) homomorphisms from

$$X(V_0 \otimes W_{K\cup\{1\}}, V_0^* \otimes V_1 \otimes W_K).$$

If $r = m-1 \notin K$, we consider the cases where $G = SL_3(2^m)$ and $G = SU_3(2^m)$ separately. For $SL_3(2^m)$, the module is isomorphic to a submodule of

$$(V_0 \otimes W_0) \otimes V_1 \otimes V_r^* \otimes W_K = V_{\{0,1\}} \otimes V_r^* \otimes W_{K\cup\{0\}},$$

so by Lemma 3.2(b) and Lemma 4.3(b), it has a simple socle isomorphic to $V_{\{0,1\}} \otimes V_r^* \otimes W_K$, so there are no maps. If $G = SU_3(2^m)$, the module is

$$V_{\{1,m-1\}} \otimes W_K \oplus V_{\{1,m-1\}} \otimes W_{K\cup\{0\}},$$

and so there are no maps.

(γ) If $r+1 \notin K \cup \{0\}$ and $r \neq m-1$, the result is clear. If $r+1 \in K$, then the module has composition factors $V_0^* \otimes V_1 \otimes W_K$, $V_0^* \otimes V_1 \otimes W_{K\cup\{r+1\}}$, those of $V_{r+2} \otimes (V_0^* \otimes V_{\{1,r+1\}} \otimes W_{K\setminus\{r+1\}})$, and those of

$$V_{r+2}^* \otimes (V_{\{0,r+1\}}^* \otimes V_1 \otimes W_{K\setminus\{r+1\}}).$$

By Lemma 3.3, none of the composition factors is $V_0 \otimes W_{K\cup\{1\}}$, so there are no (non-zero) homomorphisms from $X(V_0 \otimes W_{K\cup\{1\}}, V_0^* \otimes V_1 \otimes W_K)$. Finally, suppose $r = m-1 \notin K$. Then by Lemma 3.2(b) and Lemma 4.3(b),

$$W_0 \otimes (V_0^* \otimes V_1 \otimes W_K)$$

has socle $V_0^* \otimes V_1 \otimes W_K$, so if there were a non-zero homomorphism from $X(V_0 \otimes W_{K\cup\{1\}}, V_0^* \otimes V_1 \otimes W_K)$, there would be a factor $V_0 \otimes W_{K\cup\{1\}}$ in the second socle layer of $W_0 \otimes (V_0^* \otimes V_1 \otimes W_K)$. If this were so, then setting $J = N\setminus(K \cup \{0, 1\})$, we would have by Lemma 3.2(b),

$$\begin{aligned} \operatorname{soc}(W_0 \otimes (V_0^* \otimes V_1 \otimes W_K)) \otimes W_J &\cong V_0^* \otimes V_1 \otimes W_{N\setminus\{0,1\}} \\ &\cong \operatorname{soc}(V_0^* \otimes V_1 \otimes W_{N\setminus\{0\}}) \\ &= \operatorname{soc}((W_0 \otimes (V_0^* \otimes V_1 \otimes W_K)) \otimes W_J). \end{aligned}$$

Then we could deduce that $V_0^* \otimes V_1 \otimes W_{N\setminus\{0\}}$ has a factor $(V_0 \otimes W_{K\cup\{1\}}) \otimes W_J \cong V_0 \otimes W_{N\setminus\{0\}}$ in its second socle layer, whence

$$\operatorname{Ext}^1_{FG}(V_0 \otimes W_{N\setminus\{0\}}, V_0^* \otimes V_1 \otimes W_{N\setminus\{0,1\}}) \neq 0,$$

contradicting Lemma 5.15. This completes the proof of Lemma 5.16.

We now come to the last step in the proof of the theorem. In view of Lemma 2.1, it suffices to prove the theorem for pairs of triples satisfying Condition (1) of that lemma. It is easy to see that for such pairs of triples, Conditions (a), (b) and

(c) of the theorem are equivalent to Conditions (a), (b) and (c) of the following proposition. Since we have already calculated the extension groups for pairs of triples satisfying the latter conditions in Lemmas 5.4 and 5.14, the proof of the theorem will be completed by our next and final result.

PROPOSITION 5.17. *Let (I, J, K) and (A, B, C) be triples such that $A \subseteq J \cup K$, $B \subseteq I \cup K$ and $C \subseteq K$. Then*

$$\operatorname{Ext}^1_{FG}(V_I \otimes V_J^* \otimes W_K, V_A \otimes V_B^* \otimes W_C) = 0$$

unless a variant of one of the following holds:

(a) $K = C$, $I = \{0, 1\}$, $J = A = B = \varnothing$;

(b) $K = C \cup \{1\}$, $I = \{0\}$, $J = A = \varnothing$, $B = \{1\}$;

(c) $K = C$, $I = \{0\}$, $J = A = \{1\}$, $B = \varnothing$.

Proof. By Corollary 5.7, we may assume that $(I, J, K) \neq (A, B, C)$. Let $X = X(V_I \otimes V_J^* \otimes W_K, V_A \otimes V_B^* \otimes W_C)$. By Lemma 4.3,

$$\operatorname{hd}((V_I^* \otimes V_J) \otimes (V_A \otimes V_B^* \otimes W_C))$$

contains a factor

$$V_{(K\cap A)\cup(I\cap B)\cup(J\setminus A)} \otimes V^*_{(K\cap B)\cup(J\cap A)\cup(I\setminus B)} \otimes W_C.$$

Also, we have

$$(V_I \otimes V_J^* \otimes W_K) \otimes (V_I^* \otimes V_J) \cong \bigoplus_{K\subseteq S\subseteq I\cup J\cup K} W_S.$$

Our first step is to prove the inequality

$$\begin{aligned}(5.18)\quad d &= d(V_I \otimes V_J^* \otimes W_K, V_A \otimes V_B^* \otimes W_C)\\ &\leqslant \sum_{K\subseteq S\subseteq I\cup J\cup K} d(W_S, V_{(K\cap A)\cup(I\cap B)\cup(J\setminus A)} \otimes V^*_{(K\cap B)\cup(J\cap A)\cup(I\setminus B)} \otimes W_C).\end{aligned}$$

Suppose first that $I = N$; then by Lemma 4.1, (5.18) follows from the fact that

$$\operatorname{Hom}_{FG}(X(V_N, V_N^*), \operatorname{soc}^2(V_N \otimes V_N)) = 0,$$

which is true by Lemma 4.7. We may apply Remark 4.2 if $J = N$, so in proving (5.18) we may assume from now on that $I, J \neq N$.

We shall prove (5.18) by means of Lemma 4.1 by showing that

$$\begin{aligned}(5.19)\qquad \operatorname{Hom}_{FG}(X, (V_I \otimes V_J^*) \otimes (&V_{(K\cap A)\cup(I\cap B)\cup(J\setminus A)}\\ &\otimes V^*_{(K\cap B)\cup(J\cap A)\cup(I\setminus B)} \otimes W_C)) = 0.\end{aligned}$$

The left-hand side of (5.19) may be rewritten as

$$\begin{aligned}(5.20)\quad &\operatorname{Hom}_{FG}\Big(X, \bigoplus_{T\subseteq(I\setminus A)\cup(J\setminus B)} (V_{I\cap B} \otimes V^*_{J\cap A}) \otimes (V_{I\cap B} \otimes V^*_{J\cap A})\\ &\qquad\qquad \otimes V_{K\cap A} \otimes V^*_{K\cap B} \otimes W_C \otimes W_T\Big)\\ &\cong \bigoplus_{T\subseteq(I\setminus A)\cup(J\setminus B)} \operatorname{Hom}_{FG}(X \otimes W_T, (V_{I\cap B} \otimes V^*_{J\cap A})\\ &\qquad\qquad \otimes (V_{I\cap B} \otimes V^*_{J\cap A}) \otimes V_{K\cap A} \otimes V^*_{K\cap B} \otimes W_C).\end{aligned}$$

By Lemmas 3.1(a) and 3.3(c), we have

$$\operatorname{mass}((V_{I\cap B} \otimes V^*_{J\cap A}) \otimes (V_{I\cap B} \otimes V^*_{J\cap A})) \leqslant |I \cap B| + |J \cap A|,$$

so by Lemma 3.3(c), we have

$$\text{mass}((V_{I\cap B}\otimes V^*_{J\cap A})\otimes(V_{I\cap B}\otimes V^*_{J\cap A})\otimes V_{K\cap A}\otimes V^*_{K\cap B}\otimes W_C)\leqslant |A|+|B|+2|C|.$$

For $T=\varnothing$, since $\text{mass}(V_I\otimes V_J^*\otimes W_K)=|I|+|J|+2|K|$, we see that

$$\text{Hom}_{FG}(X,(V_{I\cap B}\otimes V^*_{J\cap A})\otimes(V_{I\cap B}\otimes V^*_{J\cap A})\otimes V_{K\cap A}\otimes V^*_{K\cap B}\otimes W_C)=0$$

unless $A=J$, $B=I$ and $K=C$. Suppose this to be the case. Then

$$(V_{I\cap B}\otimes V^*_{J\cap A})\otimes(V_{I\cap B}\otimes V^*_{J\cap A})\otimes V_{K\cap A}\otimes V^*_{K\cap B}\otimes W_C$$

becomes $(V_I\otimes V_J^*)\otimes(V_I\otimes V_J^*)\otimes W_K$. By Lemma 3.1(a) and Lemma 3.3(c),

$$\text{mass}((V_I\otimes V_J^*)\otimes(V_I\otimes V_J^*))\leqslant |I|+|J|,$$

and since $I,J\neq N$, Lemma 3.5 tells us that $(V_I\otimes V_J^*)\otimes(V_I\otimes V_J^*)$ has no composition factor $V_I\otimes V_J^*$. Then it follows from Lemma 3.3(c) that

$$(V_I\otimes V_J^*)\otimes(V_I\otimes V_J^*)\otimes W_K$$

has no composition factor $V_I\otimes V_J^*\otimes W_K$. Thus, there are no non-zero homomorphisms from X in this case either. We have shown that the term in (5.20) for $T=\varnothing$ is zero.

Suppose now that $T\neq\varnothing$. Then $X\otimes W_T$ has a submodule Y isomorphic to the direct sum of d copies of $V_A\otimes V_B^*\otimes W_{C\cup T}$, with quotient isomorphic to $(V_I\otimes V_J^*\otimes W_K)\otimes W_T=(V_I\otimes V_J^*)\otimes W_{K\cup T}$. Since $V_A\otimes V_B^*\otimes W_{C\cup T}$ is simple of mass $|A|+|B|+2|C|+2|T|>|A|+|B|+2|C|$, we obviously have

$$\text{Hom}_{FG}(Y,(V_{I\cap B}\otimes V^*_{J\cap A})\otimes(V_{I\cap B}\otimes V^*_{J\cap A})\otimes V_{K\cap A}\otimes V^*_{K\cap B}\otimes W_C)=0.$$

To show that

$$\text{Hom}_{FG}((X\otimes W_T)/Y,(V_{I\cap B}\otimes V^*_{J\cap A})\otimes(V_{I\cap B}\otimes V^*_{J\cap A})\otimes V_{K\cap A}\otimes V^*_{K\cap B}\otimes W_C)=0,$$

we observe that by Lemma 3.2(b) and Lemma 4.3(b), the head of

$$(V_I\otimes V_J^*)\otimes W_{K\cup T}$$

is $V_I\otimes V_J^*\otimes W_K$, which, since $T\neq\varnothing$, has mass greater than $|A|+|B|+2|C|$.

This establishes (5.19), and hence also (5.18), by Lemma 4.1.

We consider the terms on the right of (5.18). By Proposition 5.9, we have

$$d(W_S,V_{(K\cap A)\cup(I\cap B)\cup(J\backslash A)}\otimes V^*_{(K\cap B)\cup(J\cap A)\cup(I\backslash B)}\otimes W_C)=0$$

unless, after taking suitable conjugates, we have $S=C$ or $C\cup\{1\}$ and

$$\{(K\cap A)\cup(I\cap B)\cup(J\backslash A),(K\cap B)\cup(J\cap A)\cup(I\backslash B)\}=\{\{0,1\},\varnothing\},\quad\text{for some } 0,1\notin C.$$

Suppose these conditions hold. Since $C\subseteq K\subseteq S$, we must have $K=C$ or $K=C\cup\{1\}$. Suppose $K=C\cup\{1\}$. Then the conditions

$$(K\cap A)\cup(I\cap B)\cup(J\backslash A)=\{0,1\},\quad (K\cap B)\cup(J\cap A)\cup(I\backslash B)=\varnothing,$$

together with the assumptions $A\subseteq J\cup K$ and $B\subseteq I\cup K$, imply that $I=B$, $A=\{1\}$ and $I\cup J=\{0\}$. The possibilities for $((I,J,K),(A,B,C))$ are then

(i) $((\varnothing,\{0\},C\cup\{1\}),(\{1\},\varnothing,C))$ and

(ii) $((\{0\},\varnothing,C\cup\{1\}),(\{1\},\{0\},C))$.

The conditions

$$(K\cap A)\cup(I\cap B)\cup(J\backslash A)=\varnothing,\quad (K\cap B)\cup(J\cap A)\cup(I\backslash B)=\{0,1\}$$

are the $(IJ)(AB)$-variants of the conditions just considered, so they yield the $(IJ)(AB)$-variants of (i) and (ii).

Now (i) is a variant of (b) and Lemma 5.16(c) shows that the group of extensions for the pair of triples in (ii) is zero.

Suppose that $K=C$. Then the conditions

$$(K\cap A)\cup(I\cap B)\cup(J\backslash A)=\{0,1\}\quad(\text{respectively }\varnothing),$$
$$(K\cap B)\cup(J\cap A)\cup(I\backslash B)=\varnothing\quad(\text{respectively }\{0,1\})$$

yield $I=B$, $A=\varnothing$ and $I\cup J=\{0,1\}$ (and their $(IJ)(AB)$-variants). The possible pairs of triples $((I,J,K),(A,B,C))$ are then

(iii) $((\varnothing,\{0,1\},C),(\varnothing,\varnothing,C))$,

(iv) $((\{1\},\{0\},C),(\varnothing,\{1\},C))$,

(v) $((\{0\},\{1\},C),(\varnothing,\{0\},C))$ and

(vi) $((\{0,1\},\varnothing,C),(\varnothing,\{0,1\},C))$,

and the $(IJ)(AB)$-variants of these.

We see that (iii) is a variant of (a) and that (iv) is a variant of (c). The groups of extensions for (variants of) the pairs of triples in (v) and (vi) have been shown to be zero in Lemma 5.16(b) and Lemma 5.16(a) respectively.

The proof is finished.

Appendix: $m\leqslant 2$

Case 1: $m=1$. The module W is projective, so its multiplicity as a direct summand of a module is simply its multiplicity as a composition factor. Also, Lemma 3.1(a) is still valid. Some straightforeward calculations now yield

(i) $V\otimes W\cong W\oplus P(V)$ and

(ii) $W\otimes W\cong W\oplus W\oplus W\oplus P(V)\oplus P(V^*)\oplus P(F)$.

Since $\mathrm{Hom}_{FG}(V^*\otimes V^*,V^*)\cong\mathrm{Hom}_{FG}(V^*,F\oplus W)=0$, $V^*\otimes V^*$ is uniserial and, just as before, it is isomorphic to both a submodule and a quotient of $V\otimes W$. Also, since the composition factors of $P(V)$ are V, V^* and F, we must have $\mathrm{Ext}^1_{FG}(V,F)\neq 0$. It follows easily from these facts that $P(V)$ has the structure in the following picture:

V

V^* \

V F

V^* /

V

From (i) and (ii) we see that $\dim_F P(F)=8$, and then since $P(F)\cong P(F)^\tau$, we obtain the following structure for $P(F)$:

$$\begin{array}{ccc} & F & \\ / & & \backslash \\ V & & V^* \\ \backslash & & / \\ & F & \end{array}$$

These modules for $SL_3(2)$ are described in [**3**].

Case 2: $m = 2$. In the statement of the following proposition, the results for the unitary group are placed in parentheses.

PROPOSITION. *Let* $G = SL_3(4)$ (*respectively* $SU_3(4)$). *Then*

(a) $\mathrm{Ext}^1_{FG}(K, V_{\{0,1\}}) \cong \mathrm{Ext}^1_{FG}(V_0^*, V_1) \cong F^2\ (F)$,

(b) $\mathrm{Ext}^1_{FG}(W_1, V_{\{0,1\}}) \cong \mathrm{Ext}^1_{FG}(V_0^* \otimes W_1, V_1) \cong F\ (F)$,

(c) $\mathrm{Ext}^1_{FG}(V_1^*, V_0^* \otimes V_1) \cong F\ (F)$.

Except for variants of these, all other extensions between simple modules are trivial.

The remainder of this appendix is a proof of the proposition. We give the argument in detail for $G = SL_3(4)$ only. The calculation for $SU_3(4)$ is similar. Thus from now on, $G = SL_3(4)$. All results up to Lemma 5.1 remain valid for $m = 2$. Using (1.1), duality and τ-conjugation, Galois conjugation, the tensor identity for Ext and the projectivity of $W_N = W_{\{0,1\}}$, we find that the problem is reduced to computing the following groups of extensions:

(1) $\mathrm{Ext}^1_{FG}(F, F)$, (2) $\mathrm{Ext}^1_{FG}(F, V_0)$,
(3) $\mathrm{Ext}^1_{FG}(F, W_0)$, (4) $\mathrm{Ext}^1_{FG}(F, V_{\{0,1\}})$,
(5) $\mathrm{Ext}^1_{FG}(F, V_0 \otimes V_1^*)$, (6) $\mathrm{Ext}^1_{FG}(F, W_0 \otimes V_1)$,
(7) $\mathrm{Ext}^1_{FG}(V_0, V_0^* \otimes V_1)$, (8) $\mathrm{Ext}^1_{FG}(V_0, V_{\{0,1\}}^*)$,
(9) $\mathrm{Ext}^1_{FG}(V_0, W_0)$, (10) $\mathrm{Ext}^1_{FG}(V_0, V_1 \otimes W_0)$,
(11) $\mathrm{Ext}^1_{FG}(F, V_0 \otimes V_1^*)$, (12) $\mathrm{Ext}^1_{FG}(V_{\{0,1\}}, V_{\{0,1\}}^*)$,
(13) $\mathrm{Ext}^1_{FG}(V_{\{0,1\}}, V_0^* \otimes W_1)$, (14) $\mathrm{Ext}^1_{FG}(V_0^* \otimes V_1, V_0 \otimes V_1^*)$,
(15) $\mathrm{Ext}^1_{FG}(V_0^* \otimes V_1, V_0 \otimes W_1)$, (16) $\mathrm{Ext}^1_{FG}(W_0, W_0)$,
(17) $\mathrm{Ext}^1_{FG}(W_0, V_1 \otimes W_0)$, (18) $\mathrm{Ext}^1_{FG}(V_0 \otimes W_1, V_0^* \otimes W_1)$.

By Lemma 5.1, we see that the groups in (3), (6), (9), (10), (16) and (17) are trivial and that the group in (11) is one-dimensional. From the composition factors of $V_i \otimes V_i$ and using the groups just computed and the long exact sequence for Ext, we also see the triviality of the groups in (12) and (18). For any two modules L and M, we write $d(L, M)$ for $\dim_F \mathrm{Ext}^1_{FG}(L, M)$. Then using Lemma 4.1, one can prove without difficulty that

$$d(F, F) \leq d(W_i, W_i),$$

$$d(F, V_0) \leq d(W_1, V_0 \otimes W_1),$$

$$d(F, V_0^* \otimes V_1) \leq d(W_0, V_0^* \otimes V_1),$$

which gives vanishing in (1), (2) and (5). Using the long exact sequence again, we also obtain

$$d(V_1, V_0^* \otimes V_0^*) \leq 2d(V_1, V_0) + d(V_1, V_1^*) \leq 2d(V_1, V_0) + 2d(F, V_1) + d(F, V_0^*);$$

the right-hand terms are all zero by (2) and (5), so we have proved that the group in (8) is zero. Similarly, the inequality

$$d(V_0^* \otimes V_0^*, V_1^* \otimes V_1^*) \leq 2d(V_0^* \otimes V_0^*, V_1) + d(V_0^* \otimes V_0^*, V_0^*)$$

shows by (8), (2) and (6) that the group in (14) is trivial.

Next we consider (4). By Lemma 3.2 we have

$$P(V_{\{0,1\}}) \oplus W_N \cong V_{\{0,1\}} \otimes W_N = (V_0 \otimes W_0) \otimes (V_1 \otimes W_1).$$

Thus, by Lemma 4.3(b) we see that $P(V_{\{0,1\}})$ has a quotient which has a filtration with factors (in descending order)

$$V_{\{0,1\}}, \ (F \oplus W_0) \oplus (F \oplus W_1).$$

Therefore $d(V_{\{0,1\}}, F) \geq 2$. Now $(V_0 \otimes W_0)$ has a filtration with factors (in descending order)

$$V_0, \ V_1^*, \ V_0^* \otimes V_1, \ V_0^* \otimes V_0^*.$$

It is easy to check that $(V_0^* \otimes V_0^*) \otimes (V_1 \otimes W_1)$ and $(V_0^* \otimes V_1) \otimes (V_1 \otimes W_1)$ do not have F in their heads, and that $\mathrm{Hom}_{FG}(V_1^* \otimes (V_1 \otimes W_1), F) \cong F$. Also by considering the corresponding filtration of $V_1 \otimes W_1$, it can be seen that $\mathrm{Hom}_{FG}(J(V_0 \otimes (V_1 \otimes W_1)), F) \cong F$. This proves that $d(V_{\{0,1\}}, F) \leq 2$, so the group in (4) is 2-dimensional.

It remains to determine the groups in (7), (11), (12) and (15). For these, we need the Loewy structure of $V_0 \otimes W_0$. The argument of Lemma 5.12 no longer works because Lemma 5.11 is false for $m=2$. However, the highest weights of the composition factors are 2^2-restricted, so [**2**, Proposition 2.7] tells us that the Loewy structure will be the same as that for $\mathrm{SL}_3(16)$, namely, as stated in Lemma 5.12. We know by Lemma 4.3(b) that $V_0 \otimes W_0$ has a submodule $S \cong V_0^* \otimes V_0^*$. Let $E = J(V_0 \otimes W_0)/S$. Thus E is uniserial of length 2 with head V_1^* and socle $V_0^* \otimes V_1$. We shall need the following calculations.

LEMMA. *We have*

(a) $\mathrm{Hom}_{FG}(E \otimes W_1, V_0^* \otimes V_1) = 0$;

(b) $\mathrm{Hom}_{FG}((V_0^* \otimes V_0^*) \otimes W_1, V_0^* \otimes V_1) = 0$;

(c) $\mathrm{Hom}_{FG}(J(V_1 \otimes W_1 \otimes V_0^*), V_0) \cong F$;

(d) $\mathrm{Hom}_{FG}(J(V_1 \otimes W_1 \otimes V_0), V_0^* \otimes V_1^*) = 0$;

(e) $\mathrm{Hom}_{FG}((V_0^* \otimes V_0^*) \otimes (V_1 \otimes W_1), V_{\{0,1\}}^*) = 0$.

Proof. (a) Applying duality, τ- and Galois conjugation, and the filtration $0 \subset S \subset J(V_0 \otimes W_0) \subset V_0 \otimes W_0$, we have

$$\begin{aligned}\dim_F \mathrm{Hom}_{FG}(E \otimes W_1, V_0^* \otimes V_1) \\ &= \dim_F \mathrm{Hom}_{FG}((V_0 \otimes W_0) \otimes V_1^*, E_1^{*\tau}) \\ &\leq \dim_F \mathrm{Hom}_{FG}(V_0 \otimes V_1^*, E_1^{*\tau}) + \dim_F \mathrm{Hom}_{FG}(E \otimes V_1^*, E_1^{*\tau}) \\ &\quad + \dim_F \mathrm{Hom}_{FG}((V_0^* \otimes V_0^*) \otimes V_1^*, E_1^{*\tau}).\end{aligned}$$

Now $E_1^{*\tau}$ has head $V_0 \otimes V_1^*$ and radical V_0^*, so the first of the three terms on the right of the inequality is clearly zero, and the third term is zero by Lemma 4.6. The fact that the second term is zero follows because

$$\mathrm{Hom}_{FG}(E \otimes V_1^*, V_0^*) \cong \mathrm{Hom}_{FG}(E, V_1 \otimes V_0^*) = 0$$

and

$$\mathrm{Hom}_{FG}(E \otimes V_1^*, V_0 \otimes V_1^*) \cong \mathrm{Hom}_{FG}(E, V_0 \oplus (V_0 \otimes W_1)) = 0.$$

Part (b) is straightforward. For (c) and (d) we note that $J(V_1 \otimes W_1 \otimes V_0^*) \cong J(V_1 \otimes W_1) \otimes V_0^*$ and $J(V_1 \otimes W_1 \otimes V_0) \cong J(V_1 \otimes W_1) \otimes V_0$. The results then follow from the structure of $V_1 \otimes W_1$, using the structure of $V_0 \otimes V_0$ in (c) and that of $V_0^* \otimes V_0^* \otimes V_1^*$ in (d) (Lemma 4.6). To prove (e) we have

$$\begin{aligned}\mathrm{Hom}_{FG}(V_0^* \otimes V_0^* \otimes V_1 \otimes W_1, V_0^* \otimes V_1^*)&\\ \cong \mathrm{Hom}_{FG}(V_0^* \otimes V_1 \otimes W_1, V_1^* \oplus (W_0 \otimes V_1^*)) &= 0\end{aligned}$$

since by Lemma 3.2(b), $V_0^* \otimes V_1 \otimes W_1$ has a simple head isomorphic to $V_0^* \otimes V_1$. All parts of the lemma are proved.

We now consider (15). By Lemma 3.2, we have

$$JP(V_0 \otimes W_1) \cong J(V_0 \otimes W_N) = J(V_0 \otimes W_0) \otimes W_1.$$

Since $J(V_0 \otimes W_0)$ has a submodule isomorphic to $V_0^* \otimes V_0^*$ with quotient E, the triviality of the group in (15) follows from (a) and (b) of the lemma.

Next we turn to (7). It is clear from Lemma 4.6 that the group of extensions is not trivial, so it remains to bound the dimension. By Lemma 3.2(b) we have

$$P(V_0^* \otimes V_1) \cong (V_0^* \otimes W_0) \otimes (V_1 \otimes W_1).$$

Using the filtration of $V_0^* \otimes W_0$ with factors (in descending order) V_0^*, E^τ, $V_0 \otimes V_0$, we see that the result follows from Part (c) and the τ-twisted versions of (a) and (b) of the lemma.

Finally, we must prove the triviality of the group in (12). By Lemma 3.2(a), we have

$$P(V_{\{0,1\}}) \oplus W_N \cong (V_0 \otimes W) \otimes (V_1 \otimes W_1).$$

By considering the filtration $0 \subset S \subset J(V_0 \otimes W_0) \subset V_0 \otimes W_0$, we see that it will suffice to show that

(1) $\mathrm{Hom}_{FG}(J(V_0 \otimes (V_1 \otimes W_1)), V_{\{0,1\}}^*) = 0$,

(2) $\mathrm{Hom}_{FG}(E \otimes (V_1 \otimes W_1), V_{\{0,1\}}^*) = 0$ and

(3) $\mathrm{Hom}_{FG}((V_0^* \otimes V_0^*) \otimes (V_1 \otimes W_1), V_{\{0,1\}}^*) = 0$.

The first and third of these are Parts (d) and (e) of the lemma. To prove (2), we note, as in the proof of Lemma 5.15, that since $\mathrm{Ext}_{FG}^1(V_1^*, V_0^* \otimes V_1) \cong F$, E is a homomorphic image of $V_0^* \otimes V_0^* \otimes V_0^* \otimes V_1^*$. We have

$$\begin{aligned}&\mathrm{Hom}_{FG}((V_0^* \otimes V_0^* \otimes V_0^* \otimes V_1^*) \otimes (V_1 \otimes W_1), V_{\{0,1\}}^*)\\ &\quad\cong \mathrm{Hom}_{FG}(V_1 \otimes V_0^* \otimes W_1, V_0 \otimes W_1) \oplus \mathrm{Hom}_{FG}(V_1 \otimes V_0^* \otimes W_N, V_0 \otimes W_1) = 0,\end{aligned}$$

by Lemma 3.2(b). This proves (2) and completes the proof of the proposition.

The calculation of extensions for $\mathrm{SU}_3(4)$ was first made in a recent paper **[5]** by other methods.

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