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Extensions of Simple Modules for Special Algebraic Groups

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We compute the groups $\mathrm{Ext}^1_G(L, M)$ for any two simple rational modules for the simple algebraic groups G of type C_2 and F_4 in characteristic 2 and type G_2 in characteristic 3 by exploiting the special endomorphisms which exist for these groups. © 1994 Academic Press, Inc.

0. INTRODUCTION

This paper is about three Chevalley groups—the groups of types C_2 and F_4 over an algebraic closure of $\mathbf{F}_2$ and the group of type G_2 over an algebraic closure of $\mathbf{F}_3$. These are the only simple algebraic groups that have an endomorphism whose square is the Frobenius map for some rational structure over the prime subfield. It is well known that the subgroups of fixed points of odd powers of these special isogenies form the three infinite series of (mostly) simple finite groups discovered by Suzuki (type C_2) and Ree (types G_2 and F_4).

It was shown by Steinberg [13, Section 11] that the existence of a special isogeny τ leads to a much stronger version of his Tensor Theorem (cf. 1.4 below), which simplifies considerably the problem of determining the characters of the irreducible representations. There, an important role is played by a certain ideal in the Lie algebra of the group G, the ideal $\mathfrak{g}_\tau = \mathrm{Ker}\, d\tau$, whose representation theory as a restricted Lie algebra is the same as that of the subgroup scheme $G_\tau = \mathrm{Ker}\, \tau$.

For the same reasons, the presence of the normal subgroup scheme G_τ gives rise to simplifications in the problem of computing extensions between simply modules. If τ did not exist, one could try to calculate these using the Lyndon–Hochschild–Serre spectral sequence associated to the first Frobenius kernel G_1 (cf. [5], [1]). This reduces the problem to certain questions about simple modules with restricted highest weights. For our groups, the spectral sequence for G_τ reduces the problem to the same

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questions about a much smaller set of simple modules, those with τ-restricted weights (cf. 1.4 and 1.5 below).

In this way, we obtain simple proofs of the known results for C_2 ([17]; see also [10] for the finite groups) and type G_2 ([11], where it is derived from the result for the finite groups), as well as the calculation of extensions for F_4, which is new.

In the F_4 calculation, an additional benefit of using G_τ becomes important. Let $D \leq G$ be the subgroup generated by the long root subgroups. It is a simply connected group of type D_4. Then τ maps the Frobenius kernel D_1 onto G_τ and the kernel of this map is of multiplicative type. This relationship allows us at a critical stage to apply knowledge about D_4 in order to obtain information about G_τ, hence G.

In Section 1, we establish notation and discuss some general aspects of the methods to be used. The remaining three sections, one for each group, may be read independently of each other.

This work has been strongly influenced by the computations of Lie algebra cohomology in [8]. In the process of computing simple module extensions for G, we determine the G-module structure of the extensions of simple modules for G_τ. From this, one can also calculate the extensions of simple modules for the Lie algebra of G.

1. Preliminaries

1.1. Let G and τ be as in the Introduction and let k be an algebraic closure of $\mathbf{F}_p$, where $p = 2$ for types C_2 and F_4 and $p = 3$ for G_2. For the groups of types F_4 and G_2, the index of connection is 1 and for type C_2 it is 2. In the latter case we take G to be of simply connected type. There is no cost in making this assumption as far as cohomology is concerned, because the center of the simply connected group is a subgroup scheme of multiplicative type and it follows easily that

$$H^*(G, E) \cong H^*(\overline{G}, E), \tag{1}$$

where $\overline{G}$ is the adjoint group and E is any $\overline{G}$-module.

The k-group scheme G is obtained by base change from a split $\mathbb{Z}$-group scheme $G_{\mathbb{Z}}$ of the same type. Let $T_{\mathbb{Z}} \subset G_{\mathbb{Z}}$ be a split maximal torus and $T = (T_{\mathbb{Z}})_k$ be the corresponding maximal torus of G. Let $X(T)$ be the character group of T and $R \subset X(T)$ be the root system. Let $\mathfrak{g}_{\mathbb{Z}} = \mathrm{Lie}(G_{\mathbb{Z}})$. The $\mathrm{Lie}(G) = \mathfrak{g} = \mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} k$, and the assumption of simple connectivity means that $\mathfrak{g}_{\mathbb{Z}}$ is the span of a Chevalley basis $\{X_\alpha, \alpha \in R; H_\alpha = [X_\alpha, X_{-\alpha}], \alpha \in S\}$, where $S = \{\alpha_1, \dots, \alpha_l\}$ is a set of simple roots. We shall use the same notation for the corresponding basis elements of $\mathfrak{g}$. For each $\alpha \in R$ there is a root homomorphism x_α: $\mathbf{G}_a \to G$ with $dx_\alpha(1) = X_\alpha$ and a coroot $\alpha^\vee$: $\mathbf{G}_m \to T$ with $d\alpha^\vee(1) = H_\alpha$. Since G is simply connected, the coroots are embeddings and they span $Y = \mathrm{Hom}(\mathbf{G}_m, T)$. Under the natural pairing $X \times X \to \mathbb{Z}$ the coroot $\alpha^\vee$ becomes identified with the element $2\alpha/\langle \alpha, \alpha \rangle \in \mathbb{Z}R \otimes_{\mathbb{Z}} \mathbb{Q}$. Let $\{\omega_i\}_{i=1}^{l}$ be the set of fundamental dominant weights, so $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{i,j}$. The group G is generated by the elements $x_\alpha(t)$ for $t \in K$ and $\alpha \in R$, and T is generated by the $\alpha^\vee(t)$ for $t \in k^\times$ and $\alpha \in S$. We set $U = \langle x_\alpha(t) | \alpha \in R^-, t \in k \rangle$ and denote by B the Borel subgroup UT.

1.2. *The special isogeny* τ. The set $R^\vee$ forms a root system with simple roots $S^\vee$. In our cases, $R^\vee$ and R are isomorphic, and so we shall fix an isomorphism which sends $S^\vee$ to S and denote by α^* the image of $\alpha^\vee$. Thus, the map of R to itself sending α to α^* preserves angles and interchanges long and short roots.

The map τ is defined [14, p. 146, Theorem 28] by

$$x_\alpha(t) \mapsto \begin{cases} x_{\alpha^*}(t) & \text{if } \alpha \text{ is long} \\ x_{\alpha^*}(t^p) & \text{if } \alpha \text{ is short,} \end{cases} \tag{1}$$

and its differential $d\tau$ is given by

$$X_\alpha \mapsto \begin{cases} X_{\alpha^*} & \text{if } \alpha \text{ is long} \\ 0 & \text{if } \alpha \text{ is short.} \end{cases} \tag{2}$$

Let $G_\tau = \mathrm{Ker}\,\tau$ and $\mathfrak{g}_\tau = \mathrm{Ker}\, d\tau = \mathrm{Lie}(G_\tau)$. Since G_τ is an infinitesimal group scheme of height one, its representation theory is equivalent to that of the restricted Lie algebra $\mathfrak{g}_\tau$ [4, II, Section 7, 4.1-3], [7, I, 8.6], which is the ideal in $\mathfrak{g}$ spanned by the elements X_α and H_α, indexed by short roots.

1.3. *Subgroups generated by long and short root subgroups.* Let R_l be the subsystem of long roots and S_l be the set of simple roots for R_l which are positive in R, and define R_s and S_s similarly. Let $D = \langle x_\alpha(t) | t \in k, \alpha \in R_l \rangle$ and $\overline{D} = \langle x_\alpha(t) | t \in k, \alpha \in R_s \rangle$. Then D is a simply connected group of type $A_1 \times A_1$ when G is of type C_2, type A_2 when G is of type G_2, and type D_4 when G is of type F_4. For types G_2 and F_4, the group $\overline{D}$ is of adjoint type, while for C_2 it is the quotient of $D \cong \mathrm{SL}_2 \times \mathrm{SL}_2$ by the "diagonal" copy of $(\mathbf{G}_m)_1 \cong \mu_{(2)}$ in its scheme-theoretic center $\mu_{(2)} \times \mu_{(2)}$. Let $\mathfrak{d} = \mathrm{Lie}(D)$. Then $d\tau$ maps $\mathfrak{d}$ onto $\mathfrak{g}_\tau$ with kernel $\mathfrak{h}_\tau = \langle H_\alpha | \alpha \in S_s \rangle$. Translating to group schemes, we have a commutative diagram (see [4, II, Section 7, 4.1-3 and the proof of III, Section 6, 8.5

Proposition (b)])

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G_\tau & \longrightarrow & G_1 & \xrightarrow{\tau} & G_\tau & \longrightarrow & 1 \\ & & \uparrow \cup & & \uparrow \cup & & \| & & \\ 1 & \longrightarrow & T_\tau & \longrightarrow & D_1 & \xrightarrow{\tau} & G_\tau & \longrightarrow & 1 \end{array} \tag{1}$$

with exact rows. Since T_τ is of multiplicative type, it follows easily from the Lyndon–Hochschild–Serre spectral sequence for the bottom row of (1) that if E is any $\overline{D}$-module with D acting through τ, then we have an isomorphism of D-modules

$$H^i(G_\tau, E) \cong H^i(D_1, E), \qquad i \geq 0. \tag{2}$$

This will allow us to apply results about D_1-cohomology to G_τ-cohomology. Also, from the spectral sequence of the top row of 1.3(1) it will be possible to derive results about G_1-cohomology from results about G_τ.

1.4. *Simple modules.* We now describe the simple modules. The map τ stabilizes T, and hence acts on $X(T)$, by twisting representations. The simple G-modules are in bijection with the dominant weights $\lambda \in X_+ = X_+(T)$, and are denoted by $L(\lambda)$. Let $X_1 = \{\lambda \in X_+ \mid \langle \lambda, \alpha^\vee \rangle < p, \alpha \in S\}$ be the set of restricted weights and define the set of τ-restricted weights $X_\tau \subset X_1$ as the subset of those elements which are orthogonal to all the long simple roots. Then $\tau(X_\tau) \subset X_1$ and every $\lambda \in X_1$ has a unique expression as $\lambda^0 + \tau\lambda^1$ for $\lambda^0, \lambda^1 \in X_\tau$, so every $\lambda \in X_+$ has a τ-adic expression

$$\lambda = \sum_{i=0}^{r} \tau^k \lambda^i, \qquad \lambda^i \in X_\tau. \tag{1}$$

Then Steinberg's Tensor Product Theorem states that there is an isomorphism of G-modules

$$L(\lambda) \cong L(\lambda^0) \otimes L(\lambda^1)^{(\tau)} \otimes L(\lambda^2)^{(\tau^2)} \otimes \cdots \otimes L(\lambda^r)^{(\tau^r)}, \tag{2}$$

where the superscript τ^i indicates twisting by the endomorphism τ^i.

Let us make the notational convention that when a dominant weight is written τ-adically as in (1), or partially so as $\lambda = \lambda^0 + \tau\overline{\lambda}$, or $\lambda = \lambda^0 + \tau\lambda^1 + \tau^2\lambda''$, it shall be understood that the λ^i lie in X_τ and $\overline{\lambda}, \lambda''$, etc. belong to X_+.

Extensions of Simple G-Modules

1.5. We are interested in computing the groups $\mathrm{Ext}^1_G(L(\lambda), L(\mu))$ for $\lambda, \mu \in X_+$. The Hochschild–Serre spectral sequence for the pair (G, G_τ)

takes the form

$$\operatorname{Ext}^i_G\Big(L(\bar{\lambda}), \operatorname{Ext}^j_{G_\tau}\big(L(\lambda^0), L(\mu^0)\big)^{(\tau^{-1})} \otimes L(\bar{\mu})\Big) \Rightarrow_i \operatorname{Ext}^{i+j}(L(\lambda), L(\mu)). \tag{1}$$

Here, the superscript (τ^{-1}) indicates untwisting; if G_τ acts trivially on a G-module G, given by a representation π, then $V^{(\tau^{-1})}$ is the module defined by the right vertical arrow in the commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\pi} & \mathrm{GL}(V) \\ \downarrow & & \uparrow \\ G/G_\tau & \xrightarrow{\cong} & G \end{array} .$$

The 5-term sequence is

$$\begin{aligned} 0 &\to \operatorname{Ext}^1_G\Big(L(\bar{\lambda}), \operatorname{Hom}_{G_\tau}\big(L(\lambda^0), L(\mu^0)\big)^{(\tau^{-1})} \otimes L(\bar{\mu})\Big) \\ &\to \operatorname{Ext}^1_G(L(\lambda), L(\mu)) \\ &\to \operatorname{Hom}_G\Big(L(\bar{\lambda}), \operatorname{Ext}^1_{G_\tau}\big(L(\lambda^0), L(\mu^0)\big)^{(\tau^{-1})} \otimes L(\bar{\mu})\Big) \\ &\to \operatorname{Ext}^2_G\Big(L(\bar{\lambda}), \operatorname{Hom}_{G_\tau}\big(L(\lambda^0), L(\mu^0)\big)^{(\tau^{-1})} \otimes L(\bar{\mu})\Big) \\ &\to \operatorname{Ext}^2_G(L(\lambda), L(\mu)). \end{aligned} \tag{2}$$

1.6. In applying 1.5(2), we shall be interested in determining the structure of the G-module $\operatorname{Ext}^1_{G_\tau}(L(\lambda^0), L(\mu^0))^{(\tau^{-1})}$.

In the case $\lambda^0 = 0$, we shall follow the method of [8]. This relies on the isomorphism (cf. [7, II.12.2(3)])

$$H^1\big(G_\tau, H^0(\lambda)\big)^{(\tau^{-1})} \cong \operatorname{ind}_B^G\Big[H^1(B_\tau, \lambda)^{(\tau^{-1})}\Big] \qquad \text{for } \lambda \in X_+. \tag{1}$$

To compute the right hand side, we use the isomorphisms

$$H^1(B_\tau, \lambda) \cong \big[H^1(U_\tau, k) \otimes k_\lambda\big]^{T_\tau} \tag{2}$$

and [8, Proposition 2.1]

$$H^1(U_\tau, k) \cong (\mathfrak{u}_\tau/[\mathfrak{u}_\tau, \mathfrak{u}_\tau])^*, \tag{3}$$

where $\mathfrak{u}_\tau = \operatorname{Lie} U_\tau = \langle X_\alpha | \alpha \in R_s^- \rangle$. By considering commutators, one sees that for our systems R_s the right side of the last equation is isomorphic to $\bigoplus_{\alpha \in S_s} k_\alpha$ [8, Proposition 2.2]. Putting this all together, we obtain

$$H^1(B_\tau, \lambda) \cong \left[\bigoplus_{\alpha \in S_s} k_{\alpha+\lambda} \right]^{T_\tau}. \tag{4}$$

Having thus found $H^1(G_\tau, H^0(\mu^0))^{(\tau^{-1})}$ one may try to use the obvious short exact sequence to try to describe $H^1(G_\tau, L(\mu^0))^{(\tau^{-1})}$ (cf. 3.2, 4.5, 4.6).

1.7. Let $\tilde{\rho}$ be the half-sum of the positive short roots. Then the composite $\tilde{\rho} \circ \tau$ is the half-sum of the positive long roots. Therefore as a module for D, acting through τ, we see that $L((p-1)\tilde{\rho})$ is isomorphic to the first Steinberg module. Hence by 1.3(2) and the injectivity of this module for D_1, we have

$$\mathrm{Ext}^1_{G_\tau}\big(L(\lambda^0), L(\mu^0)\big) = 0 \qquad \text{if either } \lambda^0 \text{ or } \mu^0 \text{ is } (p-1)\tilde{\rho}. \tag{1}$$

Similarly, through 1.3(2), we can apply Andersen's result [7, II, 12.9] in the G_2 and F_4 cases to obtain

$$\mathrm{Ext}^1_{G_\tau}\big(L(\lambda^0), L(\lambda^0)\big) = 0 \qquad \text{for all } \lambda^0 \in X_\tau. \tag{2}$$

1.8. Next, we state explicitly an obvious formula, in the form in which it will be used repeatedly in computing the third term of 1.5(2). Let $\lambda^0, \mu^0, \nu^0 \in X_\tau$ and $\bar{\lambda}, \bar{\mu} \in X_+$. Then

$$\begin{aligned} &\mathrm{Hom}_G\big(L(\lambda^0) \otimes L(\tau\bar{\lambda}), L(\nu^0) \otimes L(\mu^0) \otimes L(\tau\bar{\mu})\big) \\ &\quad \cong \Big[\mathrm{Hom}_{G_\tau}\big(L(\lambda^0), L(\nu^0) \otimes L(\mu^0)\big)^{(\tau^{-1})} \otimes \mathrm{Hom}\big(L(\bar{\lambda}), L(\bar{\mu})\big)\Big]^G. \end{aligned} \tag{1}$$

This will be applied in the simple case where G acts trivially on the first factor in the right side of (1). We may also replace $L(\tau\bar{\lambda})$ and $L(\tau\bar{\mu})$ by any modules on which G_τ acts trivially.

1.9. *Extensions of simple G_1-modules.* The restrictions of G_1 of the modules $L(\lambda)$, $\lambda \in X_1$ form a complete set of nonisomorphic simple G_1-modules [7, II, 3.15]. Let $\lambda = \lambda^0 + \tau\lambda^1$ and $\mu = \mu^0 + \tau\mu^1 \in X_1$. The top group extension in 1.3(1) gives rise to a spectral sequence whose

5-term sequence is

$$\begin{aligned}
0 &\to \left[\mathrm{Ext}^1_{G_\tau}\left(L(\lambda^1), \mathrm{Hom}_{G_\tau}(L(\lambda^0), L(\mu^0))^{(\tau^{-1})} \otimes L(\mu^1)\right)\right]^{(\tau)} \\
&\to \mathrm{Ext}^1_{G_1}(L(\lambda), L(\mu)) \\
&\to \left[\mathrm{Hom}_{G_\tau}\left(L(\lambda^1), \mathrm{Ext}^1_{G_\tau}(L(\lambda^0), L(\mu^0))^{(\tau^{-1})} \otimes L(\mu^1)\right)\right]^{(\tau)} \\
&\to \left[\mathrm{Ext}^2_{G_\tau}\left(L(\lambda^1), \mathrm{Hom}_{G_\tau}(L(\lambda^0), L(\mu^0))^{(\tau^{-1})} \otimes L(\mu^1)\right)\right]^{(\tau)} \\
&\to \mathrm{Ext}^2_{G_1}(L(\lambda), L(\mu)). \qquad (1)
\end{aligned}$$

(Here we have made use of the isomorphism $G_1/G_\tau \cong G_\tau$, so that for instance

$$\mathrm{Hom}_{G_1/G_\tau}(L(\tau\lambda^1), L(\tau\mu^1)) \cong \left[\mathrm{Hom}_{G_\tau}(L(\lambda^1), L(\mu^1))\right]^{(\tau)}$$

as G-modules.)

2. C_2 in Characteristic 2

2.1. In this section, G is simply connected of type C_2 and k is of characteristic 2. Let α_1 be the short fundamental root and α_2 be the long one. Then τ: $\omega_1 \mapsto \omega_2 \mapsto 2\omega_1$ and $X_\tau = \{0, \omega_1 = \tilde{\rho}\}$. Therefore, by 1.7(1), the only $\mathrm{Ext}^1_{G_\tau}(L(\lambda^0), L(\mu^0))$ that can be nonzero is $H^1(G_\tau, k)$

LEMMA. $H^1(G_\tau, k) \cong L(\omega_2)$.

Proof. From its definition, we see that $\mathfrak{g}_\tau$ is five-dimensional and has a one-dimensional center $\langle H_{\alpha_1}\rangle$, equal to the commutator ideal, with abelian quotient spanned by the images of the X_α for short roots α. The quotient is isomorphic to $L(\omega_2)$ as a G-module, since ω_2 is the highest short root. It follows from the long exact sequence for G_τ-cohomology resulting from the short exact sequence

$$0 \to k \to \mathfrak{g}_\tau \to L(\omega_2) \to 0 \qquad (1)$$

that $L(\omega_2)$ embeds into $H^1(G_\tau, k)$. On the other hand, there is always an embedding $H^1(G_\tau, k) \hookrightarrow H^1(\mathfrak{g}_\tau, k)$ of restricted cohomology into ordinary cohomology of Lie algebras [7, I, 9.19], and $H^1(\mathfrak{g}_\tau, k) \cong (\mathfrak{g}_\tau/[\mathfrak{g}_\tau, \mathfrak{g}_\tau])^*$, so both of the embeddings above are isomorphisms.

Remark. We have $G_\tau/T_\tau \cong (L(\omega_2))_{a,1}$. From this it follows that

$$H^*(G_\tau, k) \cong H^*(G_\tau/T_\tau, k) \cong S(L(\omega_2)), \tag{2}$$

where the last isomorphism is given by [7, I, Proposition 4.27(a)].

Extensions

2.2. We can now compute the groups $\operatorname{Ext}^1_G(L(\lambda), L(\mu))$ using 1.5(1) and 1.5(2). Let $\lambda = \lambda^0 + \tau\bar{\lambda}$ and $\mu = \mu^0 + \tau\bar{\mu}$. There are three cases.

(a) If $\lambda^0 \neq \mu^0$, then one of them is $\tilde{\rho}$ so 1.5(1) yields $\operatorname{Ext}^i_G(L(\lambda), L(\mu)) = 0$ for $i \geq 0$.

(b) If $\lambda^0 = \mu^0 = \tilde{\rho}$, then 1.5(1) again degenerates, giving isomorphisms

$$\operatorname{Ext}^i_G(L(\lambda), L(\mu)) \cong \operatorname{Ext}^i_G(L(\bar{\lambda}), L(\bar{\mu})), \qquad i \geq 1. \tag{1}$$

(c) If $\lambda^0 = \mu^0 = 0$, then 1.5(2) and Lemma 2.1 give an exact sequence

$$\begin{aligned} 0 &\to \operatorname{Ext}^1_G(L(\bar{\lambda}), L(\bar{\mu})) \to \operatorname{Ext}^1_G(L(\lambda), L(\mu)) \\ &\to \operatorname{Hom}_G(L(\bar{\lambda}), L(\omega_1) \otimes L(\bar{\mu})) \to \operatorname{Ext}^2(L(\bar{\lambda}), L(\bar{\mu})). \end{aligned} \tag{2}$$

Let $\bar{\lambda} = \lambda^1 + \tau\lambda''$ and $\bar{\mu} = \mu^1 + \tau\mu''$.

(i) If $\lambda^1 = \mu^1$ then either both are 0, in which case $L(\omega_1) \otimes L(\bar{\mu})$ is irreducible and not isomorphic to $L(\bar{\lambda})$, making the third term in the sequence zero, or else $\lambda^1 = \mu^1 = \tilde{\rho}$, in which case the third term is again zero because

$$\begin{aligned} \operatorname{Hom}_G(L(\bar{\lambda}), L(\omega_1) \otimes L(\bar{\mu})) &\subseteq \operatorname{Hom}_{G_\tau}(L(\omega_1), L(\omega_1) \\ &\otimes L(\omega_1)) \otimes \operatorname{Hom}(L(\tau\lambda''), L(\tau\mu'')) = 0, \end{aligned} \tag{3}$$

since ω_1 is not a weight of $L(\omega_1) \otimes L(\omega_1)$. Therefore, if $\lambda^1 = \mu^1$, we have

$$\operatorname{Ext}^1_G(L(\lambda), L(\mu)) \cong \operatorname{Ext}^1_G(L(\bar{\lambda}), L(\bar{\mu})). \tag{4}$$

(ii) Suppose $\lambda^1 \neq \mu^1$. Then by (a) applied to $(\bar{\lambda}, \bar{\mu})$ in place of (λ, μ), the first and fourth terms of the 5-term sequence are zero and we obtain

$$\begin{aligned} \operatorname{Ext}^1_G(L(\lambda), L(\mu)) &\cong \operatorname{Hom}_G(L(\omega_1 + \tau\lambda''), L(\omega_1 + \tau\mu'')) \\ &\cong \begin{cases} k & \text{if } \lambda'' = \mu'', \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{5}$$

We summarize these results (cf. [17], [10]).

TABLE I

(λ, μ)	0	ω_1	ω_2	$\omega_1 + \omega_2$
0	$L(2\omega_1)$	0	k	0
ω_1	0	$L(2\omega_1)$	0	0
ω_2	k	0	0	0
$\omega_1 + \omega_2$	0	0	0	0

PROPOSITION. *Let* $\lambda = \Sigma_i \tau^i \lambda^i$ *and* $\mu = \Sigma_j \tau^j \mu^j$. *Then* $\mathrm{Ext}^1_G(L(\lambda), L(\mu)) = 0$ *unless* $\lambda - \mu = \pm \tau^s \omega_1$ *with* $s \geq 1$ *and* $\lambda^{s-1} = \mu^{s-1} = 0$, *in which case the space of extensions is one-dimensional.*

2.3. The extensions of simple G_1-modules can be computed from 1.9(1), using 2.1 and 2.2(3) (cf. [8, Section 5]).

PROPOSITION. *The G-modules* $\mathrm{Ext}^1_{G_1}(L(\lambda), L(\mu))$ *for* λ, $\mu \in X_1$ *are given in Table I.*

2.4. Let $\overline{G} = G/Z(G)$ be the adjoint group. Its simple modules are the $L(\lambda)$ with $\lambda \in \mathbb{Z}_+R = \tau X_+$. The extensions between these $\overline{G}$-modules are the same as when they are viewed as G-modules, by 1.1(1). However, it is not as simple to pass from G_τ to $\overline{G}_\tau$ or from G_1 to $\overline{G}_1$ (cf. [2, 6.4]. We have exact sequences

$$1 \to G_\tau/Z(G) \to \overline{G}_\tau \to Z(G/G_\tau) \to 1,$$
$$1 \to Z(G) \to G_\tau \to G_\tau/Z(G) \to 1, \qquad (1)$$

where $Z(G/G_\tau) \cong Z(G) = T_\tau$. The unipotent subgroup $G_\tau/Z(G)$ acts trivially on the simple $\overline{G}_\tau$-modules, so these are the trivial module and the other one-dimensional $Z(G/G_\tau)$-module $\overline{\omega}_2$. The restriction of $L(\omega_2)$ to $\overline{G}_\tau$ is isomorphic to a direct sum of 4 copies of $\overline{\omega}_2$. Also, from (1), 1.3(1), and

$$1 \to G_1/Z(G) \to \overline{G}_1 \to Z(G/G_1) \to 1 \qquad (2)$$

it is not difficult to see that the simple $\overline{G}_1$-modules are k, $L(\omega_2)$, the one-dimensional $Z(G/G_1)$-module $\overline{2\omega_1}$ with $L(2\omega_1) \cong \oplus^4 \overline{2\omega_1}$, and $L(\omega_2) \otimes \overline{2\omega_1}$. For any $\overline{G}_\tau$-module E, the Hochschild–Serre 5-term sequences for (1) yield

$$H^1(\overline{G}_\tau, E)^{(\tau^{-1})} \cong \left[H^1(G_\tau/Z(G), E)^{(\tau^{-1})}\right]^{T_\tau} \cong \left[H^1(G_\tau, E)^{(\tau^{-1})}\right]^{T_\tau}. \quad (3)$$

TABLE II

$(\tau\lambda, \tau\mu)$	0	ω_2	$2\omega_1$	$2\omega_1 + \omega_2$
0	0	k	$L(2\omega_1) \otimes L(2\omega_1)$	0
ω_2	k	0	0	$L(2\omega_1) \otimes L(2\omega_1)$
$2\omega_1$	$L(2\omega_1) \otimes L(2\omega_1)$	0	0	k
$2\omega_1 + \omega_2$	0	$L(2\omega_1) \otimes L(2\omega_1)$	k	0

Now we can deduce the $\overline{G}_\tau$-extensions from 2.1, and then, by the appropriate version of 1.9(1), we also obtain the $\overline{G}_1$-extensions.

PROPOSITION. (a) *Let $\lambda^1, \mu^1 \in X_\tau$. Then as $\overline{G}$-modules,*

$$\mathrm{Ext}^1_{\overline{G}_\tau}(L(\tau\lambda^1), L(\tau\mu^1)) \cong \begin{cases} L(\omega_2) \otimes L(\omega_2) & \text{if } \lambda^1 \neq \mu^1, \\ 0 & \text{if } \lambda^1 = \mu^1. \end{cases}$$

(b) *The $\overline{G}$-modules $\mathrm{Ext}^1_{\overline{G}_1}(L(\tau\lambda), L(\tau\mu))$ for $\lambda = \lambda^1 + \tau\lambda^2$ and $\mu = \mu^1 + \tau\mu^2 \in X_1$ are given in Table II.*

3. G_2 IN CHARACTERISTIC 3

3.1. Let α_1 and α_2 be the short and long fundamental roots, respectively. Then τ: $\omega_1 \mapsto \omega_2 \mapsto 3\omega_1$ and $X_\tau = \{0, \omega_1, 2\omega_1 = 2\tilde{\rho}\}$. Since ω_1 is the largest short root and $\mathfrak{g}_\tau$ is simple we have $L(\omega_1) \cong \mathfrak{g}_\tau$, and since ω_2 is the largest root, it follows that $\mathfrak{g}/\mathfrak{g}_\tau \cong L(\omega_2)$. The module $L(2\tilde{\rho})$ is a 27-dimensional, simple, injective module for G_τ (see 1.7).

3.2. *G-module structure of G_τ-extensions.* By 1.7(1) and (2), the only nonzero group $\mathrm{Ext}^1_{G_\tau}(L(\lambda^0), L(\mu^0))$ is $H^1(G_\tau, L(\omega_1))$.

LEMMA. $H^1(G_\tau, L(\omega_1) \cong L(\omega_2)$

Proof. From their dimensions, we see that $L(\omega_1) \cong H^0(\omega_1)$, so 1.6(1) gives an isomorphism of G-modules $H^1(G_\tau, L(\omega_1))^{(\tau^{-1})} \cong \mathrm{ind}_B^G[H^1(B_\tau, \omega_1)^{(\tau^{-1})}]$. Now $S_s = \{\alpha_1, \alpha_2 + \alpha_1\}$, so by 1.6(4),

$$H^1(B_\tau, \omega_1) \cong \left[k_{\alpha_1 + \omega_1} \oplus k_{\alpha_2 + \alpha_1 + \omega_1}\right]^{T_\tau}. \qquad (1)$$

We have $\alpha_1 + \omega_1 = 3\omega_1 - \omega_2 = \tau(\omega_2 - \omega_1)$ and $\alpha_2 + \alpha_1 + \omega_1 = \omega_2 = \tau\omega_1$. Since $\omega_2 - \omega_1$ is not dominant, we obtain $\mathrm{ind}_B^G[H^1(B_\tau, \omega_1)^{(\tau^{-1})}] \cong H^0(\omega_1)$.

3.3

LEMMA. (a) $\operatorname{Hom}_{G_\tau}(L(2\omega_1), L(\omega_1) \otimes L(2\omega_1)) \cong k$.

(b) $\operatorname{Hom}_{G_\tau}(L(2\omega_1), L(\omega_1) \otimes L(\omega_1)) \cong k$.

(c) $\operatorname{Hom}_{G_\tau}(L(\omega_1), L(\omega_1) \otimes L(\omega_1)) \cong k$.

Proof. Since $L(2\omega_1) = L(2\tilde{\rho})$ is an injective (hence also projective) G_τ-module, the dimensions of the spaces in (a) and (b) are simply the multiplicities of $L(2\omega_1)$ as a G_τ-composition factor of the tensor products and this can be found by routine weight calculations. The same kind of calculation also shows that the multiplicity of $L(\omega_1)$ as a G_τ-composition factor of $L(\omega_1) \otimes L(\omega_1)$ is 2, so this is an upper bound for the dimension of $\operatorname{Hom}_{G_\tau}(L(\omega_1), L(\omega_1) \otimes L(\omega_1))$. Therefore G acts trivially on this space so it equals $\operatorname{Hom}_G(L(\omega_1), L(\omega_1) \otimes L(\omega_1))$. We claim that this space is one-dimensional. It can be seen by weight calculations that $S^2(L(\omega_1))$ has no G-composition factors $L(\omega_1)$ and that the G-composition factors of $\wedge^2(L(\omega_1))$ are $L(\omega_1)$ (twice) and $L(\omega_2)$. Since $\operatorname{Hom}_G(L(\omega_2), L(\omega_1) \otimes L(\omega_1)) = 0$, the claim follows from the self-duality of $\wedge^2(L(\omega_1))$.

Extensions

3.4. We can now apply 1.5(2). Let $\lambda + \lambda^0 + \bar{\lambda}$ and $\mu = \mu^0 + \bar{\mu}$. If $\lambda^0 = \mu^0$ then the third term of 1.5(2) is zero, giving

$$\operatorname{Ext}^1_G(L(\lambda), L(\mu)) \cong \operatorname{Ext}^1_G(L(\bar{\lambda}), L(\bar{\mu})). \tag{1}$$

Thus we are reduced to the case $\lambda^0 \neq \mu^0$. Then the first and fourth terms of 1.5(2) are zero and we have

$$\operatorname{Ext}^1_G(L(\lambda), L(\mu)) \cong \operatorname{Hom}_G\Big(L(\bar{\lambda}), \operatorname{Ext}^1_{G_\tau}(L(\lambda^0), L(\mu^0))^{(\tau^{-1})} \otimes L(\bar{\mu})\Big). \tag{2}$$

If either λ^0 and μ^0 is $2\omega_1$, then the right hand side of (2) is zero. It remains to consider the case where $\{\lambda^0, \mu^0\} = \{0, \omega_1\}$. By the self-duality of $L(\omega_1)$ and the Tensor Product Theorem, we may assume $\lambda^0 = \omega_1$ and $\mu^0 = 0$. Then, by Lemma 3.2, the isomorphism (2) becomes

$$\operatorname{Ext}^1_G(L(\lambda), L(\mu)) \cong \operatorname{Hom}_G(L(\bar{\lambda}), L(\omega_1) \otimes L(\bar{\mu})). \tag{3}$$

Write $\bar{\lambda} = \lambda^1 + \tau\lambda''$ and $\bar{\mu} = \mu^1 + \tau\mu''$. We now consider the various possibilities for λ^1 and μ^1. If $\lambda^1 = \mu^1 = 0$, or if $\{\lambda^1, \mu^1\} = \{0, 2\omega_1\}$, then the right hand side of (3) is zero, by 1.8(1). For all other combinations of λ^1

TABLE III

(λ, μ)	(0, 0)	(0, 1)	(0, 2)	(1, 0)	(1, 1)	(1, 2)	(2, 0)	(2, 1)	(2, 2)
(0, 1)	0	$L(2\omega_1)$	0	0	k	0	0	0	0
(0, 1)	$L(2\omega_1)$	0	0	k	k	k	0	0	0
(0, 2)	0	0	0	0	k	k	0	0	0
(1, 0)	0	k	0	0	$L(2\omega_1)$	0	0	0	0
(1, 1)	k	k	k	$L(2\omega_1)$	0	0	0	0	0
(1, 2)	0	k	k	0	0	0	0	0	0
(2, 0)	0	0	0	0	0	0	0	$L(2\omega_1)$	0
(2, 1)	0	0	0	0	0	0	$L(2\omega_1)$	0	0
(2, 2)	0	0	0	0	0	0	0	0	0

and μ^1, it follows from Lemma 3.3 and 1.8(1) that the right hand side of (3) is equal to k if $\lambda'' = \mu''$ and is zero otherwise.

We summarize these results (cf. [11]).

PROPOSITION. *Let* $\lambda = \sum_i \tau^i \lambda^i$ *and* $\mu = \sum_j \tau^j \mu^j$. *Then* $\operatorname{Ext}^1_G(L(\lambda), L(\mu)) = 0$ *unless* $\lambda - \mu = \pm\tau^s \omega_1 \pm \tau^{s+1}\nu$ *for some* $s \geq 0$, *where* $\nu \in \{0, \omega_1, 2\omega_1 - \omega_1\}$ *and* $\lambda^{s+1} \neq 0$ *if* $\nu = 0$. *In this case the space of extensions is one-dimensional.*

3.5. We can compute the extensions of simple G_1-modules by using 3.2 and Lemma 3.3 in 1.9(1) (cf. [8, Section 5]).

PROPOSITION. *The G-modules* $\operatorname{Ext}^1_{G_1}(L(\lambda), L(\mu))$ *for* $\lambda, \mu \in X_1$ *are given in Table* III, *in which* $a\omega_1 + b\omega_2 \in X_1$ *is denoted by* (a, b).

4. F_4 CHARACTERISTIC 2

4.1. We may take R to be the set $\{\pm e_i, \pm e_i \pm e_j, \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4) \mid 1 \leq i, j \leq 4, i \neq j\}$, where the e_i form an orthonormal base for $\mathbb{Q}^4$. Then a set of simple roots is $S = \{\alpha_1 = (1, -1, 0, 0),\ \alpha_2 = (0, 1, -1, 0),\ \alpha_3 = (0, 0, 1, 0),\ \alpha_4 = \frac{1}{2}(-1, -1, -1, 1)\}$, and the corresponding fundamental dominant weights are $\omega_1 = (1, 0, 0, 1)$, $\omega_2 = (1, 1, 0, 2)$, $\omega_3 = \frac{1}{2}(1, 1, 1, 3)$, and $\omega_4 = (0, 0, 0, 1)$. The linear map $\alpha_i^\vee \mapsto \alpha_{5-i}$ is an isomorphism of root systems from $R^\vee$ with R which we shall use to identify the two. Therefore the map $\alpha \mapsto \alpha^*$ of 1.2 interchanges α_1 with α_4 and α_2 with α_3.

We have sets of positive simple roots

$$\begin{aligned} S_1 = \{&\beta_1 = (0, 1, -1, 0), \beta_2 = (1, -1, 0, 0), \\ &\beta_3 = (-1, 0, 0, 1), \beta_4 = (0, 1, 1, 0)\} \end{aligned} \tag{1}$$

and

$$S_s = \{\bar{\beta}_1 = (0,0,1,0), \bar{\beta}_2 = \tfrac{1}{2}(-1,-1,-1,1), \bar{\beta}_3 = (1,0,0,0), \bar{\beta}_4 = (0,1,0,0)\} \tag{2}$$

for the subsystems R_l and R_s of long and short roots, with the notation chosen so that $\beta_i^* = \bar{\beta}_i$. With this notation, we have τ: $\omega_4 \mapsto \omega_1 \mapsto 2\omega_4$ and $\omega_3 \mapsto \omega_2 \mapsto 2\omega_3$. Thus,

$$X_\tau = \{0, \omega_4, \omega_3, \omega_3 + \omega_4 = \tilde{\rho}\}. \tag{3}$$

The module $L(\omega_4)$ is $\mathfrak{g}_\tau$ since the latter is simple and ω_4 is the largest short root. Since ω_1 is the largest root, the Weyl module $V(\omega_1)$ is $\mathfrak{g}$ and $\mathfrak{g}/\mathfrak{g}_\tau \cong L(\omega_1)$. The module $L(\omega_3 + \omega_4) = L(\tilde{\rho})$ is a 2^{12}-dimensional injective module for G_τ (See 1.7). The Weyl modules for ω_4 and $\tilde{\rho}$ are easily seen to be simple by considering dimensions. For ω_3, the Jantzen sum formula [7, II, 8.19] shows that the radical of $V(\omega_3)$ is isomorphic to $k \oplus L(\omega_4)$, and the simple module $L(\omega_3)$ is 246-dimensional. The (formal) characters of all these modules were first given in [16]. Therefore, using the Tensor Product Theorem and Weyl's character formula, one knows in principle the composition multiplicities of simple modules in Weyl modules and tensor product of simple modules or Weyl modules. We shall freely use such information for G and related groups. It can either be found in or derived from the tables in [3].

4.2. *Good filtrations.* In this paragraph, G will denote any reductive group over an algebraically closed field k. An ascending filtration of a G-module is said to be *good* if the subquotients are isomorphic to induced modules $H^0(\lambda)$ for various $\lambda \in X_+$. There is also the dual notion of a Weyl filtration. The important facts for our purpose about a finite-dimensional G-module M with a good filtration are the following:

(1) The multiplicity of $H^0(\lambda)$ as a subquotient is $\dim_k \operatorname{Hom}_G(V(\lambda), M)$.

(2) If $H^0(\lambda)$ and $H^0(\mu)$ are both good filtration factors and $\lambda \not> \mu$, then M has a good filtration in which the factor $H^0(\mu)$ appears above the factor $H^0(\lambda)$.

(3) If the module M' also has a good filtration, then so does $M \otimes M'$.

A proof of (1) can be found in [7, II, 4.18], and (2) follows from a standard property of Weyl modules [7, II, 2.14]. The deeper fact (3) is proved in [6, 7.3.1] with a few exceptions and in general in [9]. Given a module with a good filtration, the multiplicities of the subquotients can be determined from the weight multiplicities in the module. Thus, for modules of the form $M = H^0(\lambda) \otimes H^0(\mu)$, it is routine to calculate $\dim_k \mathrm{Hom}_G(V(\nu), M)$ for any $\nu \in X_+$.

Representations of D

4.3. We shall need a number of facts concerning modules for the subgroup $D = \langle x_\alpha(t) | t \in k,\ \alpha \in R_1 \rangle$, is simply connected group of type D_4. Let $\tilde{\omega}_i$, $1 \le i \le 4$ be the fundamental weights for the simple roots S_1 in R_1. To distinguish between D-modules and G-modules, we shall denote simple modules and Weyl modules for D by $\tilde{L}(\tilde{\omega}), \tilde{V}(\tilde{\omega})$, etc. The group D has a "triality" automorphism which fixes $x_{\beta_2}(t)$ and cycles $x_{\beta_1}(t)$, $x_{\beta_3}(t)$, and $x_{\beta_4}(t)$. The simple modules $\tilde{L}(\omega_i)$ are permuted accordingly. For $i = 1, 3, 4$, these simple modules are 8-dimensional, and if D is identified with Spin(8, k), then these are the natural orthogonal module and the two half-spin modules. They are equal to the Weyl modules of the same highest weights. The adjoint module $\mathfrak{d} = \tilde{V}(\tilde{\omega}_2)$ is 28-dimensional and the center of $\mathfrak{d}$ forms the radical, isomorphic to $k \oplus k$.

We shall be particularly interested in describing the D-module structure of some G-modules. We shall only consider the action of D through $D \xrightarrow{\tau} \bar{D} \subset G$, as in 1.3(2), never through restriction. If λ is a weight of a G-module, then it corresponds to the weight $\tilde{\lambda}$ of the associated D-module given by

$$\langle \tilde{\lambda}, \beta_i^\vee \rangle = \langle \lambda, \bar{\beta}_i^\vee \rangle, \qquad 1 \le i \le 4. \tag{1}$$

Of course, the simple modules with highest weights in $X_\mathfrak{t}$ are all simple as D-modules; we have already seen in 1.7 that $L(\bar{\rho}) \cong \tilde{St}_1$, and from the highest weights we see also that $L(\omega_4) \cong \tilde{L}(\tilde{\omega}_2)$ and $L(\omega_3) \cong \tilde{L}(\tilde{\omega}_1 + \tilde{\omega}_3 + \tilde{\omega}_4)$.

In the following lemma and thereafter, we use the standard notation $[M : L]$ to mean the multiplicity of the simple module L as a composition factor of the module M, which will always have finite dimension.

LEMMA. (a) $\tilde{V}(\tilde{\omega}_4) \otimes \tilde{V}(\tilde{\omega}_4)$ *has a filtration with factors (in descending order)* k, $\tilde{V}(\tilde{\omega}_2), \tilde{V}(2\tilde{\omega}_4)$.

(b) $\tilde{V}(\tilde{\omega}_1) \otimes \tilde{V}(\tilde{\omega}_3)$ *has a filtration with factors (in descending order)* $\tilde{V}(\tilde{\omega}_4), \tilde{V}(\tilde{\omega}_1 + \tilde{\omega}_3)$.

(c) $\tilde{V}(\tilde{\omega}_1 + \tilde{\omega}_3) \otimes \tilde{V}(\tilde{\omega}_4)$ *has a filtration with factors (in descending order)* $\tilde{V}(\tilde{\omega}_2), \tilde{V}(2\tilde{\omega}_1) \oplus \tilde{V}(2\tilde{\omega}_3), \tilde{V}(\tilde{\omega}_1 + \tilde{\omega}_3 + \tilde{\omega}_4)$.

(d) $[\tilde{V}(\tilde{\omega}_2 + 2\tilde{\omega}_1) : \tilde{L}(2\tilde{\omega}_2)] = 1$.

(e) $[\tilde{V}(\tilde{\omega}_2 + 2\tilde{\omega}_1) : \tilde{L}(\tilde{\omega}_1 + \tilde{\omega}_3 + \tilde{\omega}_4)] = 1$.

(f) $[\tilde{V}(2\tilde{\omega}_2) : \tilde{L}(\tilde{\omega}_1 + \tilde{\omega}_3 + \tilde{\omega}_4)] = 1$.

(g) $\tilde{V}(\tilde{\omega}_1 + \tilde{\omega}_3 + \tilde{\omega}_4)$ *has composition factors* $\tilde{L}(\tilde{\omega}_1 + \tilde{\omega}_3 + \tilde{\omega}_4), (3)\tilde{L}(\tilde{\omega}_2), \tilde{L}(2\tilde{\omega}_1), \tilde{L}(2\tilde{\omega}_3), \tilde{L}(2\tilde{\omega}_4), (2)k$. (*Multiplicities are given in parentheses.*)

(h) $\tilde{V}(2\tilde{\omega}_1)$ *has composition factors* $\tilde{L}(2\tilde{\omega}_1), \tilde{L}(\tilde{\omega}_2), k$.

(i) $\operatorname{rad}\tilde{V}(\tilde{\omega}_1 + \tilde{\omega}_3) \cong \tilde{L}(\tilde{\omega}_4)$.

Proof. In view of the discussion in the previous two paragraphs, the verifications of (a)–(h) are routine since we have described all of the simple modules involved, up to twisting by various endomorphisms. Then (i) can be obtained using (b), the self-duality of $\tilde{V}(\tilde{\omega}_1) \otimes \tilde{V}(\tilde{\omega}_3)$, and the Jantzen sum formula for $\tilde{V}(\tilde{\omega}_1 + \tilde{\omega}_3)$.

4.4. The calculation to prove (i) above gives the character of $\tilde{L}(\tilde{\omega}_1 + \tilde{\omega}_3)$, which was essentially the only simple module in the lemma whose character was not yet described. It has dimension 48. With this information, we can apply 4.3(1) to obtain further information about the way D acts on G-modules in the next lemma, whose proof is straightforward.

LEMMA. (a)

$$L(\omega_1) \cong k \oplus k \oplus \tilde{L}(2\tilde{\omega}_1) \oplus \tilde{L}(2\tilde{\omega}_3) \oplus \tilde{L}(2\tilde{\omega}_4)$$

as D-modules.

(b) $L(\omega_2)$ *is a self-dual D-module with composition factors* $\tilde{L}(2(\tilde{\omega}_1 + \tilde{\omega}_3)), \tilde{L}(2(\tilde{\omega}_1 + \tilde{\omega}_4)), \tilde{L}(2(\tilde{\omega}_3 + \tilde{\omega}_4)), (2)\tilde{L}(2\tilde{\omega}_1), (2)\tilde{L}(2\tilde{\omega}_3), (2)\tilde{L}(2\tilde{\omega}_4), (2)\tilde{L}(2\tilde{\omega}_2), (2)k$.

G-Module Structure of G_τ-Extensions

4.5. Our aim is to determine the structure of the G-modules $\operatorname{Ext}^1_{G_\tau}(L(\lambda^0), L(\mu^0))$ for $\lambda^0, \mu^0 \in X_\tau$, which appear in 1.5(2). By 1.7(1) and (2) we are reduced to studying the three modules $H^1(G_\tau, L(\omega_4))$, $H^1(G_\tau, L(\omega_3))$, and $\operatorname{Ext}^1_{G_\tau}(L(\omega_3), L(\omega_4))$.

LEMMA. (a) $H^1(G_\tau, L(\omega_4))^{(\tau^{-1})} \cong L(\omega_4)$.

(b) $H^1(G_\tau, H^0(\omega_3))^{(\tau^{-1})} \cong H^0(\omega_1)$.

Proof. Since $L(\omega_4) \cong H^0(\omega_4)$, we have

$$H^1(G_\tau, L(\omega_4))^{(\tau^{-1})} \cong \operatorname{ind}_B^G\left[\left(H^1(B_\tau, \omega_4)\right)^{(\tau^{-1})}\right] \tag{1}$$

by 1.6(1) and

$$H^1(B_\tau, \omega_4) \cong \bigoplus_{i=1}^{4} \left[k_{\bar\beta_i+\omega_4}\right]^{T_\tau} \tag{2}$$

by 1.6(4). Of the weights $\bar\beta_i + \omega_4$, only $\bar\beta_3 + \omega_4 = \omega_1 = \tau\omega_4$ is dominant. Therefore,

$$H^1(G_\tau, L(\omega_4))^{(\tau^{-1})} \cong H^0(\omega_4) \cong L(\omega_4), \tag{3}$$

proving (a). The calculation for (b) is very similar.

4.6.

Lemma. $H^1(G_\tau, L(\omega_3)) \cong k \oplus L(2\omega_4)$.

Proof. Since $H^0(\omega_3)/L(\omega_3) \cong k \oplus L(\omega_4)$, we can use Lemma 4.5 in the long exact sequence of G_τ-cohomology to obtain the exact sequence

$$0 \to k \to H^1(G_\tau, L(\omega_3))^{(\tau^{-1})} \to H^0(\omega_1) \to L(\omega_4). \tag{1}$$

From the descriptions of $V(\omega_1)$ and $V(\omega_4)$ in 4.1, it follows easily that $\mathrm{Ext}^1_G(H^0(\omega_1), k) = 0$, so the above exact sequence shows

$$k \oplus L(2\omega_4) \subseteq H^1(G_\tau, L(\omega_3)) \subseteq k \oplus H^0(\omega_1)^{(\tau)}, \tag{2}$$

and so it remains to decide between the two possibilities. Suppose for a contradiction that $H^1(G_\tau, L(\omega_3)) \cong k \oplus H^0(\omega_2)^{(\tau)}$. Then, since as a D-module (with D acting via τ),

$$H^1(G_\tau, L(\omega_3)) \cong H^1\big(D_1, \tilde L(\tilde\omega_1 + \tilde\omega_3 + \tilde\omega_4)\big), \tag{3}$$

by 1.3(2), this module has a filtration with bottom factor $L(2\omega_4) \cong \tilde L(2\tilde\omega_2)$ and top factor $L(\omega_1) \cong k \oplus k \oplus \tilde L(2\tilde\omega_1) \oplus \tilde L(2\tilde\omega_3) \oplus \tilde L(2\tilde\omega_4)$. Since one can see from the Weyl modules that $\mathrm{Ext}^1_D(\tilde L(\tilde\omega_1), \tilde L(\tilde\omega_2)) = 0$, it follows that

$$\mathrm{Hom}_D\Big(\tilde L(\tilde\omega_1), H^1\big(D_1, \tilde L(\tilde\omega_1 + \tilde\omega_3 + \tilde\omega_4)\big)^{(-1)}\Big) \neq 0, \tag{4}$$

where the superscript -1 indicates untwisting by Frobenius. Then, from the Hochschild–Serre 5-term sequence for the pair (D, D_1), we deduce that

$$\mathrm{Ext}^1_D\big(\tilde L(2\tilde\omega_1), \tilde L(\tilde\omega_1 + \tilde\omega_3 + \tilde\omega_4)\big) \neq 0, \tag{5}$$

and further, by a standard property of Weyl modules [7, II.2.14], that

$$\operatorname{Hom}_D\big(\tilde{L}(2\tilde{\omega}_1), \tilde{H}^0(\tilde{\omega}_1 + \tilde{\omega}_3 + \tilde{\omega}_4)/\tilde{L}(\tilde{\omega}_1 + \tilde{\omega}_3 + \tilde{\omega}_4)\big) \neq 0. \quad (6)$$

Then, because of the triality automorphism of D, we can replace $\tilde{L}(2\tilde{\omega}_1)$ in the above by $\tilde{L}(2\tilde{\omega}_3)$ and $\tilde{L}(2\tilde{\omega}_4)$, so there is an embedding

$$\tilde{L}(2\tilde{\omega}_1) \oplus \tilde{L}(2\tilde{\omega}_3) \oplus \tilde{L}(2\tilde{\omega}_4) \hookrightarrow \overline{H} = \tilde{H}^0(\tilde{\omega}_1 + \tilde{\omega}_3 + \tilde{\omega}_4)/\tilde{L}(\tilde{\omega}_1 + \tilde{\omega}_3 + \tilde{\omega}_4). \quad (7)$$

Consider the module $V = \tilde{L}(\tilde{\omega}_1) \otimes L(\tilde{\omega}_3) \otimes L(\tilde{\omega}_4)$. Since $\tilde{L}(\omega_i) \cong \tilde{H}^0(\omega_i) \cong \tilde{V}(\omega_i)$ for $i = 1, 3, 4$, it follows that V has both good and Weyl filtrations (See 4.2). From the Weyl filtrations, we obtain (using Lemma 4.3(a)–(c), (h)) a filtration of V with factors (in descending order)

$$k, \tilde{V}(\tilde{\omega}_2) \oplus \tilde{V}(\tilde{\omega}_2), \tilde{V}(2\tilde{\omega}_1) \oplus \tilde{V}(2\tilde{\omega}_3) \oplus \tilde{V}(2\tilde{\omega}_4), \tilde{V}(\tilde{\omega}_1 + \tilde{\omega}_3 + \tilde{\omega}_4).$$

From the good filtrations, we know that V has a homomorphic image isomorphic to $\tilde{H}^0(\tilde{\omega}_1 + \tilde{\omega}_3 + \tilde{\omega}_4)$. Now, since the highest weight $\tilde{\omega}_1 + \tilde{\omega}_3 + \tilde{\omega}_4$ of V occurs with multiplicity one, the homomorphism onto $\tilde{H}^0(\tilde{\omega}_1 + \tilde{\omega}_3 + \tilde{\omega}_4)$ is unique up to scalar and, dually, V has only one submodule isomorphic to $\tilde{V}(\tilde{\omega}_1 + \tilde{\omega}_3 + \tilde{\omega}_4)$. Moreover, the uniqueness of the weight implies that the natural map into $\tilde{H}^0(\tilde{\omega}_1 + \tilde{\omega}_3 + \tilde{\omega}_4)$ factors through $V/\operatorname{rad}\tilde{V}(\tilde{\omega}_1 + \tilde{\omega}_3 + \tilde{\omega}_4)$, and hence that there is a surjection ϕ: $\overline{V} = V/\tilde{V}(\tilde{\omega}_1 + \tilde{\omega}_3 + \tilde{\omega}_4) \to \overline{H}$. We now consider the filtration on $\overline{V}$ induced by the one on V with factors given above. We see that $\overline{V}$ has a submodule $\overline{V}_1 \cong \tilde{V}(2\tilde{\omega}_1) \oplus \tilde{V}(2\tilde{\omega}_3) \oplus \tilde{V}(2\tilde{\omega}_4)$ and that the quotient $\overline{V}/\overline{V}_1$ has no composition factors $\tilde{L}(2\tilde{\omega}_i)$ for $i = 1$, 3, or 4. Therefore, the embedding (7) has its image in $\phi(\overline{V}_1)$. From the structure of $\overline{V}_1$ it follows that $\operatorname{rad}\overline{V}_1 \subset \operatorname{Ker}\phi$. Then, by considering composition factors, this leads us to conclude that the multiplicity of $\tilde{L}(\tilde{\omega}_2)$ in $\overline{H} = \phi(\overline{V})$ is at most 2. But this contradicts the fact that $[\tilde{H}^0(\tilde{\omega}_1 + \tilde{\omega}_3 + \tilde{\omega}_4): \tilde{L}(\tilde{\omega}_2)] = 3$ (Lemma 4.3(g)). The lemma is proved.

4.7. We turn to $\operatorname{Ext}^1_{G_\tau}(L(\omega_4), L(\omega_3))$. From this point on we shall require many facts about multiplicities of simple modules in composition series and multiplicities of Weyl modules in Weyl filtrations, all of which can be derived by routine calculations which we omit. The composition factors of tensor products of τ-restricted simple modules can be found in Table 2 of [12].

For $\lambda' \in X_\tau$, we denote the injective hull of the simple G_τ-module $L(\lambda')$ by $Q(\lambda')$.

LEMMA. *We have the following G_τ-isomorphisms.*

(a) $L(\tilde{\rho}) \otimes L(\omega_4) \cong Q(\omega_3) \oplus L(\tilde{\rho}) \oplus L(\tilde{\rho})$.

(b) $\mathrm{soc}_{G_\tau}(L(\omega_4) \otimes L(\omega_4)) \cong k \oplus L(\omega_4)$.

(c) $\mathrm{soc}_{G_\tau}(L(\omega_4) \otimes L(\omega_3)) \cong L(\tilde{\rho}) \oplus L(\omega_3)$.

Proof. The multiplicities of k in the socles are obvious. Since $L(\tilde{\rho})$ is injective (and projective), so is $L(\tilde{\rho}) \otimes L(\omega_4)$, and the multiplicity if $Q(\lambda')$ as a direct summand is equal to the G_τ-composition multiplicity $[L(\omega_4) \otimes L(\lambda'){:}\ L(\tilde{\rho})]_{G_\tau}$. This gives (a) and also gives the multiplicities of $(L(\tilde{\rho})$ in (b) and (c). For the other parts, we use the G_τ-multiplicities $[L(\omega_4) \otimes L(\omega_4){:}\ L(\omega_4)]_{G_\tau} = 4$ and $[L(\omega_4) \otimes (\omega_4){:}\ L(\omega_3)]_{G_\tau} = 2$. Since these numbers are smaller than the dimension of the smallest nontrivial G-module, it follows that $\mathrm{Hom}_{G_\tau}(L(\omega_4), L(\omega_4) \otimes L(\omega_4)) \cong \mathrm{Hom}_G(L(\omega_4), L(\omega_4) \otimes L(\omega_4))$ and $\mathrm{Hom}_{G_\tau}(L(\omega_3), L(\omega_4) \otimes L(\omega_4)) \cong \mathrm{Hom}_G(L(\omega_3), L(\omega_4) \otimes L(\omega_4))$. To see that $\mathrm{Hom}_G(L(\omega_4), L(\omega_4) \otimes L(\omega_4))$ is one-dimensional, one computes the multiplicity of $H^0(\omega_4)$ in a good filtration of its tensor square (See 4.2). The fact that $\mathrm{Hom}_G(L(\omega_3), L(\omega_4) \otimes L(\omega_4)) = 0$ is proved in [12, proof of (5) in "Completion of the Proof"], but we will sketch the argument here. By 1.5(2) and Lemma 4.5(a), this fact is equivalent to the vanishing $\mathrm{Ext}^1_G(L(\omega_2), L(\omega_1 + \omega_4))$. This can be seen from the Jantzen sum formula [7, II, 8.19] for $V(\omega_2)$ (cf. [12, Lemma 3.8]). The first layer in the Jantzen filtration has composition factors $L(\omega_1 + \omega_4), (2)L(\omega_1)$, and possibly k. The layers of the Jantzen filtration are self-dual, so if the Ext group above were not trivial, this layer would have to be semisimple, leading to the conclusion that $\dim_k \mathrm{Ext}^1_G(L(\omega_2), L(\omega_1)) = 2$. However, from 1.5(2) and 1.7(2) we have $\mathrm{Ext}^1_G(L(\omega_2), L(\omega_1)) \cong \mathrm{Ext}^1_G(L(\omega_3), L(\omega_4))$, which, by the structure of $V(\omega_3)$, is one-dimensional.

This proves (b) and shows that $L(\omega_4)$ is not in the socle in (c). Therefore, in a complementary direct summand to $L(\tilde{\rho})$ in $L(\omega_3) \otimes L(\omega_4)$, the only possible simple submodules are isomorphic to $L(\omega_3)$, and by self-duality the same is true of simple quotients. Since $[L(\omega_3) \otimes L(\omega_4){:}\ L(\omega_3)]_{G_\tau} = 2$, the fact that $L(\omega_3)$ and $L(\tilde{\rho})$ are not the only composition factors of $L(\omega_3) \otimes L(\omega_4)$ implies that $L(\omega_3)$ occurs with multiplicity one in $\mathrm{soc}\, L(\omega_3) \otimes L(\omega_4)$, proving (c).

4.8. Next we consider the possible composition factors of $\mathrm{Ext}^1_{G_\tau}(L(\omega_4), L(\omega_3))$. Lemma 4.7 (a) has the consequence

$$\mathrm{Ext}^1_{G_\tau}(L(\omega_4), L(\omega_3)) \cong \mathrm{Hom}_{G_\tau}(L(\omega_4), [L(\tilde{\rho}) \otimes L(\omega_4)]/\mathrm{soc}[L(\tilde{\rho}) \otimes L(\omega_4)]). \quad (1)$$

First, this shows that every composition factor of $\mathrm{Ext}^1_{G_\tau}(L(\omega_4), L(\omega_3))$ is a G_τ-trivial composition factor of $\mathrm{Hom}(L(\omega_4), L(\tilde{\rho}) \otimes L(\omega_4)) \cong L(\tilde{\rho}) \otimes L(\omega_4) \otimes L(\omega_4)$. Second, we note that, for any $\lambda' \in X_\tau$ and any G-module M, the evaluation map

$$\mathrm{Hom}_{G_\tau}(L(\lambda'), M) \otimes L(\lambda') \to M \tag{2}$$

is injective. Indeed, it follows from 1.8 (with $\nu^0 = 0$, $\mu^0 = \lambda'$, and $L(\tau\overline{\mu})$ replaced by $\mathrm{Hom}_{G_\tau}(L(\lambda'), M)$) that a simple submodule of the tensor product must be of the form $L(\tau\nu) \otimes L(\lambda')$, where $L(\tau\nu)$ is a simple submodule of $\mathrm{Hom}_{G_\tau}(L(\lambda'), M)$, and so in particular it cannot lie in the kernel of the evaluation map. Applying this to (1), we see that for each composition factor $L(\tau\nu)$ of $\mathrm{Ext}^1_{G_\tau}(L(\omega_4), L(\omega_3))$, we must have a composition factor $L(\tau\nu) \otimes L(\omega_4)$ of $L(\tilde{\rho}) \otimes L(\omega_4)$. With these two criteria in mind, inspection of the composition factors of $L(\tilde{\rho}) \otimes L(\omega_4) \otimes L(\omega_4)$ and $L(\tilde{\rho}) \otimes L(\omega_4)$ reveals that the only possible composition factors of $\mathrm{Ext}^1_{G_\tau}(L(\omega_4), L(\omega_3))$ are

$$k,\ L(\omega_1),\ L(2\omega_4),\ \text{and}\ L(\omega_2). \tag{3}$$

(For the moment, we say nothing about multiplicities.)

4.9.

LEMMA. $\mathrm{Ext}^1_{G_\tau}(L(\omega_4), L(\omega_3)) \cong k$.

Proof. Let $E = \mathrm{Ext}^1_{G_\tau}(L(\omega_4), L(\omega_3))$. From the structure of $V(\omega_3)$ and 1.5(2) we obtain

$$k \cong \mathrm{Ext}^1_G(L(\omega_3), L(\omega_4)) \cong \mathrm{Hom}_G(k, E), \tag{1}$$

showing that k appears in soc E with multiplicity 1.

We claim next that $L(\omega_1)$ does not appear in soc E. By 1.5(2), this is equivalent to showing that $\mathrm{Ext}^1(L(\omega_1 + \omega_4), L(\omega_3)) = 0$. By considering composition factors, we see that $2\omega_4$ is a maximal weight of $\mathrm{rad}\, V(\omega_1 + \omega_4)$. Also, since $V(\omega_1 + \omega_4) \subseteq V(\omega_1) \otimes V(\omega_4)$ and $L(\omega_1 + \omega_4) \cong L(\omega_1) \otimes L(\omega_4)$ it follow that $\mathrm{rad}\, V(\omega_1 + \omega_4) \subseteq \mathrm{rad}\, V(\omega_1) \otimes V(\omega_4) \cong L(\omega_4) \otimes L(\omega_4)$. Since the unique line of weight $2\omega_4$ in $L(\omega_4) \otimes L(\omega_4)$ generates a submodule isomorphic to $V(2\omega_4)$, we may conclude that $\mathrm{rad}\, V(\omega_1 + \omega_4)$ has a submodule isomorphic to $V(2\omega_4)$. Then, since $[V(\omega_1 + \omega_4): L(\omega_3)] = 1 = [V(2\omega_4): L(\omega_3)]$, the claim is proved.

The next step is to show $\mathrm{Hom}_G(L(2\omega_4), E) = 0$, which by 1.5(2) is equivalent to

$$\mathrm{Ext}^1_G(L(3\omega_4), L(\omega_3)) = 0. \tag{2}$$

It is easily checked that $V(3\omega_3)$ has composition factors $L(3\omega_3)$, $L(\omega_2)$, $(2)L(\omega_1+\omega_4)$, $L(2\omega_4)$, $L(\omega_3)$, $(3)L(\omega_1)$, $L(\omega_4)$, and $(2)k$. Therefore, ω_2 is maximal among the weights of rad $V(3\omega_4)$, so the one-dimensional ω_2 weight space generates a nonzero homomorphic image V of $V(\omega_2)$. In order to prove (2), it is enough to prove $[V\colon L(\omega_3)] \neq 0$. Now $V(3\omega_4) \subseteq V(2\omega_4) \otimes V(\omega_4)$ and $[V(2\omega_4) \otimes V(\omega_4)\colon L(\omega_2)] = 1$. Since $V(2\omega_4)$ has a composition factor $L(\omega_3)$, and $[L(\omega_3) \otimes L(\omega_4) : L(\omega_2)] = 1$, it follows that $L(\omega_3) \otimes L(\omega_4)$ contains a nonzero homomorphic image of V. Lemma 4.7(c) and the fact that $[V(\omega_2)\colon L(\bar{\rho})] = 0$ now imply that this homomorphic image has a submodule isomorphic to $L(\omega_3)$. This proves (2).

We have proved so far that, of the four possible composition factors 4.8(3) of E, the simple modules $L(\omega_1)$ and $L(2\omega_4)$ do not appear in the socle and k appears once. Since neither $L(\omega_1)$ nor $L(2\omega_4)$ extends the trivial module, it follows that if either of them were a composition factor of E, then E would necessarily have a composition factor $L(\omega_2)$ as well, in the first or second socle. Thus, the lemma will be proved once we show that E has no submodule E_1 with $k \subset E_1$ and $E_1/k \cong L(\omega_2)$.

Suppose that E_1 exists. We shall derive a contradiction from considering the D-module structure of E. From Lemma 4.4(b) and what we know about extensions from the structures of $\tilde{V}(\tilde{\omega}_i)$, $1 \leq i \leq 4$, we see that as a D-module, $L(\omega_2)^{(-1)}$ has a submodule isomorphic to $\tilde{L}(\tilde{\omega}_1)$, and hence so does $E^{(-1)}$, since $\operatorname{Ext}^1_D(\tilde{L}(\tilde{\omega}_1), k) = 0$. Therefore, by the Hochschild–Serre 5-term sequence for (D, D_1), we have

$$\operatorname{Ext}^1_D\big(\tilde{L}(\tilde{\omega}_2 + 2\tilde{\omega}_1), \tilde{L}(\tilde{\omega}_1 + \tilde{\omega}_3 + \tilde{\omega}_4)\big) \cong \operatorname{Hom}_D\big(\tilde{L}(\tilde{\omega}_1), E^{(-1)}\big) \neq 0. \tag{3}$$

We shall now calculate $\operatorname{Ext}^1_D(\tilde{L}(\tilde{\omega}_2 + 2\tilde{\omega}_1), \tilde{L}(\tilde{\omega}_1 + \tilde{\omega}_3 + \tilde{\omega}_4))$ in a different way, showing that it is zero, which will contradict (3). By considering the composition factors of $\tilde{V}(\tilde{\omega}_2) \otimes \tilde{V}(2\tilde{\omega}_1)$ and its submodule $\tilde{V}(\tilde{\omega}_2 + 2\tilde{\omega}_1)$, we see that the weight $2\tilde{\omega}_2$ appears with multiplicity one in the radicals of both and is maximal there. Therefore rad $\tilde{V}(\tilde{\omega}_2 + 2\tilde{\omega}_1)$ contains a nonzero homomorphic image V of $\tilde{V}(2\tilde{\omega}_2)$. Now $\tilde{V}(2\tilde{\omega}_1)$ has a composition factor $\tilde{L}(\tilde{\omega}_2)$, and the subquotient $\tilde{L}(\tilde{\omega}_2) \otimes \tilde{L}(\tilde{\omega}_2)$ of rad$[\tilde{V}(\tilde{\omega}_2) \otimes \tilde{V}(2\tilde{\omega}_1)]$ certainly has a weight $2\tilde{\omega}_2$, so this weight space generates a homomorphic image $\bar{V}$ of V inside $\tilde{L}(\tilde{\omega}_2) \otimes \tilde{L}(\tilde{\omega}_2)$. It is clear that $\bar{V}$ is equal to the image of the composition

$$\phi\colon \tilde{V}(2\tilde{\omega}_2) \subset \tilde{V}(\tilde{\omega}_2) \otimes \tilde{V}(\tilde{\omega}_2) \twoheadrightarrow \tilde{L}(\tilde{\omega}_2) \otimes \tilde{L}(\tilde{\omega}_2). \tag{4}$$

Since rad $\tilde{V}(\tilde{\omega}_2) \cong k \oplus k$, the only possible composition factors of Ker ϕ are k and $\tilde{L}(\tilde{\omega}_2)$, none of which is $\tilde{L}(\tilde{\omega}_1 + \tilde{\omega}_3 + \tilde{\omega}_4)$. Therefore, by

Lemma 4.3(e) and (f),

$$1 = \left[\tilde{V}(\tilde{\omega}_2 + 2\tilde{\omega}_1) : \tilde{L}(\tilde{\omega}_1 + \tilde{\omega}_3 + \tilde{\omega}_4)\right] \geq \left[V : (\tilde{\omega}_1 + \tilde{\omega}_3 + \tilde{\omega}_4)\right]$$
$$\geq \left[\bar{V} : \tilde{L}(\tilde{\omega}_1 + \tilde{\omega}_3 + \tilde{\omega}_4)\right] = \left[\tilde{V}(2\tilde{\omega}_2) : \tilde{L}(\tilde{\omega}_1 + \tilde{\omega}_3 + \tilde{\omega}_4)\right] = 1, \quad (5)$$

which shows that $[\tilde{V}(\tilde{\omega}_2 + 2\tilde{\omega}_1)/\mathrm{rad}^2\, \tilde{V}(\tilde{\omega}_2 + 2\tilde{\omega}_1) : \tilde{L}(\tilde{\omega}_1 + \tilde{\omega}_3 + \tilde{\omega}_4)] = 0$, contradicting (3), and completing the proof of the lemma.

Extensions

4.10. With the results of the last section, we can now apply 1.5(2). Write $\lambda = \lambda^0 + \tau\bar{\lambda}$, $\mu = \mu^0 + \tau\bar{\mu}$.

If $\lambda^0 = \mu^0$, then, by 1.5(2) and 1.7(2), we obtain

$$\mathrm{Ext}^1_G(L(\lambda), L(\mu)) \cong \mathrm{Ext}^1_G(L(\bar{\lambda}), L(\bar{\mu})). \quad (1)$$

Thus, we are reduced to the case $\lambda^0 \neq \mu^0$. Then the first and fourth terms of 1.5(2) vanish, leaving

$$\mathrm{Ext}^1_G(L(\lambda), L(\mu)) \cong \mathrm{Hom}_G\left(L(\bar{\lambda}), \mathrm{Ext}^1_{G_\tau}(L(\lambda^0), L(\mu^0))^{(\tau^{-1})} \otimes L(\bar{\mu})\right). \quad (2)$$

It is immediate from this that if one of λ^0 or μ^0 is $\tilde{\rho}$, then $\mathrm{Ext}^1_G(L(\lambda), L(\mu)) = 0$.

Three cases remain:

(i) $\{\lambda^0, \mu^0\} = \{0, \omega_4\}$

(ii) $\{\lambda^0, \mu^0\} = \{0, \omega_3\}$

(iii) $\{\lambda^0, \mu^0\} = \{\omega_3, \omega_4\}$.

(iii) By Lemma 4.9 and (2) we have

$$\mathrm{Ext}^1_G(L(\lambda), L(\mu)) \cong \begin{cases} k & \text{if } \bar{\lambda} = \bar{\mu}, \\ 0 & \text{otherwise}. \end{cases} \quad (3)$$

Let $\bar{\lambda} = \lambda^1 + \tau\lambda''$ and $\bar{\mu} = \mu^1 + \mu''$.

(i) Then, by Lemma 4.5(a),

$$\mathrm{Ext}^1_G(L(\lambda), L(\mu)) \cong \mathrm{Hom}_G(L(\bar{\lambda}), L(\omega_4) \otimes L(\bar{\mu})), \quad (4)$$

and by 1.8(1) and Lemma 4.7, this space is zero unless $\lambda'' = \mu''$, in which case its dimension is given according to the Table IV.

TABLE IV

(λ^1, μ^1)	0	ω_4	ω_3	$\omega_3 + \omega_3$
0	0	1	0	0
ω_4	1	1	0	0
ω_3	0	0	1	1
$\omega_3 + \omega_4$	0	0	1	2

(ii) Substituting the result of Lemma 4.6 into (2) yields

$$\mathrm{Ext}^1_G(L(\lambda), L(\mu)) \cong \mathrm{Hom}_G\big(L(\bar{\lambda}), L(\bar{\mu}) \oplus L(\omega_1) \otimes L(\bar{\mu})\big). \quad (5)$$

We consider the space $\mathrm{Hom}_G(L(\bar{\lambda}), L(\omega_1) \otimes L(\bar{\mu}))$. If $\lambda^1 \neq \mu^1$, then even the space of G_τ-maps is zero. If $\lambda^1 = \mu^1$, then the space is isomorphic to $\mathrm{Hom}_G(L(\lambda''), L(\omega_4) \otimes L(\mu''))$, by 1.8(1), and we are in the same situation as in (4), but with $\bar{\lambda}$ and $\bar{\mu}$ replaced by λ'' and μ'', and the dimension is given in Table IV. For the dimension in (5) we must add 1 to the diagonal entries to account for the extra summand $L(\bar{\mu})$.

Thus, we have shown that if $\lambda = \lambda^0 + \tau\lambda^1 + \tau^2\lambda^2 + \tau^3\lambda'''$ and $\mu = \mu^0 + \tau\mu^1 + \tau^2\mu^2 + \tau^3\mu'''$, with $\{\lambda^0, \mu^0\} = \{0, \omega_3\}$, then $\mathrm{Ext}^1_G(L(\lambda), L(\mu)) = 0$ unless $\lambda^1 = \mu^1$ and $\lambda''' = \mu'''$ and, in this case, the dimension is given in Table V.

We have proved the following result.

Proposition. *Let* $\lambda = \Sigma_i \tau^i \lambda^i$ *and* $\mu = \Sigma_j \tau^j \mu^j$. *Then* $\mathrm{Ext}^1_G(L(\lambda), L(\mu)) = 0$ *unless*

$$\lambda - \mu = \tau^m(\lambda^m - \mu^m) + \tau^{m+1}(\lambda^{m+1} - \mu^{m+1}) + \tau^{m+2}(\lambda^{m+2} - \mu^{m+2}),$$

where $m \geq 0$, $\lambda^m \neq \mu^m$ *and* λ^m, $\mu^m \neq \omega_3 + \omega_4$. *If these conditions hold then the dimension of* $\mathrm{Ext}^1_G(L(\lambda), L(\mu))$ *is given as follows.*

(i) *If* $\lambda^m - \mu^m = \pm\omega_4$, *then the dimension is zero unless* $\lambda^{m+2} = \mu^{m+2}$, *in which case it is given in Table IV, on replacing* (λ^1, μ^1) *by* $(\lambda^{m+1}, \mu^{m+1})$.

TABLE V

(λ^2, μ^2)	0	ω_4	ω_3	$\omega_3 + \omega_4$
0	1	1	0	0
ω_4	1	2	0	0
ω_3	0	0	2	1
$\omega_3 + \omega_4$	0	0	1	3

(ii) If $\lambda^m - \mu^m = \pm\omega_3$, then the dimension is zero unless $\lambda^{m+1} = \mu^{m+1}$, in which case it is given in Table V, on replacing (λ^2, μ^2) by $(\lambda^{m+2}, \mu^{m+2})$.

(iii) If $\lambda^m - \mu^m = \pm(\omega_3 - \omega_4)$, then the dimension is one if $\lambda - \tau^m\lambda^m = \mu - \tau^m\mu^m$ and zero otherwise.

4.11. We can combine our results on G_τ-extensions (4.5(a), 4.6, 4.9) with results in 1.9(1) to compute the G_1-extensions (cf. [8, Section 5]).

TABLE VI

	(0, 0)	(0, 4)	(0, 3)	$(0, \tilde{\rho})$	(4, 0)	(4, 4)	(4, 3)	$(4, \tilde{\rho})$
(0, 0)	0	$L(2\omega_4)$	$k \oplus L(2\omega_1)$	0	0	k	0	0
(0, 4)	$L(2\omega_4)$	0	k	0	k	k	0	0
(0, 3)	$k \oplus L(2\omega_1)$	k	0	0	0	0	k	k
$(0, \tilde{\rho})$	0	0	0	0	0	0	k	$k \oplus k$
(4, 0)	0	k	0	0	0	$L(2\omega_4)$	$k \oplus L(2\omega_1)$	0
(4, 4)	k	k	0	0	$L(2\omega_4)$	0	k	0
(4, 3)	0	0	k	k	$k \oplus L(2\omega_1)$	k	0	0
$(4, \tilde{\rho})$	0	0	k	$k \oplus k$	0	0	0	0
(3, 0)	$k \oplus L(2\omega_4)$	0	0	0	k	0	0	0
(3, 4)	0	$k \oplus L(2\omega_4)$	0	0	0	k	0	0
(3, 3)	0	0	$k \oplus L(2\omega_4)$	0	0	0	k	0
$(3, \tilde{\rho})$	0	0	0	$k \oplus L(2\omega_4)$	0	0	0	k
$(\tilde{\rho}, 0)$	0	0	0	0	0	0	0	0
$(\tilde{\rho}, 4)$	0	0	0	0	0	0	0	0
$(\tilde{\rho}, 3)$	0	0	0	0	0	0	0	0
$(\tilde{\rho}, \tilde{\rho})$	0	0	0	0	0	0	0	0

	(3, 0)	(3, 4)	(3, 3)	$(3, \tilde{\rho})$	$(\tilde{\rho}, 0)$	$(\tilde{\rho}, 4)$	$(\tilde{\rho}, 3)$	$(\tilde{\rho}, \tilde{\rho})$
(0, 0)	$k \oplus L(2\omega_4)$	0	0	0	0	0	0	0
(0, 4)	0	$k \oplus L(2\omega_4)$	0	0	0	0	0	0
(0, 3)	0	0	$k \oplus L(2\omega_4)$	0	0	0	0	0
$(0, \tilde{\rho})$	0	0	0	$k \oplus L(2\omega_4)$	0	0	0	0
(4, 0)	k	0	0	0	0	0	0	0
(4, 4)	0	k	0	0	0	0	0	0
(4, 3)	0	0	k	0	0	0	0	0
$(4, \tilde{\rho})$	0	0	0	k	0	0	0	0
(3, 0)	0	$L(2\omega_4)$	$k \oplus L(2\omega_1)$	0	0	0	0	0
(3, 4)	$L(2\omega_4)$	0	k	0	0	0	0	0
(3, 3)	$k \oplus L(2\omega_1)$	k	0	0	0	0	0	0
$(3, \tilde{\rho})$	0	0	0	0	0	0	0	0
$(\tilde{\rho}, 0)$	0	0	0	0	0	$L(2\omega_4)$	$k \oplus L(2\omega_1)$	0
$(\tilde{\rho}, 4)$	0	0	0	0	$L(2\omega_4)$	0	k	0
$(\tilde{\rho}, 3)$	0	0	0	0	$k \oplus L(2\omega_1)$	k	0	0
$(\tilde{\rho}, \tilde{\rho})$	0	0	0	0	0	0	0	0

PROPOSITION. *The G-modules* $\mathrm{Ext}^1_{G_1}(L(\lambda), L(\mu))$ *for* $\lambda, \mu \in X_1$ *are given in Table* VI, *in which, for example, the symbol* $(3, \tilde{\rho})$ *stands for* $L(\omega_3 + \tau\tilde{\rho})$.

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