4.3.10. We shall use the symbol "0" to denote both the scalar zero and the zero $n \times n$ matrix. Suppose $M \in M_{n\times n}(C)$ is nilpotent. Then for some positive integer $k$ we have $M^k = 0$. Then

$$0 = \det(0) = \det(M^k) = \det(M)^k, \tag{1}$$

where the last equality is by the multiplicative property of determinants. The above equation shows that the $k$-th power of the scalar $\det(M)$ is 0, which forces $\det(M) = 0$.

5.1.17.
(a) We have $T(T(A)) = (A^t)^t = A$. Thus if $\lambda$ is an eigenvalue and $A$ is an eigenvector for $\lambda$, we have

$$A = T(T(A)) = T(\lambda A) = \lambda(T(A) = \lambda^2 A. \tag{2}$$

Since $A$ is a nonzero vector, it follows that $\lambda^2 = 1$, so $\lambda = \pm 1$.
(b) For $\lambda = 1$, the eigenvectors are the matrices such that $A^t = A$, namely the symmetric matrices. For $\lambda = -1$, the eigenvectors are the matrices such that $A^t = -A$, namely the skew-symmetric matrices.
(c) In $M_{2\times 2}(R)$, a basis for the space of symmetric matrices is

$$\beta_1 = \{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\} \tag{3}$$

and the space of skew-symmetric matrices has basis

$$\beta_{-1} = \{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\} \tag{4}$$

Then $\beta = \beta_1 \cup \beta_{-1}$ is the required basis in which $[T]_\beta^\beta$ is diagonal.
(d) Let $E_{ij} \in M_{n\times n}(R)$ denote matrix with a single nonzero entry 1 in row $i$, column $j$. The $E_{ij}$ are the elements of the standard basis of $M_{n\times n}(R)$.

For $i < j$, let $S_{ij} = E_{ij} + E_{ji}$. Consider the set

$$\beta_1 = \{E_{ii} : i = 1 \dots n\} \cup \{S_{ij} : 1 \le i < j \le n\}. \tag{5}$$

This set has $n(n+1)/2$ elements. It is clear that the elements are symmetric.

For $i < j$, let $R_{ij} = E_{ij} - E_{ji}$. Consider also the set

$$\beta_{-1} = \{R_{ij} : 1 \le i < j \le n\}. \tag{6}$$

This set has $n(n-1)/2$ elements. It is clear that the elements are skew-symmetric.

We claim that the union $\beta$ of the two sets above is a basis of $M_{n\times n}(R)$. Then it will clear that $[T]_\beta^\beta$ is diagonal. To prove our claim, we note that $\beta$ has $\frac{n(n+1)}{2} + \frac{n(n-1)}{2} = n^2$ elements, so it suffices to show that $\beta$ generates $M_{n\times n}(R)$. From the equations

$$S_{ij} + R_{ij} = 2E_{ij}, \qquad S_{ij} - R_{ij} = 2E_{ji}, \tag{7}$$

and the fact that each $E_{ii}$ belongs to $\beta_1$, it follows that all the standard basis elements $E_{ij}$ are in span of $\beta$, so our claim is proved.