# The Divisor Matrix, Dirichlet Series and $\mathrm{SL}(2, \mathbf{Z})$

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# Overview

The Divisor matrix

- $D = (d_{i,j})_{i,j \in \mathbf{N}}$ defined by

$$d_{i,j} = \begin{cases} 1, & \text{if } i \text{ divides } j, \\ 0 & \text{otherwise.} \end{cases}$$

## The matrix ring $\mathcal{A}$

- $E :=$ free $\mathbf{Q}$-module with basis $\{e_n\}_{n\in\mathbf{N}}$.
- $\mathrm{End}_{\mathbf{Q}}(E) \cong \mathcal{A} :=$ the space of matrices $A = (a_{i,j})_{i,j\in\mathbf{N}}$, with rational entries, such that each column has only finitely many nonzero entries.
- $E^* \cong \mathbf{Q}^{\mathbf{N}}$, sequences of rational numbers, $f \in E^*$ is identified with the sequence $(f(e_n))_{n\in\mathbf{N}}$. Write $f(e_n)$ as $f(n)$ for short.
- $\mathcal{A}$ acts on the right of $E^*$ :

$$(fA)(n) = \sum_{m\in\mathbf{N}} a_{m,n} f(m), \qquad f \in \mathbf{Q}^{\mathbf{N}}, A \in \mathcal{A}.$$

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## Dirichlet Space and Dirichlet Ring

- $$\mathcal{DS} := \{f \in \mathbf{Q}^{\mathbf{N}} \mid (\exists C, c > 0)(\forall n)(|f(n)| \leq Cn^c)\}$$
- $f \in \mathcal{DS}$ if and only if the Dirichlet series $\sum_n f(n)n^{-s}$ converges for some complex number $s$.
- $\mathcal{DR} :=$ the subring of $\mathcal{A}$ consisting of all elements which leave $\mathcal{DS}$ invariant.

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## Sufficient conditions for $\mathcal{DR}$

*Let $A = (a_{i,j})_{i,j\in\mathbf{N}} \in \mathcal{A}$. Suppose that there exist positive constants $C$ and $c$ such that the following hold.*

(i) $a_{i,j} = 0$ *whenever* $i > Cj^c$.

(ii) *For all $i$ and $j$ we have* $|a_{i,j}| \leq Cj^c$.

*Then $A \in \mathcal{DR}$. Moreover, these conditions define a subring of $\mathcal{DR}$.*

- $\mathcal{DR}$ is not closed under inversion.

$$B = \begin{bmatrix} 1 & -1 & -1 & -1 & \cdots \\ 0 & 1 & -1 & -1 & \cdots \\ 0 & 0 & 1 & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \in \mathcal{DR},$$

$$B^{-1} = \begin{bmatrix} 1 & 1 & 2 & 4 & 8 & \cdots \\ 0 & 1 & 1 & 2 & 4 & \cdots \\ 0 & 0 & 1 & 1 & 2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \notin \mathcal{DR},$$

- $DR$ is not closed under conjugation by $\mathcal{A}^{\times}$.

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad C = \mathrm{diag}(1, 2, 4, 8, \ldots),$$

$$C^{-1}BC = \begin{bmatrix} 1 & 2 & 4 & 8 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \notin \mathcal{DR}.$$

- $D \in \mathcal{DR}$.
- $D$ acts on Dirichlet series as multiplication by the Riemann zeta function:

$$\sum_{n=1}^{\infty} \frac{(fD)(n)}{n^s} = \sum_{n=1}^{\infty} \frac{\left(\sum_{d|n} f(d)\right)}{n^s} = \zeta(s) \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

(in some half-plane).
- The inverse of $D$ is given by

$$d'_{i,j} = \begin{cases} \mu(j/i), & \text{if } i \text{ divides } j, \\ 0 & \text{otherwise.} \end{cases}$$

where $\mu$ is the Möbius function.

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# Overview

- Which subgroups of $\mathcal{DR}^\times$ contain $D$?
- Thompson (2006) showed $\langle D \rangle \leq Dih_\infty \leq \mathcal{DR}^\times$.
- The divisor matrix is a (locally) unipotent element of $\mathcal{DR}^\times$.
- Is there a subgroup of $\mathcal{DR}^\times$ isomorphic to $\mathrm{SL}(2, \mathbf{Z})$ which contains $D$ as a standard unipotent element?

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# Overview

## $\mathrm{SL}(2, \mathbf{Z})$

- $G := \mathrm{SL}(2, \mathbf{Z})$ is generated by the matrices

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \text{and} \quad T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

- $S$ has order 4 and $R := ST$ has order 6.
- $G = \langle S, R \mid S^4, R^6, S^2 = R^3 \rangle$

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## Main Theorem

*There exists a representation* $\rho : \mathrm{SL}(2, \mathbf{Z}) \to \mathcal{A}^\times$ *with the following properties.*

(a) *The underlying* $\mathbf{Q}\,\mathrm{SL}(2, \mathbf{Z})$*-module* $E$ *has an ascending filtration*

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots$$

*of* $\mathbf{Q}\,\mathrm{SL}(2, \mathbf{Z})$*-submodules such that for each* $i \in \mathbf{N}$*, the quotient module* $E_i/E_{i-1}$ *is isomorphic to the standard 2-dimensional* $\mathbf{Q}\,\mathrm{SL}(2, \mathbf{Z})$*-module.*

(b) $\rho(T) = D$.

(c) $\rho(Y)$ *is an integer matrix for every* $Y \in \mathrm{SL}(2, \mathbf{Z})$.

(d) $\rho(\mathrm{SL}(2, \mathbf{Z})) \subseteq \mathcal{DR}$.

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Module-theoretic steps in proof

- Find a "Jordan canonical form" of *D*.
- For each Jordan block construct an integral representation of $\mathrm{SL}(2, \mathbf{Z})$ satisfying the filtration condition and so that $T$ is represented by a matrix similar to the Jordan block.
- Form the direct sum. Then we will have a module satisying (a)-(c).
- Part (d) is not a module-theoretic statement, since $\mathcal{DR}$ is not closed under conjugation in $\mathcal{A}^{\times}$, so the main difficulty is to end up with the right *matrix* representation.

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Matrix part of proof

- Choose the right J.C.F of $D$. Order is important.
- Find the transition matrices from $D$ to its J.C.F. explicitly to show that they belong to $\mathcal{DR}$.
- For each Jordan block, find a matrix representation $\tau_1$ which gives the right module and find the transition matrices taking $\tau_1(T)$ to the standard Jordan block.
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# Overview

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$$A_k(m) := \{(m_1, m_2, \ldots, m_k) \in (\mathbf{N}\backslash\{1\})^k \mid m_1 m_2 \cdots m_k = m\}$$

- $\alpha_k(m) := |A_k(m)|$
- $\alpha_k(1) = 0$, $\alpha_k(m) = 0$ if $m < 2^k$ and $\alpha_k(2^k) = 1$.
- $(\zeta(s) - 1)^k = \sum_{n=2}^{\infty} \frac{\alpha_k(n)}{n^s}$.

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## Some identities

- $$\left(\sum_{d|m} \alpha_{k-1}(d)\right) = \alpha_k(m) + \alpha_{k-1}(m).$$
- $$\sum_{i=1}^{k-1} (-1)^{k-1-i} \sum_{d|m} \alpha_i(d) = \alpha_k(m) + (-1)^k \alpha_1(m).$$
- $$\sum_{k=1}^{v(m)} (-1)^k \alpha_k(m) = \begin{cases} \mu(m) & \text{if } m \neq 1, \\ 0 & \text{if } m = 1. \end{cases}$$
- Pick $c$ ($\approx 1.7286$) with $\zeta(c) = 2$. Then $\alpha_k(m) \leq m^c$ for all $k$ and $m$.

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Relation to $D$

- The $(1, m)$ entry of $(D - I)^k$ is equal to $\alpha_k(m)$
- More generally,

$$(d, m) \text{ entry of } (D - I)^k = \begin{cases} 0 & \text{if } d \nmid m, \\ \alpha_k(m/d) = (1, m/d) \text{ entry, if } d \mid m. \end{cases}$$

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# Overview

- $D$ is unitriangular. What is its JCF ?
- $$J := (J_{i,j})_{i,j \in \mathbf{N}}, \qquad J_{i,j} = \begin{cases} 1, & \text{if } j \in \{i, 2i\}, \\ 0 & \text{otherwise.} \end{cases}$$
- Think of $J$ as being the direct sum of infinite Jordan bocks, one for each odd integer.
- Let $Z := (z(i,j))_{i,j \in \mathbf{N}}$ be the matrix described in the following way. The odd rows have a single nonzero entry, equal to 1 on the diagonal. Let $i = 2^k d$ with $d$ odd. Then the $i^{\text{th}}$ row of $Z$ is equal to the $d^{\text{th}}$ row of $(D - I)^k$.

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- Let $Z := (z(i,j))_{i,j \in \mathbf{N}}$ be the matrix described in the following way. The odd rows have a single nonzero entry, equal to 1 on the diagonal. Let $i = 2^k d$ with $d$ odd. Then the $i^{\text{th}}$ row of $Z$ is equal to the $d^{\text{th}}$ row of $(D - I)^k$.

- $D$ is unitriangular. What is its JCF ?
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## Lemma

*The matrix $Z$ has the following properties:*

(a) $ZDZ^{-1} = J$.

(b) $z(i,j) = \delta_{i,j}$, *if $i$ is odd.*

(c) *If $i = d2^k$, where $d$ is odd and $k \geq 1$, then*

$$z(i,j) = \begin{cases} \alpha_k(j/d) & \text{if } d \mid j, \\ 0 & \text{otherwise.} \end{cases}$$

(d) $z(im, jm) = z(i,j)$ *whenever $m$ is odd.*

(e) *$Z$ is upper unitriangular.*

(f) $Z \in \mathcal{DR}$.

- We want $Z^{-1} \in \mathcal{DR}$.

**Theorem**

*Let $X$ be the diagonal matrix with $(i,i)$ entry equal to $(-1)^{v_2(i)}$, for $i \in \mathbf{N}$. Then $Z^{-1} = XZX$.*

- This theorem was discovered with the aid of a computer.
- It is proved by using the identities for the $\alpha_k(m)$.
- Consider the submatrix of entries whose row and column indices are powers of 2. Then $z(2^k, 2^\ell) = \alpha_k(2^\ell) = \binom{\ell-1}{k-1}$, so the equations for this submatrix reduce to the orthogonality relation for binomial coefficients:

$$\sum_{\ell=k}^{m} (-1)^{\ell+k} \binom{\ell-1}{k-1} \binom{m-1}{\ell-1} = \delta_{k,m}.$$

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Matrix part of proof

- Choose the right J.C.F of $D$. Order is important.
- Find the transition matrices from $D$ to its J.C.F. explicitly to show that they belong to $\mathcal{DR}$.
- For each Jordan block, find a matrix representation $\tau_1$ which gives the right module and find transition matrices $P$, $P^{-1}$ so that $P\tau_1(T)P^{-1}$ is the standard Jordan block.
- These have to be chosen in such a way that when the Jordan blocks are assembled, the result lies in $\mathcal{DR}$.

# Overview

- Let $J_\infty$ be the "infinite Jordan block"

$$(J_\infty)_{i,j} = \begin{cases} 1, & \text{if } j = i \text{ or } j = i+1, \\ 0 & \text{otherwise.} \end{cases}$$

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## Theorem

*There exists a representation* $\tau : \mathrm{SL}(2, Z) \to \mathcal{A}^{\times}$ *with the following properties.*

(a) *Let* $E_i$ *be the subspace of* $E$ *spanned by* $\{e_1, \ldots, e_{2i}\}$, $i \in \mathbf{N}$. *Then*

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots$$

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(b) $\tau(T) = J_\infty$.

(c) $\tau(Y)$ *is an integer matrix for every* $Y \in \mathrm{SL}(2, \mathbf{Z})$.

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Remarks

- When the Jordan blocks are interleaved, the $(i, j)$ entry of $\tau(Y)$ will be the $(2^{i-1}d, 2^{j-1}d)$ entries of the matrix of $Y$ on the direct sum, for every odd number $d$, so the exponential bound (d) is the condition to end up in $\mathcal{DR}$.
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- This representation is unique up to $\mathbf{Q}G$-isomorphism.

- Define $\{b_n\}_{n\geq 0}$ recursively by

$$b_0 = b_1 = 1, \quad b_n + \sum_{\substack{i,j\geq 1 \\ i+j=n}} b_i b_j = 0 \quad \text{for all } n \geq 2.$$

- These are Catalan numbers with signs:

$$b_m = (-1)^{m-1} \frac{1}{m} \binom{2(m-1)}{m-1}, \quad (m \geq 2), \quad b_1 = b_0 = 1.$$

- $g(t) \in \mathbf{Q}[[t]]$ defined by

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- The recurrence relations can be expressed as:

$$g(t)^2 + g(t) = t.$$

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$$\tilde{J} = \left[\begin{array}{c:c:c:c:c} B_0 & B_1 & B_2 & B_3 & \dots \\ \hdashline 0 & B_0 & B_1 & B_2 & \dots \\ \hdashline 0 & 0 & B_0 & B_1 & \dots \\ \hdashline 0 & 0 & 0 & B_0 & \dots \\ \hdashline \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right],$$

$$\tilde{S} = \mathrm{diag}(S, S, \dots),$$

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where $X^{(n)} \in M_2(\mathbf{Q})$ are repeated down the diagonals.
- $\tilde{S}$, $\tilde{J}$ and $\tilde{R}$ all belong to $\mathcal{U}$.
- There is a $\mathbf{Q}[[t]]$-algebra isomorphism

$$\gamma : \mathcal{U} \to M_2(\mathbf{Q}[[t]]) = M_2(\mathbf{Q})[[t]], \qquad U \mapsto \sum_{n=0}^{\infty} X^{(n)} t^n$$

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- We have:

$$\gamma(\tilde{S}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

$$\gamma(\tilde{J}) = \begin{bmatrix} 1 & 1 + g(t) \\ g(t) & 1 + t \end{bmatrix},$$

$$\gamma(\tilde{R}) = \begin{bmatrix} g(t) & 1 + t \\ -1 & -1 - g(t) \end{bmatrix}.$$

## Lemma

(a) $\tilde{S}^2 = -I$.

(b) $\tilde{R}^2 + \tilde{R} + I = 0$.

(c) *There exists a representation* $\tau_1$ *of* $\mathrm{SL}(2, \mathbf{Z})$ *such that* $\tau_1(S) = \tilde{S}$ *and* $\tau_1(T) = \tilde{J}$.

## Jordan form of $\tilde{J}$

- Find $P$ such that $P\tilde{J}P^{-1} = J_\infty$.
- For $n \in \mathbf{N}$, set $n^{\text{th}}$ row of $P$:= first row of $(\tilde{J} - I)^{n-1}$ ($(\tilde{J} - I)^0 = I$.)
- $$P(\tilde{J} - I) = (J_\infty - I)P.$$

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- To compute the powers of $\tilde{J} - I$ explicitly, use the isomorphism $\gamma$.
- Let $\tilde{H} = \gamma(\tilde{J} - I)$. Then

$$\tilde{H} = \begin{bmatrix} 0 & 1 + g(t) \\ g(t) & t \end{bmatrix}.$$

- Compute the powers of $\tilde{J} - I$ by diagonalizing $\tilde{H}$.
- The entries of $P$ are coefficients of the powers of $t$ in the top rows of the $\tilde{H}^n$.
- For $\ell \geq 3$ and $s \geq 0$, we have

$$p_{\ell,2s+1} = \binom{s-1}{\ell - s - 2},$$

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- $P^{-1}$ is found in a similar, but slightly more elaborate procedure.
- The entries of $P^{-1} = (q_{i,j})_{i,j\in\mathbf{N}}$ are: $q_{i,1} = \delta_{i,1}$ and $q_{i,2} = \delta_{i,2}$, for $i \in \mathbf{N}$. For $m \geq 3$ and $s \geq 0$, we have

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- For all $i$ and $j$ we have $|q_{i,j}| \leq 2^{3j}$ and $|p_{i,j}| \leq 2^{2j}$.
- So $\tau : \mathrm{SL}(2,\mathbf{Z}) \to \mathcal{A}^{\times}$ , $Y \mapsto P\tau_1(Y)P^{-1}$ gives the desired representation for each Jordan block of the divisor matrix.
- The proof the main theorem is complete.
- $\rho$ can be extended to a representation $\mathrm{GL}(2,\mathbf{Z}) \to \mathcal{DR}^{\times}$ (but not with integral coefficients).
- There is a group $B \cong$ upper triangular subgroup of $\mathrm{SL}(2,\mathbf{Q})$ such that $D \in B \leq \mathcal{DR}^{\times}$.

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# Overview

- Consider the matrix $\rho(-S)$, $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.
- Its top row $(a_1, a_2, \ldots)$ lies in $\mathcal{DS}$, with associated Dirichlet series

$$\varphi(s) := \sum_{n=1}^{\infty} a_n n^{-s}.$$

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$$a_n = \rho(-S)_{1,n} = \alpha_1(n) + \sum_{\ell \geq 4} (-1)^{\ell} \alpha_{\ell-1}(n) \sum_{k=2}^{\lfloor \frac{\ell}{2} \rfloor} b_k \binom{\ell - k - 2}{k - 2}.$$

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## The cubic equation relating $\zeta(s)$ and $\varphi(s)$

### Theorem

*In the half-plane* $\mathrm{Re}(s) > \max(1, \sigma_c)$, *we have*

$$(\zeta(s) - 1)\varphi(s)^2 + \zeta(s)\varphi(s) - \zeta(s)(\zeta(s) - 1) = 0.$$

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Orbit of $\zeta(s)$

- We have $1^{-s}.\rho(T) = \zeta(s)$ and $1^{-s}.\rho(-S) = \varphi(s)$.
- Let $\mathbf{Q}(\zeta(s), \varphi(s))$ be the subfield of the field of meromorphic functions of the half-plane $\mathrm{Re}(s) > \max(1, \sigma_c)$ generated by the functions $\zeta(s)$ and $\varphi(s)$.

**Theorem**

*For any element $Y \in G = \mathrm{SL}(2, \mathbf{Z})$ the Dirichlet series $1^{-s}.Y$ converges for $\mathrm{Re}(s) > \max(1, \sigma_c)$ and the sum function belongs to $\mathbf{Q}(\zeta(s), \varphi(s))$.*

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# Overview

## Redheffer's Matrix

- $R(n)$ is the $n \times n$ matrix:

$$R(n)_{i,j} = \begin{cases} 1, & \text{if } i \text{ divides } j \text{ or } j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

- Redheffer (1977): $\det R(n) = \sum_{m \leq n} \mu(m)$,
- The Riemann Hypothesis is equivalent to:

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Pictures of the divisor matrix
Cover of the current Notices of AMS from D. Cox's article "Visualizing the sieve of Eratosthones" .

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