Suzuki Groups

Graduate Algebra Seminar,
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1. Geometry of symplectic 3-space

We describe Tits' construction [1] of the Suzuki groups.

- V be a 4-diml. vector space, coordinates x_i, $i = 0, 1, 2, 3$.
- W 2-diml. subspace, $\wedge^2 W$ is a point of $\mathbb{P}(\wedge^2 V)$.
- If W is spanned by (a_0, a_1, a_2, a_3) and (b_0, b_1, b_2, b_3) then the "Plücker" coordinates. of W are $(p_{01} : p_{02} : p_{03} : p_{12} : p_{13} : p_{23})$, with $p_{ij} = a_i b_j - a_j b_i$.
- These coordinates satisfy the quadratic form

$$p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 0. \tag{1}$$

and form the *Klein Quadric* $\widehat{Q}$

Assume that V has a nonsingular alternating bilinear form and x_i are symplectic coordinates so that the matrix of the form is $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$.

- A 2-subspace is t.i. iff $p_{03} + p_{12} = 0$.

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- The t.i. 2-subspaces form the intersection $Q = \widehat{Q} \cap H$ of $\widehat{Q}$ with the hyperplane H of the above equation.
- The equation of Q is

$$p_{01}p_{23} - p_{02}p_{13} - p_{03}^2 = 0. \tag{2}$$

2. The isogeny τ

Suppose the field of V is $k = \overline{\mathbf{F}}_2$.

- $z = (0:0:1:1:0:0) \in H \setminus Q$ is the radical of the (alternating) bilinear form associated with (2).
- z is the common point of intersection of every tangent hyperplane to Q in H.
- Projection $H \to V_1 = H/z$ gives a bijection $Q \to \mathbb{P}(V_1)$.
- $$\alpha : \{\text{2-diml tot. isotropic subspaces of } V\} \cong \mathbb{P}(V_1).$$
- The alternating form induced on V_1 is nonsingular.
- $y_0 = \overline{p}_{01}$, $y_1 = \overline{p}_{02}$, $y_2 = \overline{p}_{13}$, $y_3 = \overline{p}_{23}$ are symplectic coords for V_1.

V_1 is a lot like V !

- Identify V with V_1 by their symplectic coordinates.
- This fixes an isomorphism $\mathrm{Sp}(V) \cong \mathrm{Sp}(V_1)$.
- Under this identification, the induced action on V_1 induces an endomorphism τ of $\mathrm{Sp}(V)$.

- $x = (a_0 : a_1 : a_2 : a_3)$. Assume for simplicity $a_0 \neq 0$.
- $x^\perp$ is spanned by x, $(0 : a_0 : 0 : a_2)$ and $(0 : 0 : a_0 : a_1)$.
- The set of t.i. 2-subspaces which contain x form an isotropic line in Q, spanned by $(a_0^2 : 0 : a_0a_2 : a_0a_2 : a_0a_3 + a_1a_2 : a_2^2)$ and $(0 : a_0^2 : a_0a_1 : a_0a_1 : a_1^2 : a_0a_3 + a_1a_2)$.
- This line maps to the t.i. line spanned by $(a_0^2 : 0 : a_0a_3 + a_1a_2 : a_2^2)$ and $(0 : a_0^2 : a_1^2 : a_0a_3 + a_1a_2)$.
- $\beta : \mathbb{P}(V) \to \{\text{t.i. lines of } \mathbb{P}(V)\}$.
- Compute Plučker coordinates: $\alpha(\beta(x)) = (a_0^2 : a_1^2 : a_2^2 : a_3^2)$.
- Conclude that β is a bijection and τ^2 is the Frobenius map, given by squaring all matrix entries.
- τ is an *isogeny* of algebraic groups.

3. The groups $G(n)$

- $G(n) =$ the subgroup of $\mathrm{Sp}(V)$ fixed by τ^n.
- $G(2t) \cong \mathrm{Sp}(4, 2^t)$.
- $G(2m+1) = \mathrm{Sz}(2^{2m+1})$, *Suzuki* groups.

For $x = (a_0 : a_1 : a_2 : a_3)$, set $x^{(2^i)} = ({a_0}^{2^i} : {a_1}^{2^i} : {a_2}^{2^i} : {a_3}^{2^i})$. Then $G(2m+1)$ preserves the set

$$\mathcal{T} = \mathcal{T}(2m+1) = \{x \mid x = x^{(2^{2m+1})}, x^{(2^{m+1})} \in \beta(x)\}.$$

This is called the *Tits ovoid*. It consists of $(0:0:0:1)$ and the points $(1:x:y:z)$ satisfying
$$z = xy + x^{2^{m+1}+2} + y^{2^{m+1}}. \tag{3}$$

- $|\mathcal{T}| = q^2 + 1$, where $q = 2^{2m+1}$.
- The action of $\mathrm{Sz}(2^{2m+1})$ on $\mathcal{T}$ is doubly transitive.
- $|\mathrm{Sz}(2^{2m+1})| = q^2(q-1)(q^2+1)$.
- $\mathrm{Sz}(2^{2m+1})$ is a simple group, for $m \geq 1$. ($\mathrm{Sz}(2)$ is isomorphic to the Frobenius group of order 20.)

References

[1] J. Tits, Les Groupes simples de Suzuki et de Ree, Sem. Bourbaki, Exp. 210, (1961).