Write your proofs using *complete English sentences* as well as mathematical formulae.
Work which the grader cannot follow may receive a grade of zero.
In this exam F denotes a field and $\mathbb{R}$ denotes the field of real numbers.

Name: ____________________

1. (8 points) In $M_{2\times 2}(\mathbb{R})$, consider the set

$$X = \{\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 \\ 4 & -2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\}$$

of four matrices. Find a subset of X which is a basis for the span of X.

2. Let the linear map $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be given by $T(f(x)) = f(x) + f'(x) + f''(x)$.
   (a) (4 points) By computing with the matrix of T or otherwise, prove that T is invertible.
   (b) (4 points) Compute $T^{-1}(x^2 - x + 2)$.

3. (8 points) Determine whether the matrix

$$A = \begin{pmatrix} -1 & -2 & 2 & 2 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \in M_{4\times 4}(\mathbb{R})$$

is diagonalizable or not. If so, find a basis of $\mathbb{R}^4$ consisting of eigenvectors of A.

4. Give a proof or a counterexample for each of the following statements.
   (a) (2 points) The determinant is a linear map from $M_{n\times n}(F)$ to F.
   (b) (2 points) The union of two subspaces of a vector space is a subspace.
   (c) (2 points) The sum of two surjective linear maps is surjective.
   (d) (2 points) If $T : V \to W$ and $U : W \to Z$ are linear maps such that T is injective and U is surjective, then UT is either injective or surjective.

5. (8 points) Use the Gram-Schmidt procedure to find an orthonomal basis of the subspace of $\mathbb{R}^4$ spanned by $(1, -1, 0, 0)$, $(0, 0, 1, -1)$, $(0, 1, -1, 0)$.