THE ELEMENTARY DIVISORS OF THE INCIDENCE MATRICES OF POINTS AND LINEAR SUBSPACES IN $\mathbb{P}^n(\mathbb{F}_p)$.

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Abstract. The elementary divisors of the incidence matrices between points and linear subspaces of fixed dimension in $\mathbb{P}^n(\mathbb{F}_p)$ are computed.

Introduction

Let V be an $(n+1)$-dimensional vector space over $\mathbb{F}_p$. Let $\mathcal{L}_r$ denote the set of r-dimensional subspaces of V. Then $\mathcal{L}_1$ is the set of points of the projective space $\mathbb{P}(V)$ and $\mathcal{L}_n$ the set of hyperplanes. The group $G = \mathrm{GL}(V)$ acts transitively on each of the sets $\mathcal{L}_r$. Between any two of these sets we have an incidence relation given by inclusion of subspaces. This information can be encoded in an incidence matrix, a $0-1$ matrix which can be read in any commutative ring. Thus, it is natural to ask for the elementary divisors of this matrix as an integer matrix. In this paper we shall be concerned with the cases in which one of the sets is $\mathcal{L}_1$. The incidence relation can be interpreted as the map

$$\mathbb{Z}^{\mathcal{L}_r} \longrightarrow \mathbb{Z}^{\mathcal{L}_1} \tag{1}$$

between the associated $\mathbb{Z}G$-permutation modules which sends an r-subspace to the (formal) sum of the 1-subspaces it contains. This homomorphism has a finite cokernel and finding the elementary divisors of the incidence matrix is equivalent to finding a cyclic decomposition of the cokernel. The problem falls naturally into the two separate parts of describing the p-torsion and the p'-torsion.

The p'-torsion can be obtained as a corollary of James' theory [6] of cross-characteristic modular representations of $\mathrm{GL}(n,q)$, where q is a power of p. It is a cyclic group of order $\frac{p^r-1}{p-1}$, the number of 1-subspaces in an r-subspace. To see this consider the map $\epsilon : \mathbb{Z}^{\mathcal{L}_1} \longrightarrow \mathbb{Z}$ sending each 1-subspace to 1. The image of the incidence map (1) is mapped by ϵ onto $\frac{p^r-1}{p-1}\mathbb{Z}$. The result will therefore follow if we show that the intersection of the image of (1) with $\mathrm{Ker}\,\epsilon$ has index a

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power of p in $\operatorname{Ker}\epsilon$. Since $\operatorname{Ker}\epsilon$ is a pure subgroup of $\mathbb{Z}^{\mathcal{L}_1}$, this is equivalent to the statement that for each prime $l \neq p$, the reduction mod l of $\operatorname{Ker}\epsilon$ is in the image of the reduction mod l of the map (1). It is this last fact which has been proved in [6, Theorem 13.3 and 11.1, Submodule Theorem]. The same argument works with p replaced by q, showing that the cokernel of the map from r-spaces to 1-spaces of a finite vector space over $\mathbb{F}_q$ is the product of a cyclic group of order $(q^r-1)/(q-1)$ with a p-group.

In this paper we concentrate on the p-torsion.

Let d_i be the coefficient of $t^{i(p-1)}$ in the expansion of $(\sum_{j=0}^{p-1} t^j)^{n+1}$. Explicitly,

$$d_i = \sum_{j=0}^{\lfloor \frac{i(p-1)}{p} \rfloor} (-1)^j \binom{n+1}{j} \binom{n+i(p-1)-jp}{n}, \tag{2}$$

where the upper limit of the sum is the integer part of $\frac{i(p-1)}{p}$. We can now state our main result.

Theorem 1. *The p-elementary divisors of the incidence matrix between $\mathcal{L}_1$ and $\mathcal{L}_r$ are p^{r-i} with multiplicity d_i for $1 \leq i \leq r-1$.*

The case $r = n$ of point-hyperplane incidence was first computed by Black and List [1]. I thank R. Liebler for this reference and for useful discussions.

Our approach is to study the $\mathbb{Z}G$-module structure of the cokernels of the incidence maps. In §1, we recall some well known facts about eigenvalues of incidence matrices. In §2, we localize at p and examine closely the permutation modules and incidence maps over $\mathbb{Z}_p$ and $\mathbb{F}_p$. The principal objects of study are certain $\mathbb{Z}_p$-forms M_r in $\mathbb{Q}_pG$-modules isomorphic to the nontrivial component of $\mathbb{Q}_p^{\mathcal{L}_1}$. The submodule structure of the mod p reductions $\overline{M}_r$ of these lattices is the essential ingredient in the proof Theorem 1, which is given in §3.

§1. Incidence maps

Here we collect together some standard facts about incidence matrices in a form convenient for our use later. Let $\eta_{r,s} : \mathbb{Z}^{\mathcal{L}_r} \longrightarrow \mathbb{Z}^{\mathcal{L}_s}$ be the map sending an r-subspace to the (formal) sum of all s-subspaces incident with it. This is obviously a homomorphism of $\mathbb{Z}G$-modules. Then with respect to the bases $\mathcal{L}_r$ and $\mathcal{L}_s$ the matrices of $\eta_{r,s}$ and $\eta_{s,r}$ are transposes of each other.

Let $x \in \mathcal{L}_1$. Then in the equation

$$\eta_{r,1}\eta_{1,r}(x) = \sum_{x'} a_{xx'} x' \tag{3}$$

the coefficients are

$$a_{x,x'} = |\{h \in \mathcal{L}_r \mid x, x' \subseteq h\}| = \begin{cases} \begin{bmatrix} n-1 \\ r-2 \end{bmatrix}_p & \text{if } x \neq x' \\ \begin{bmatrix} n \\ r-1 \end{bmatrix}_p & \text{if } x = x'. \end{cases} \tag{4}$$

In (4) we are using the p-binomial coefficients

$$\begin{bmatrix} m \\ s \end{bmatrix}_p = \prod_{i=1}^{s} \frac{p^{m-i+1} - 1}{p^i - 1} \tag{5}$$

This is the number of s-dimensional subspaces in an m-dimensional vector space over $\mathbb{F}_p$.

Thus, the matrix $A = (a_{x,x'})$ can be written as

$$A = p^{r-1} \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_p I + \begin{bmatrix} n-1 \\ r-2 \end{bmatrix}_p J, \tag{6}$$

where I is the identity and J is the matrix with all entries 1. It follows that the eigenvalues of $\eta_{r,1}\eta_{1,r}$ are $p^{r-1}\begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_p$, with multiplicity $|\mathcal{L}_1| - 1$ and $\begin{bmatrix} n \\ r-1 \end{bmatrix}_p \begin{bmatrix} r \\ 1 \end{bmatrix}_p$, with multiplicity one, with $\mathbf{1}_1$ as eigenvector.

In the case $r = n$ the above considerations yield the order of the cokernel of $\eta_{n,1}$. If we fix bases of $\mathbb{Z}^{\mathcal{L}_1}$ and $\mathbb{Z}^{\mathcal{L}_n}$ then $\eta_{n,1}$ is represented by a square matrix. The absolute value of the determinant of such a matrix does not depend on the bases chosen and is equal to the order of the cokernel. From the preceding paragraph we obtain

$$|\det(\eta_{n,1})| = \frac{p^n - 1}{p - 1} . p^{\frac{(n-1)(|\mathcal{L}_1|-1)}{2}}. \tag{7}$$

We consider the exponent of p in (7). Since $d_i = d_{n-i+1}$ and $\sum_{i=i}^{n} d_i = |\mathcal{L}_1| - 1$, we obtain

$$\nu_p(\det(\eta_{n,1})) = \frac{(n-1)(|\mathcal{L}_1| - 1)}{2} = \sum_{i=1}^{n} (n-i) d_i \tag{8}$$

We should point out that for other values of r the order of the cokernel of $\eta_{r,1}$ will not be equal to the square root of the determinant of the adjacency map $\eta_{r,1}\eta_{1,r}$ and requires a different method to compute it. (See Lemma 7 below.)

The following simple observation will be used frequently. Let $r \geq s \geq t$. Then it is easy to check that we have a commutative diagram:

$$\begin{tikzcd} \mathbb{Z}^{\mathcal{L}_r} \arrow[rr, "{\begin{bmatrix} r-t \\ s-t \end{bmatrix}_p \eta_{r,t}}"] \arrow[dr, "\eta_{r,s}"'] & & \mathbb{Z}^{\mathcal{L}_t} \\ & \mathbb{Z}^{\mathcal{L}_s} \arrow[ur, "\eta_{s,t}"'] & \end{tikzcd} \tag{9}$$

2.1. The modules Y_r.

Let $\mathbb{Z}_p$ denote the ring of p-adic integers and $\mathbb{Q}_p$ the field of p-adic numbers. For a $\mathbb{Z}_p$-module A we shall use the notations $\mathbb{Q}_pA = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} A$ and $\overline{A} = \mathbb{F}_p \otimes_{\mathbb{Z}_p} A = A/pA$.

Let $\mathbb{Z}_p^{\mathcal{L}_r}$ denote the $\mathbb{Z}_pG$-permutation module with basis $\mathcal{L}_r$ and let

$$Y_r = \{y = \sum_{z \in \mathcal{L}_r} b_z z \in \mathbb{Z}_p^{\mathcal{L}_r} \ \mid \ \sum_z b_z = 0\} \tag{10}$$

Since p does not divide $|\mathcal{L}_r|$, we have

$$\mathbb{Z}_p^{\mathcal{L}_r} = \mathbb{Z}_p\mathbf{1}_r \oplus Y_r, \qquad \text{and} \qquad \mathbb{F}_p^{\mathcal{L}_r} = \mathbb{F}_p\mathbf{1}_r \oplus \overline{Y}_r. \tag{11}$$

We shall continue to use $\eta_{r,s}$ for the incidence map over $\mathbb{Z}_p$ and we will use $\overline{\eta}_{r,s}$ for the maps over $\mathbb{F}_p$. Since $\eta_{r,s}(Y_r) \subseteq Y_s$ and, consequently, $\overline{\eta}_{r,s}(\overline{Y}_r) \subseteq \overline{Y}_s$, we will use the same notations for these restricted maps.

Let S_i be the degree $i(p-1)$ component of the graded G-algebra $S^*(V)/(V^p)$, the quotient of the symmetric algebra on V by the ideal generated by p-th powers. We observe that the dimension of S_i is equal to the number d_i in Theorem 1 (and hence equal also to the dimension of S_{n-i+1}). The following result, which is central to our method, is a reformulation of the well known structure of generalized projective Reed-Muller codes.

Theorem 2 (cf. [2, Theorem A]). *The $\mathbb{F}_pG$-module $\overline{Y}_1$ is uniserial with a composition series*

$$0 = W_{n+1} \subset W_n \subset \cdots \subset W_1 = \overline{Y}_1 \tag{12}$$

such that for each $i = 1, \ldots, n$ the simple factor W_i/W_{i+1} is isomorphic to S_i.

Theorem 2 also gives the structure of $\overline{Y}_n$ by geometric duality; G has an outer automorphism (inverse-transpose) of order two which interchanges the stablizers of points with the stabilizers of hyperplanes, and interchanges S_i with S_{n-i+1}. Therefore, it follows from Theorem 2 that the module $\overline{Y}_n$ is uniserial, with the same composition factors as $\overline{Y}_1$, but in the opposite order. Thus we have

$$0 = U_0 \subset U_1 \subset \cdots \subset U_n = \overline{Y}_n, \qquad \text{with } \ U_i/U_{i-1} \cong S_i \cong W_i/W_{i+1}. \tag{13}$$

We need in addition the following fact.

Lemma 1. *For each r, $\overline{Y}_r$ has a unique maximal submodule, with simple quotient isomorphic to S_r.*

Proof. Let L be a simple module. By Frobenius reciprocity,

$$\mathrm{Hom}_{\mathbb{F}_pG}(\mathbb{F}_p^{\mathcal{L}_r}, L) \cong \mathrm{Hom}_{\mathbb{F}_pG_r}(\mathbb{F}_p, L) \tag{14}$$

where G_r is the stabilizer of an r-space. Thus if L is a simple quotient of $\mathbb{F}_p^{\mathcal{L}_r}$, it contains a nonzero vector fixed by G_r. The general theory of modular representations of finite groups of Lie type [3, 7, 9] then says that such a vector exists only in the trivial module and in the simple module of *highest weight* $(p-1)\omega_r$, where ω_r is the r-th fundamental weight (the highest weight of $\wedge^r(V)$). Moreover, the fixed vector is the highest weight vector so is unique up to scalars. A routine computation of weights in S_r shows that its highest weight is indeed $(p-1)\omega_r$, so $L \cong S_r$, proving the lemma.

Lemma 2.

(a) $\operatorname{Hom}_{\mathbb{Z}_pG}(Y_1, Y_r) \cong \mathbb{Z}_p$, *for all* r.
(b) $\mathbb{Q}_pY_1$ *is simple and each* $\mathbb{Q}_pY_r$ *has a unique simple summand isomorphic to* $\mathbb{Q}_pY_1$.
(c) $\mathbb{Q}_pY_n \cong \mathbb{Q}_pY_1$ *and* $\operatorname{Ker}\eta_{r,1} = \operatorname{Ker}\eta_{r,n}$. *The latter module is a pure submodule of* Y_r.

Proof. Since the stabilizer of a 1-dimensional subspace of V has two orbits on $\mathcal{L}_r$, we have

$$\operatorname{rank}_{\mathbb{Z}_p} \operatorname{Hom}_{\mathbb{Z}_pG}(\mathbb{Z}_p^{\mathcal{L}_1}, \mathbb{Z}_p^{\mathcal{L}_r}) = 2, \tag{15}$$

and (a) now follows from (11). Part (b) follows immediately from (a), as does the first part of (c), since $\dim_{\mathbb{Q}_p} \mathbb{Q}_pY_n = \dim_{\mathbb{Q}_p} \mathbb{Q}_pY_1$. Kernels of homomorphisms between $\mathbb{Z}_p$-lattices are always pure. Therefore, to show $\operatorname{Ker}\eta_{r,1} = \operatorname{Ker}\eta_{r,n}$ it is enough to show the equality of $\mathbb{Q}_p \operatorname{Ker}\eta_{r,1}$ and $\mathbb{Q}_p \operatorname{Ker}\eta_{r,n}$. But the latter are the kernels of the maps over $\mathbb{Q}_p$, and their equality follows from (b) and the first part of (c).

2.2. The modules M_r.

Let $M_r = Y_r/\operatorname{Ker}\eta_{r,1}$, $(1 \le r \le n)$. We have $M_1 = Y_1$ and $M_n = Y_n$. Since $\operatorname{Ker}\eta_{r,1}$ is pure, M_r is a $\mathbb{Z}_p$-form in the $\mathbb{Q}_pG$-module $\mathbb{Q}_pY_r/\mathbb{Q}_p\operatorname{Ker}\eta_{r,1} \cong \mathbb{Q}_pY_1$ so, by a general principle [4, I, Theorem 17.7] all the mod p reductions $\overline{M}_r$ have the same composition factors S_i, $1 \le i \le n$ as $\overline{Y}_1$ and $\overline{Y}_n$.

The commutativity of the diagram (9) implies that for $r \ge s$, $\eta_{r,s}(\operatorname{Ker}\eta_{r,1}) \subseteq \operatorname{Ker}\eta_{s,1}$. Thus we have induced $\mathbb{Z}_pG$-module homomorphisms $\mu_{r,s} : M_r \longrightarrow M_s$ and $\overline{\mu}_{r,s} : \overline{M}_r \longrightarrow \overline{M}_s$ and commutative diagrams

$$\begin{tikzcd} M_r \arrow[rr, "{\begin{bmatrix} r-t \\ s-t \end{bmatrix}_p \mu_{r,t}}"] \arrow[dr, "\mu_{r,s}"'] & & M_t \\ & M_s \arrow[ur, "\mu_{s,t}"'] & \end{tikzcd} \qquad \begin{tikzcd} \overline{M}_r \arrow[rr, "{\begin{bmatrix} r-t \\ s-t \end{bmatrix}_p \overline{\mu}_{r,t}}"] \arrow[dr, "\overline{\mu}_{r,s}"'] & & \overline{M}_t \\ & \overline{M}_s \arrow[ur, "\overline{\mu}_{s,t}"'] & \end{tikzcd} . \tag{16}$$

The next result gives the submodule structure of the $\overline{M}_r$.

Lemma 3. *Assume $2 \leq r \leq n-1$*

(a) *$\overline{M}_r$ has a unique maximal submodule with simple quotient S_r.*

(b) *The maximal submodule of $\overline{M}_r$ is the direct sum of two uniserial modules $J_r^+ = \operatorname{Ker} \overline{\mu}_{r,n}$, which maps isomorphically under $\overline{\mu}_{r,1}$ to W_{r+1}, and $J_r^- = \operatorname{Ker} \overline{\mu}_{r,1}$, which maps isomorphically under $\overline{\mu}_{r,n}$ to U_{r-1}.*

Proof. The property of $\overline{M}_r$ in (a) is inherited from $\overline{Y}_r$. To prove (b), we observe that by (a), the image of the nonzero homomorphism $\overline{\mu}_{r,1}$ in $\overline{Y}_1$ must be the module W_r. It follows that the kernel J_r^- must have composition factors $S_{r-1}, \dots, S_1$. Likewise, $\overline{\mu}_{r,n}$ has image U_r and kernel J_r^+, with composition factors $S_{r+1}, \dots, S_n$. Since the kernels have no composition factors in common, it follows that each map is injective when restricted to the kernel of the other. Thus, J_r^- is isomorphic to a submodule of U_r, which from inspection of composition factors must be U_{r-1}. Similiarly $J_r^+ \cong W_{r+1}$ and the lemma is proved.

It may help to visualize these modules as shown in the picture (for $n = 5$) below.

$$\overline{M}_1\colon \begin{array}{c} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{array} \qquad \overline{M}_2\colon \begin{array}{ccc} & S_2 & \\ S_1 & & S_3 \\ & & S_4 \\ & & S_5 \end{array} \qquad \overline{M}_3\colon \begin{array}{ccc} & S_3 & \\ S_2 & & S_4 \\ S_1 & & S_5 \end{array} \qquad \overline{M}_4\colon \begin{array}{ccc} & S_4 & \\ S_3 & & S_5 \\ S_2 & & \\ S_1 & & \end{array} \qquad \overline{M}_5\colon \begin{array}{c} S_5 \\ S_4 \\ S_3 \\ S_2 \\ S_1 \end{array}$$

For notational convenience, we make the conventions that $J_n^- = U_{n-1}$, $J_n^+ = 0$, $J_1^- = 0$ and $J_1^+ = W_2$.

Lemma 4. *For $1 \leq s < r \leq n$, we have $\operatorname{Ker} \overline{\mu}_{r,s} = J_r^-$.*

Proof. By the commutativity of the maps in (16) it suffices to show $J_r^- \subseteq \operatorname{Ker} \overline{\mu}_{r,r-1}$. The image of $\overline{M}_r$ in $\overline{M}_{r-1}$ has a unique maximal submodule with top composition factor S_r, so by Lemma 3 must be J_{r-1}^+. Since the latter has no composition factors in common with J_r^-, by Lemma 3, the result follows.

§3 Proof of Theorem 1

The proof of Theorem 1 will proceed as follows. We shall construct a basis of M_r containing a subset $\{x_{ij}\}$ $(1 \leq i \leq r-1,\ 1 \leq j \leq d_i)$ of $\sum_{i=1}^{r-1} d_i$ elements such that for each i, the d_i elements x_{ij} are mapped under $\mu_{r,1}$ into $p^{r-i} Y_1$. This will establish that $\mu_{r,1}$ can be represented by a matrix which is the product AD of two square matrices, the matrix D being diagonal with elementary divisors as stated in Theorem 1.

Finally, we shall show $\det A$ is a unit, so Theorem 1 will be proved.

Let $2 \leq r \leq n$. In the isomorphism $J_r^- \cong U_{r-1}$, let $K_{r,i}$ be the submodule of J_r^- mapping to U_i, $(1 \leq i \leq r-1)$. We have $K_{r,r-1} = J_r^-$ and $K_{r,i}/K_{r,i-1} \cong S_i$.

Lemma 5. *There exists submodules $F_{r,i}$ $(1 \leq i \leq r-1)$ in M_r with the properties*

(1) *$F_{r,i}$ maps onto $K_{r,i}$ under the reduction map $M_r \longrightarrow \overline{M}_r$;*

(2) *$F_{r,i}$ has a unique maximal submodule with quotient S_i*

(3) *$F_{r,i} \subseteq F_{r,j}$ whenever $i < j$.*

Proof. We start by making the conventions that $K_{r,r} = \overline{M}_r$, and $F_{r,r} = M_r$. Then (1) and (3) hold trivially for $i = r$. We will construct modules $F_{r,i}$ for $1 \le i \le r-1$ which satisfy (1) and (3) by iteration. Suppose $F_{r,i+1}$ has already been constructed satisfying (1) and (3). Let P_i be the projective cover of S_i in the category of $\mathbb{Z}_p G$-modules. Consider the following diagram.

$$\begin{tikzcd}
& P_i \arrow[dl] \arrow[d] \arrow[dr] & & \\
& K_{r,i} \arrow[r] \arrow[d, "\cap"] & S_i \arrow[r] & 0 \\
F_{r,i+1} \arrow[r] & K_{r,i+1} \arrow[r] & 0 &
\end{tikzcd} \tag{17}$$

By the projectivity of P_i the homomorphism $P_i \longrightarrow S_i$ may be lifted to a homomorphism $P_i \longrightarrow K_{r,i}$, which must be surjective, by the uniseriality of $K_{r,i}$. Then the composite $P_i \longrightarrow K_{r,i} \subset K_{r,i+1}$ can be lifted to a homomorphism $P_i \longrightarrow F_{r,i+1}$, as shown. Let $F_{r,i}$ be the image of P_i in $F_{r,i+1}$. Then (1) and (3) hold by construction. Finally, we note that for $1 \le i \le r-1$, $F_{r,i}$ is a quotient of P_i and hence (2) holds since P_i has this property [4, I, Theorem 13.5]. The lemma is proved.

We can certainly find d_i elements $\overline{x}_{ij}$ of $K_{r,i}$ with the property that their images in S_i form a basis. By Lemma 5, we pick preimages $x_{ij} \in F_{r,i}$ of these elements. We do this for all i with $1 \le i \le r-1$, obtaining a set $\mathfrak{X}_r$ of $\sum_{i=1}^{r-1} d_i$ elements whose images $\overline{x}_{ij}$ form a basis of J_r^-. Therefore, by Nakayama's Lemma, the set $\mathfrak{X}_r$ can be extended to a basis of M_r. Since $x_{ij} \in F_{r,i}$, the next result shows that this basis will have the property mentioned above.

Lemma 6. *$\mu_{r,1}(F_{r,i}) \subset p^{r-i} Y_1$ for $1 \le i \le r-1$.*

Proof. We shall descend inductively starting at $i = r-1$. We have a commutative diagram

$$\begin{tikzcd}
F_{r,r-1} \arrow[r, "\subset"] \arrow[d] & M_r \arrow[r, "\mu_{r,1}"] \arrow[d] & Y_1 \arrow[d] \\
K_{r,r-1} \arrow[r, "\subset"] & \overline{M}_r \arrow[r, "\overline{\mu}_{r,1}"] & \overline{Y}_1
\end{tikzcd} \tag{18}$$

Since $K_{r,r-1} = J_r^- = \ker \overline{\mu}_{r,1}$, the commutativity of the diagram shows that $\mu_{r,1}(F_{r,r-1}) \subseteq pY_1$, So the result holds for $i = r-1$. Assume the lemma holds for $F_{r,i}$. Then we have the following maps.

$$F_{r,i-1} \subset F_{r,i} \xrightarrow{\mu_{r,1}} p^{r-i} Y_1 \longrightarrow p^{r-i} Y_1 / p^{r-i+1} Y_1 \cong \overline{Y}_1 \tag{19}$$

By property (2) of Lemma 5, the image of $F_{r,i}$ in $\overline{Y}_1$ has a unique simple quotient, isomorphic to S_i. The image must therefore be zero or W_i. Likewise, the image of

$F_{r,i-1}$ must be zero or W_{i-1}, but the latter is impossible since $W_{i-1} \not\subseteq W_i$. Thus, $\mu_{r,1}(F_{r,i-1}) \subseteq p^{r-i+1}Y_1$, completing the induction.

Lemma 6 shows that $\mu_{r,1}$ is represented using the above basis of M_r by a product AD of square matrices, where D is a diagonal matrix with the elementary divisors specified in Theorem 1. Thus $\nu_p(\det(D)) = \sum_{i=1}^{r-1}(r-i)d_i$.

Therefore the following computation will conclude the proof of Theorem 1.

Lemma 7. $\nu_p(\det(\mu_{r,1})) = \sum_{i=1}^{r-1}(r-i)d_i$.

Proof. By Lemma 4, for all $s = 2, \dots, n$ we have $J_s^- = \operatorname{Ker} \overline{\mu}_{s,s-1}$ so, since the image of $\mathfrak{X}_s$ in $\overline{M}_s$ lies in J_s^-, we have $\mu_{s,s-1}(\mathfrak{X}_s) \subseteq pM_{s-1}$. Since $\mathfrak{X}_s$ forms part of a basis of M_s it follows that $\nu_p(\det(\mu_{s,s-1})) \geq |\mathfrak{X}_s| = \sum_{i=1}^{s-1} d_i$.

Next, it will be helpful to consider the following commutative diagram of homomorphisms.

$$\begin{array}{ccccccccccc}
Y_n & \longrightarrow & Y_{n-1} & \longrightarrow \cdots \longrightarrow & Y_s & \xrightarrow{\eta_{s,s-1}} & Y_{s-1} & \longrightarrow \cdots \longrightarrow & Y_1 \\
\| & & \downarrow & & \downarrow & & \downarrow & & \| \\
Y_n & \longrightarrow & M_{n-1} & \longrightarrow \cdots \longrightarrow & M_s & \xrightarrow{\mu_{s,s-1}} & M_{s-1} & \longrightarrow \cdots \longrightarrow & Y_1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\overline{Y}_n & \longrightarrow & \overline{M}_{n-1} & \longrightarrow \cdots \longrightarrow & \overline{M}_s & \xrightarrow{\overline{\mu}_{s,s-1}} & \overline{M}_{s-1} & \longrightarrow \cdots \longrightarrow & \overline{Y}_1
\end{array} \tag{20}$$

By (9), the composite of the maps in the top row of (20) differs from $\eta_{n,1}$ by a unit in $\mathbb{Z}_p$.

Therefore, using (8),

$$\sum_{i=2}^{n}(n-i)d_i = \nu_p(\det(\eta_{n,1})) = \sum_{s=2}^{n} \nu_p(\det(\mu_{s,s-1})) \geq \sum_{s=2}^{n}\sum_{i=1}^{s-1} d_i = \sum_{i=2}^{n}(n-i)d_i, \tag{21}$$

so we must have equality throughout. Hence, $\nu_p(\det(\mu_{s,s-1})) = \sum_{i=1}^{s-1} d_i$ and, using (16) we have

$$\nu_p(\det(\mu_{r,1})) = \sum_{s=2}^{r} \nu_p(\det(\mu_{s,s-1})) = \sum_{s=2}^{r}\sum_{i=1}^{s-1} d_i = \sum_{i=1}^{r-1}(r-i)d_i. \tag{22}$$

The proof is finished.

Concluding Remarks.

1. If we replace the $\mathbb{F}_p$-vector space V by one over a finite extension $\mathbb{F}_q$, then, as we have already pointed out in the introduction, the problem of determining the elementary divisors of the incidence matrices for r-spaces versus 1-spaces is reduced

to finding those which are powers of p. It is known [8, §1] that every conceivable power of p occurs as an elementary divisor. Also, Theorem 2, which is crucial in our proof of Theorem 1 has been generalized to all q in [2].

2. It is natural to consider the same problems for incidences of r-subspaces with s-subspaces. In this direction, little is known about the p-torsion, but when $l \neq p$, the ranks of all such incidence maps over a field of characteristic l have been computed in [5], using [6].

3. Although Theorem 1 has been stated as a numerical result, a closer look at the proof reveals very detailed information about the $\mathbb{Z}_p G$-module structure of the cokernel of $\eta_{r,1}$. Indeed, the method used here was partly inspired by the technique of Weyl module filtrations [6, II.4.19] in the representation theory of reductive groups, to which it bears some resemblance.

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