# THE PERMUTATION MODULES FOR $\mathrm{GL}(n+1, \mathbb{F}_q)$ ACTING ON $\mathbb{P}^n(\mathbb{F}_q)$ AND $\mathbb{F}_q^{n+1}$

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ABSTRACT

The paper studies the permutation representations of a finite general linear group, first on finite projective space and then on the set of vectors of its standard module. In both cases the submodule lattices of the permutation modules are determined. In the case of projective space, the result leads to the solution of certain incidence problems in finite projective geometry, generalizing the rank formula of Hamada. In the other case, the results yield as a corollary the submodule structure of certain symmetric powers of the standard module for the finite general linear group, from which one obtains the submodule structure of all symmetric powers of the standard module of the ambient algebraic group.

## 0. *Introduction and statements of results*

### 0.1. *The permutation module on projective space*

Let $P = \mathbb{P}^n(\mathbb{F}_q)$, $n$-dimensional projective space over the field of $q = p^t$ elements. A certain class of problems in finite geometry concerns the determination of the $\mathbb{F}_p$-ranks of the incidence relations between the set $P$ of points and various other sets of geometric objects. The most significant result in this direction has been the formula of Hamada [**10**] for the rank of the incidence between $P$ and the set of $r$-dimensional linear subspaces of $P$. This formula is valid for all possible choices of the parameters $n$, $p$, $t$ and $r$. Hamada's work was motivated by questions in coding theory; the incidence matrices are generator matrices of codes closely related to the Reed–Muller codes (see [**1**, **18**]).

Problems of the type mentioned above can also be phrased as questions in group representation theory. The set $P$ admits a natural action of the group $G = \mathrm{GL}(n+1, q)$, making the vector space $\mathbb{F}_p^P$ with basis $P$ into a permutation module. If $Q$ is another set of subsets of $P$, permuted by $G$ and with $\mathbb{F}_p^Q$ as its permutation module, then the incidence relation between $P$ and $Q$ defines an $\mathbb{F}_p G$-module map

$$\mathbb{F}_p^Q \longrightarrow \mathbb{F}_p^P, \quad \beta \longmapsto \sum_{p \in \beta} p \tag{1}$$

and the rank in question is the dimension of the image of this map, which is often called the code of the incidence relation. For technical reasons, it is more convenient to work over an algebraic closure $k$ of $\mathbb{F}_q$ and to consider the map

$$k^Q \longrightarrow k^P. \tag{2}$$

Of course, the rank will be unaffected.

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Informally, we may think of the image of the incidence map as the submodule of $k^P$ generated by the objects in $Q$. Thus incidence problems lead us to the study of the $kG$-submodule lattice of $k^P$. In searching for a uniform method for treating all such incidence problems, one approach is to try to describe this lattice in enough detail so that the submodule generated by a given element can be identified and its dimension read off. It is known that, in general, it is very difficult to determine the structures of modules for finite groups. However, the problem under consideration has two favorable special features which turn out to be decisive. First, the module $k^P$ is multiplicity free, that is, no two composition factors in a Jordan–Hölder series are isomorphic. This rather obvious fact (Lemma 2.1) has the heuristically important consequence that the submodule lattice is finite. Second, we may adopt the dual viewpoint and regard $k^P$ as the space of functions on $P$, which is the set of $\mathbb{F}_q$-rational points of $\mathbb{P}^n(k)$. The homogeneous coordinate ring $k[X_0, \ldots, X_n]$ has a natural action of the algebraic group $\mathrm{GL}(n+1, k)$ and restriction of functions defines a map of $G$-algebras from $k[X_0, \ldots, X_n]^{\mathbb{F}_q^\times}$ onto $k^P$. This point of view enables us to carry out many explicit computations in $k^P$, and at the same time provides a useful point of contact with the representation theory of reductive groups.

The first principal result of this paper will be a full description of the $kG$-submodule lattice of $k^P$.

In $k^P$, viewed now as functions, we have a $kG$-decomposition

$$k^P = k1 \oplus Y_P, \tag{3}$$

where $k1$ is the space of constant functions and $Y_P$ is the kernel of the map $k^P \longrightarrow k$, $f \longmapsto |P|^{-1} \sum_{p \in P} f(p)$, which splits the inclusion map of $k1$ in $k^P$.

Thus the essential point is to understand the structure of $Y_P$. We can now state our theorem.

THEOREM A. *Let $\mathscr{H}$ denote the set of $t$-tuples $(s_0, \ldots, s_{t-1})$ of integers satisfying (for $j = 0, \ldots, t-1$) the following:*

(1) $1 \leqslant s_j \leqslant n$.

(2) $0 \leqslant ps_{j+1} - s_j \leqslant (p-1)(n+1)$ (*subscripts* mod $t$).

*Let $\mathscr{H}$ be partially ordered in the natural way: $(s'_0, \ldots, s'_{t-1}) \leqslant (s_0, \ldots, s_{t-1})$ if and only if $s'_j \leqslant s_j$ for all $j$.*

*Then the following hold:*

(a) *The module $k^P$ is multiplicity free and has composition factors $L(s_0, \ldots, s_{t-1})$ parametrized by the set $\mathscr{H} \cup \{(0, \ldots, 0)\}$.*

(b) *For $(s_0, \ldots, s_{t-1}) \in \mathscr{H}$, let $\lambda_j = ps_{j+1} - s_j$. Then the simple $kG$-module $L(s_0, \ldots, s_{t-1})$ is isomorphic to the twisted tensor product*

$$\bigotimes_{j=0}^{t-1} (\bar{S}^{\lambda_j})^{(p^j)},$$

*where $\bar{S}^\lambda$ denotes the component of degree $\lambda$ in the truncated polynomial ring $\bar{S} = k[X_0, \ldots, X_n]/(X_i^p)_{i=0}^n$ and the superscripts $(p^j)$ indicate twisting by powers of the Frobenius map.*

(c) *For each submodule $M$ of $Y_P$, let $\mathscr{H}_M \subseteq \mathscr{H}$ be the set of its composition factors. Then $\mathscr{H}_M$ is an ideal of the partially ordered set $(\mathscr{H}, \leqslant)$, that is, if $(s_0, \ldots, s_{t-1}) \in \mathscr{H}_M$ and $(s'_0, \ldots, s'_{t-1}) \leqslant (s_0, \ldots, s_{t-1})$, then $(s'_0, \ldots, s'_{t-1}) \in \mathscr{H}_M$.*

(d) *The mapping* $M \longmapsto \mathscr{H}_M$ *defines a lattice isomorphism between the submodule lattice of* $Y_P$ *and the lattice of ideals, ordered by inclusion, of the partially ordered set* $(\mathscr{H}, \leqslant)$.

Remarks. (1) The set $\mathscr{H}$ is essentially the same as the set of tuples introduced in [**10**]; much of the present work has been motivated by the desire to understand the algebraic structure behind his rank formula. In §8, we will give a new proof of this formula, as a corollary of one of the main steps in the proof of our theorem. Our proof is more conceptual, though less elementary, than the original, and does not rely on the earlier result of K. Smith [**19**] on which [**10**] is based.

(2) The statements about composition factors are well known and follow from various published results after suitable reformulation. (See §2 below.) The twisted tensor product formula in part (b) of Theorem A is a special case of Steinberg's tensor product theorem.

(3) Stabilization of module structure: It is interesting to note that condition (2) in the definition of $\mathscr{H}$ in Theorem A is automatically satisfied when $t = 1$ (that is, $q = p$), or when $p \geqslant n$. Thus, in both of these cases, the submodule lattice of $Y_P$ is isomorphic to the lattice of ideals in the $t$-fold product of the integer interval $[1, n]$. In particular, it does not depend on $p$. Of course, in the case $t = 1$ the submodules of $k^P$ are well understood by coding theorists, as they are generalizations of the Reed–Muller codes.

In order to apply Theorem A, one needs to be able to read off the submodule generated by a given element. Now from parts (a) and (b) of Theorem A, we see that $k^P$ has a basis of monomials coming from the composition factors $L(s_0, \ldots, s_{t-1})$ and it is clear what is meant by the $t$-tuple in $\mathscr{H} \cup \{(0, \ldots, 0)\}$ of such a monomial. (The precise definitions are given in §§1, 2 and 3.)

For $f \in k^P$, let $\mathscr{H}_f \subseteq \mathscr{H} \cup \{(0, \ldots, 0)\}$ denote the set of tuples of the basis monomials appearing with nonzero coefficients in the expression for $f$.

Theorem B. *The* $kG$*-submodule of* $k^P$ *generated by* $f$ *is the smallest submodule having all the* $L(s_0, \ldots, s_{t-1})$ *for* $(s_0, \ldots, s_{t-1}) \in \mathscr{H}_f$ *as composition factors.*

In conjunction with Theorem A, this result enables one to read off the composition factors of the submodule generated by $f$. The dimensions or, more precisely, the characters of the composition factors are given in §2.

### 0.2. *The permutation module on vectors*

Let $V(q) = \mathbb{F}_q^{n+1}$ be the standard module for $G$. We may identify the permutation $kG$-module on the set $V(q)$ with the space of functions $k[V(q)] = k[X_0, \ldots, X_n]/(X_i^q - X_i)_{i=0}^n$.

The $kG$-module structure of this ring is very closely related to several other questions which have been studied by other authors. The submodule lattice of $k[X_0, \ldots, X_n]$ with respect to the algebraic group $\mathrm{GL}(n+1, k)$ has been determined by Doty [**5**] and Krop [**12**, **13**, **15**]. The latter also shows that this submodule lattice is the same as that for the multiplicative semigroup $M(n+1, k)$ of all matrices over $k$. Kovacs [**11**] has determined the $kG$-module structure of the ring $k[X_0, \ldots, X_n]/(X_i^q)$, from which the previous results can be deduced. Recently, Kuhn [**16**] has determined the submodule lattice of the module $k[V(q)]$ with respect to the semigroup $M(n+1, q)$ of matrices over $\mathbb{F}_q$ and is able to deduce the earlier results with the help of a theorem of Krop [**14**, Theorem 1], which gives a condition under which the submodule lattices of certain modules with respect to the actions of $G$, $M(n+1,q)$ and $\mathrm{GL}(n+1,k)$ are identical. This condition does not apply to the module $k[V(q)]$ however (in fact, the $kG$- and $M(n+1,q)$-submodule lattices are different), so it does not seem possible to deduce the $kG$-submodule lattice directly from these known facts. Nevertheless, much of the preparatory work we do in investigating the $kG$-module structure of $k[V(q)]$ will follow or parallel the 'standard' lines established by these earlier papers. A survey of the literature on these related problems and further interesting connections may be found in [**6**].

The $kG$-module $k[V(q)]$ decomposes as a direct sum of isotypic components with respect to the center $Z(G) \cong \mathbb{F}_q^\times$. Thus

$$k[V(q)] = \bigoplus_{[d]\in\mathbb{Z}/(q-1)\mathbb{Z}} A[d], \tag{4}$$

where $[d]$ is the character $t \longmapsto t^{-d}$ of $\mathbb{F}_q^\times$. Explicitly, $A[d]$ is the span of the images of monomials of degree congruent to $d \bmod q-1$. Now the module $A[0]$ differs from $k^P$ only by a trivial summand (see §1.1 below), so its structure is given by Theorem A. The methods used to prove that theorem can be extended to give the structures of the summands $A[d]$ for $[d] \neq [0]$. Fix $d$ with $0 < d < q-1$ and let its $p$-adic expression be

$$d = d_0 + d_1 p + \ldots + d_{t-1}p^{t-1}, \quad (0 \leqslant d_j \leqslant p-1). \tag{5}$$

We have the following variation of Theorem A.

THEOREM C. *Let $\mathscr{H}[d]$ denote the set of t-tuples $(r_0, \ldots, r_{t-1})$ of integers satisfying (for $j = 0, \ldots, t-1$) the following:*

(1) $0 \leqslant r_j \leqslant n$.

(2) $0 \leqslant d_j + pr_{j+1} - r_j \leqslant (p-1)(n+1)$ (*subscripts* mod $t$).

*Let $\mathscr{H}[d]$ be partially ordered in the natural way: $(r'_0, \ldots, r'_{t-1}) \leqslant (r_0, \ldots, r_{t-1})$ if and only if $r'_j \leqslant r_j$ for all $j$.*

*Then the following hold:*

(a) *The module $A[d]$ is multiplicity free and has composition factors $L[d](r_0, \ldots, r_{t-1})$ parametrized by the set $\mathscr{H}[d]$.*

(b) *For $(r_0, \ldots, r_{t-1}) \in \mathscr{H}[d]$, let $\lambda_j = d_j + pr_{j+1} - r_j$. Then the simple kG-module $L[d](r_0, \ldots, r_{t-1})$ is isomorphic to the twisted tensor product*

$$\bigotimes_{j=0}^{t-1} (\bar{S}^{\lambda_j})^{(p^j)}.$$

(c) *For each submodule $M$ of $A[d]$, let $\mathscr{H}[d]_M \subseteq \mathscr{H}[d]$ be the set of its composition factors. Then $\mathscr{H}[d]_M$ is an ideal of the partially ordered set $(\mathscr{H}[d], \leqslant)$.*

(d) *The mapping $M \longmapsto \mathscr{H}[d]_M$ defines a lattice isomorphism between the submodule lattice of $A[d]$ and the lattice of ideals, ordered by inclusion, of the partially ordered set $(\mathscr{H}[d], \leqslant)$.*

REMARKS. (1) Let us compare Theorem C with Theorem A. By Theorem C, the module $A[d]$ has a simple socle and a simple head, since the set $\mathscr{H}[d]$ has unique minimal and maximal elements $(0, \ldots, 0)$ and $(n, \ldots, n)$ respectively. The module $A[0]$, on the other hand, is decomposable and it is the nontrivial summand $Y_P$ which has a simple head and socle. The submodule lattice of $Y_P$ is given in terms of the set $\mathscr{H}$ of which the unique maximal element is $(n, \ldots, n)$, as for $\mathscr{H}[d]$, but the unique minimal element is $(1, \ldots, 1)$, not $(0, \ldots, 0)$. Thus, in a certain sense, Theorem C is the generic case and Theorem A the degenerate case. The proof of Theorem C requires only slight modifications to the arguments used to prove Theorem A. The details are given in §9.

(2) We may also compare our results with those of [**16**]. Let $M$ be the semigroup of all $(n+1)\times(n+1)$ matrices over $\mathbb{F}_q$. As is known [**14**], the composition factors of $k[V(q)]$ are the same for $kG$ or $kM$, and they have been known for a long time (see §2 below). It is of interest to compare the submodule lattices. One finds that the lattices are the same for $A[d]$, $[d] \neq [0]$, while for $[d] = [0]$, the only difference is that the one-dimensional top factor in the degree filtration splits off for $kG$ but not for $kM$. The smallest case of this phenomenon is noted in [**16**, Remark 5.5].

(3) Let $G_\alpha$ be a maximal parabolic subgroup stabilizing a one-dimensional subspace in $V(q)$. Then the module $A[d]$ can be viewed as the $kG$-module induced from the one-dimensional $kG_\alpha$-module whose restriction to $Z(G)$ affords $[d]$. The simplicity of the heads and socles of $Y_P$ and $A[d]$ can also be proved as a special case of more general results about such induced modules for finite groups of Lie type, due to Curtis [**4**].

(4) Structure of symmetric powers: In §10, we will describe in detail the following application of Theorem C in the study of polynomial representations of the algebraic group $\mathrm{GL}(n+1, k)$. Let $S^d$ denote the space of homogeneous polynomials of degree $d$ in the variables $X_0, \ldots, X_n$. This space admits a natural action of $\mathrm{GL}(n+1, k)$. Since $d < q-1$, we have an embedding of $kG$-modules $S^d \hookrightarrow A[d]$. The image has a very simple description in terms of $\mathscr{H}[d]$, so Theorem C gives the submodule structure of $S^d$ for the finite groups $\mathrm{GL}(n+1, q)$. It is then shown that the submodule lattice is the same with respect to all the finite groups $\mathrm{GL}(n+1, p^N)$ with $p^N - 1 > d$, and hence is equal to the submodule lattice for the algebraic group $\mathrm{GL}(n+1, k)$. Thus Theorem C generalizes the work of Doty [**5**] and Krop [**12**, **13**], mentioned earlier, in which the latter structure was first described.

## 1. *Preliminaries*

1.1.

The following notation will be used throughout. Let $q = p^t$ be a prime power, $k$ an algebraic closure of $\mathbb{F}_q$, $V(q) = \mathbb{F}_q^{n+1}$, $V = k^{n+1}$, $G = \mathrm{GL}(n+1, q)$, $P = \mathbb{P}^n(\mathbb{F}_q)$. Our main object of study is the space $k^P$ of functions from $P$ to $k$. Let $X_0, \ldots, X_n$ be coordinates of $V$. Restriction of functions defines a surjective ring homomorphism

$$k[X_0, \ldots, X_n] \longrightarrow k[V(q)] \tag{6}$$

from the polynomial ring onto the ring of functions on $V(q)$, with kernel generated by the elements $X_i^q - X_i$, $i = 0, \ldots, n$.

Further restriction to $V(q)\setminus\{0\}$ gives a surjective ring homomorphism

$$k[V(q)] \longrightarrow k[V(q)\setminus\{0\}] \tag{7}$$

with kernel spanned by the characteristic function $\delta_0$ of $\{0\}$, which is the image of $\prod_{i=0}^n (1 - X_i^{q-1})$. Now the inclusion map $k\delta_0 \subset k[V(q)]$ has a $G$-splitting given by the evaluation map $f \longmapsto f(0)$, so it follows that the map (7) is $G$-split and we have an isomorphism of $kG$-modules

$$k[X_0, \ldots, X_n]/(X_i^q - X_i)_{i=0}^n \cong k\delta_0 \oplus k[V(q)\setminus\{0\}]. \tag{8}$$

The functions on $V(q) \setminus \{0\}$ which descend to $P$ are precisely those which are invariant under scalar multiplication by $\mathbb{F}_q^\times$, so we have

$$k^P \cong \left(k[X_0, \dots, X_n]/(X_i^q - X_i, 0 \leqslant i \leqslant n, \prod_{i=0}^{n}(1 - X_i^{q-1}))\right)^{\mathbb{F}_q^\times}. \tag{9}$$

Moreover, since $\mathbb{F}_p^\times$ acts semisimply on $k[X_0, \dots, X_n]$, the subring $k[X_0, \dots, X_n]^{\mathbb{F}_q^\times}$, consisting of linear combinations of the monomials with total degree divisible by $q-1$, maps onto $k^P$.

Next, consider the inclusion map $k1 \longrightarrow k^P$ of the constant functions. This map has a $G$-splitting given by $f \longmapsto \sum_{p \in P} f(p)$, since $|P| \equiv 1 \pmod p$. Thus we have

$$k^P = k1 \oplus Y_P, \tag{10}$$

where $Y_P$ is the kernel of the latter map.

1.2.

We now consider bases for the various modules. It is clear that the images under mapping (6) of the monomials $\prod_{i=0}^{n} X_i^{b_i}$ with $0 \leqslant b_i \leqslant q-1$ form a basis of $k[V(q)]$, and also that among these, those with $\sum_i b_i \equiv 0 \pmod{q-1}$ form a basis of $k[V(q)]^{\mathbb{F}_q^\times}$. Then, because of the extra relation given by $\delta_0$, we see that the images in $k^P$ of the latter set of monomials will form a basis of $k^P$ if we exclude the monomial $\prod_{i=0}^{n} X_i^{q-1}$. Let $x_i$ be the image of $X_i$ in $k^P$. By the *monomial basis* of $k^P$ we shall mean the set

$$\left\{\prod_{i=0}^{n} x_i^{b_i} \mid 0 \leqslant b_i \leqslant q-1,\ \sum_i b_i \equiv 0 \pmod{q-1},\ (b_0, \dots, b_n) \neq (q-1, \dots, q-1)\right\} \tag{11}$$

and will refer to its elements as *basis monomials*.

In all cases, we will refer to the number $\sum_i b_i$ as the *degree* of the basis monomial.

LEMMA 1.1. *The nonconstant basis monomials form a basis of $Y_P$.*

*Proof.* Consider the basis monomial $x = \prod_{i=0}^{n} x_i^{b_i} \neq 1$. Suppose first that all of the $b_i$ are either 0 or $q-1$. Then $x$ takes the value 0 on the union of a nonempty subset of the coordinate hyperplanes of $P$, and the value 1 on its complement. The complement has cardinality divisible by $q$, since $x$ does not involve all $n+1$ variables, so $x \in Y_P$. Now suppose some $b_i$, say $b_0$, is neither 0 nor $q-1$. Then

$$(q-1) \sum_{a \in P} x(a) = \left(\sum_{\mu_0 \in \mathbb{F}_q^\times} \mu_0^{b_0}\right)\left(\sum_{(\mu_1, \dots, \mu_n) \in \mathbb{F}_q^n} \prod_{i=1}^{n} \mu_i^{b_i}\right) = 0, \tag{12}$$

since the first factor on the right is 0. Therefore, $x \in Y_P$. □

We shall call this the *monomial basis* of $Y_P$.

## 2. *Composition factors, formal characters and truncated polynomial rings*

The first objective in this section is to describe the composition factors of the modules $k[V(q)]$ and $k^P$. This is done by comparing these modules with modules for the algebraic group $\mathrm{GL}(n+1, k)$, for which composition factors are determined by the action of a maximal torus. These results are just variations and reformulations of the known results in [**14**, Theorem 3.2; **11**].

Then we give a formula for the characters of the composition factors in order to relate the composition factors to Hamada's work later on in §8.

2.1.

We recall that a $kG$-module is *multiplicity-free* if no two of its composition factors are isomorphic. The following result is well known, and follows immediately from the fact that $G$ has a cyclic subgroup which acts transitively on $V(q)\setminus\{0\}$.

LEMMA 2.1 *$k[V(q)\setminus\{0\}]$ is a multiplicity-free $kG$-module.*

It follows that $k^P$ is multiplicity-free. With regard to module structure, the most important consequence of this result is that the set of submodules of a multiplicity-free module is finite. Moreover, a submodule is completely determined by the isomorphism type of its *head* (maximal semisimple quotient).

2.2.

Let $A = k[V(q)] = k[X_0,\dots,X_n]/(X_i^q - X_i)_{i=0}^n$ and $B = k[X_0,\dots,X_n]/(X_i^q)_{i=0}^n$. The natural filtration $F_r = \{f \in k[X_0,\dots,X_n]\colon \deg(f) \leqslant r\}$ $(r \geqslant 0)$ is a filtration of $k[X_0,\dots,X_n]$ by $\mathrm{GL}(n+1,k)$-modules.

This filtration induces filtrations on $B$ and $A$, the first being one of $\mathrm{GL}(n+1,k)$-modules, and the second one of $kG$-modules. We consider the associated graded modules $\operatorname{gr} B$ and $\operatorname{gr} A$. We use the same notation $[f]$ to denote the symbol in the corresponding graded module of an element $f$ of either $B$ or $A$.

LEMMA 2.2. *The graded modules $\operatorname{gr} B$ and $\operatorname{gr} A$ are isomorphic $kG$-modules.*

*Proof.* Let $y_i$ and $z_i$ denote the images of $X_i$ in $B$ and $A$ respectively. Then the symbols $[\prod_i y_i^{b_i}]$ and $[\prod_i z_i^{b_i}]$ with $0 \leqslant b_i \leqslant q-1$ and $\sum_i b_i = r$ form bases for the degree $r$ components of the respective graded modules, and the action of $G$ on these modules is induced from the action on $k[X_0,\dots,X_n]$. Let $m = \prod_i X_i^{b_i} \in k[X_0,\dots,X_n]$ be of degree $r$ with each $b_i \leqslant q-1$. For $g \in G$ we may write

$$\mathrm{gm} = \sum_{\beta} c_\beta X^\beta + f, \tag{13}$$

where the multi-indices $\beta = (\beta_0,\dots,\beta_n)$ satisfy $\beta_i \leqslant q-1$ and $f$ is the sum of the remaining terms in gm. Since gm is homogeneous of degree $r$ and since each monomial in $f$ contains at least one variable with exponent at least $q$, the images of $f$ in both $B$ and in $A$ lie in the images of $F_{r-1}$. It follows that the map $[\prod_i y_i^{b_i}] \longmapsto [\prod_i z_i^{b_i}]$ is a $kG$-isomorphism of the graded modules. □

The lemma implies that $B$ and $A$ have the same $kG$-composition factors. The point of this is that the $kG$-module structure on $B$ is the restriction of a $\mathrm{GL}(n+1,k)$-module structure. The composition factors are therefore determined by the weights with respect to the diagonal subgroup $T$ (a maximal torus) of $\mathrm{GL}(n+1,k)$.

2.3.

Let $\bar{S} = \oplus_{\lambda=0}^{(n+1)(p-1)} \bar{S}^{\lambda}$ denote the truncated polynomial ring $S(V^*)/(V^{*(p)}) \cong k[X_0,\dots,X_n]/(X_i^p)_{i=0}^n$, and let $\bar{S}^{(p^j)}$ denote the same ring but with the variables $X_i$ replaced by their $p^j$th powers. These graded algebras all have the structure of modules for $\mathrm{GL}(n+1,k)$, with $\bar{S}^{(p^j)}$ being isomorphic to the $j$th Frobenius twist of $\bar{S}$. Further, it is well known and easy to check by direct computation that the homogeneous components $\bar{S}^{\lambda}$ are simple $\mathrm{GL}(n+1,k)$-modules which remain irreducible for the finite group $\mathrm{SL}(n+1,p)$. To be precise, let $\omega_i$, $1 \leqslant i \leqslant n$, denote the fundamental weights for $\mathrm{SL}(n+1,k)$, chosen so that $\omega_i$ is the highest weight of the $i$th exterior power of the natural module $V$. Write $\lambda = (p-1)r+b$, with $0 \leqslant r \leqslant n+1$, $0 \leqslant b < p-1$. Then the highest weight of $\bar{S}^{\lambda}$ is

$$\begin{cases} b\omega_n & r=0 \\ (p-1-b)\omega_{n+1-r} + b\omega_{n-r} & 1 \leqslant r \leqslant n-1 \\ (p-1-b)\omega_1 & r=n \\ 0 & r=n+1. \end{cases} \tag{14}$$

In particular, the coefficients of the fundamental weights in the above expressions for the highest weights are at most $p-1$, so the $\bar{S}^{\lambda}$ are *restricted* simple modules for $\mathrm{SL}(n+1,k)$. As is well known, the restrictions of the restricted simple modules to $\mathrm{SL}(n+1,p)$ form a complete set of nonisomorphic simple modules for $\mathrm{SL}(n+1,p)$.

Therefore, by Steinberg's tensor product theorem [**20**, Theorems 1.1 and 1.3], the modules

$$S(\lambda_0,\dots,\lambda_{t-1}) = \bigotimes_{j=0}^{t-1} (\bar{S}^{\lambda_j})^{(p^j)} \tag{15}$$

are simple $\mathrm{SL}(n+1,k)$-modules which remain simple for $\mathrm{SL}(n+1,p^t)$. Therefore they are also simple as $\mathrm{GL}(n+1,k)$-modules and as $kG$-modules.

Note that $\bar{S}^{(p^j)}$ has a basis consisting of the images of the monomials $\prod_i X_i^{a_{ij}p^j}$ with $a_{ij} \leqslant p-1$. We use the same notation for the images. It is natural to refer to $p^j \sum_i a_{ij}$ as the degree of such a basis element, with $p^j a_{ij}$ being the degree in the variable $X_i$. This leads to a gradation of the module $\otimes_{j=0}^{t-1} \bar{S}^{(p^j)}$.

LEMMA 2.3. *The map* $\otimes_{j=0}^{t-1} \bar{S}^{(p^j)} \longrightarrow \mathrm{gr}\, B$ *given by*

$$\prod_i X_i^{a_{i0}} \otimes \prod_i X_i^{a_{i1}p} \otimes \dots \otimes \prod_i X_i^{a_{i(t-1)}p^{t-1}} \longmapsto [\prod_i y_i^{\sum_j a_{ij}p^j}]$$

*is an isomorphism of* (*graded*) *T-modules.*

*Proof.* This is clear, since a basis is mapped to a basis, with corresponding elements having the same degree in each variable. □

The lemma gives a $T$-isomorphism between two $\mathrm{GL}(n+1,k)$ modules, which means that these modules have the same $\mathrm{GL}(n+1,k)$ composition factors. Thus we have found the $\mathrm{GL}(n+1,k)$ composition factors of $B$, and hence also the $G$-

composition factors of $A$, by Lemma 2.2. Since $\mathbb{F}_q^\times$ acts semisimply, the composition factors of $A^{\mathbb{F}_q^\times}$ are precisely those $S(\lambda_0, \dots, \lambda_{t-1})$ for which $q-1$ divides the total degree $\lambda_0 + \lambda_1 p + \dots + \lambda_{t-1} p^{t-1}$. The composition factors for $k^P$ and $Y_P$ are then obtained by removing the trivial module, twice for $Y_P$, by equations (8) and (10). We summarize this discussion.

THEOREM 2.1 ([**11**, p. 211; **16**, Proposition 1.1], cf. [**12**, Theorem 3.2]). (a) *The composition factors of $B$ and $A$ are the modules $S(\lambda_0, \dots, \lambda_{t-1})$ for $0 \leqslant \lambda_j \leqslant (n+1)(p-1)$.*

(b) *The composition factors of $k^P$ (and respectively $Y_P$) are the modules $S(\lambda_0, \dots, \lambda_{t-1})$ for $0 \leqslant \lambda_j \leqslant (n+1)(p-1)$ satisfying the following conditions:*

(1) $\sum_{j=0}^{t-1} \lambda_j p^j \equiv 0 \pmod{q-1}$.

(2) $(\lambda_0, \dots, \lambda_{t-1}) \neq ((n+1)(p-1), \dots, (n+1)(p-1))$ (*respectively nor* $(0, \dots, 0)$).

2.4.

We next derive a formula [**7**, Proposition 3.12(2)] for the characters of the modules $\bar{S}^\lambda$ in terms of ordinary symmetric powers and exterior powers. In [**7**] the formula is proved using generating functions. The dimension, namely the number of monomials of degree at most $p-1$ in each variable is standard in combinatorics and appears for example in [**10**, Lemma 2.6].

LEMMA 2.4 [**7**, Proposition 3.12(2)]. *Let $0 \leqslant \lambda \leqslant (n+1)(p-1)$. Then in the Grothendieck group of $\mathrm{GL}(n+1,k)$-modules we have*

$$\bar{S}^\lambda = \sum_{i=0}^{\lfloor \lambda/p \rfloor} (-1)^i \wedge^i (V^{*(p)}) \otimes S^{\lambda - ip}(V^*).$$

(*Here, $\lfloor . \rfloor$ signifies the integer part.*)

*Proof.* The sequence $X_0^p, \dots, X_n^p$ is a regular sequence [**17**, p. 127] in the polynomial ring $S(V^*)$. Hence the Koszul complex (see [**3**, VIII.4.3; **17**, p. 127]) $K_*(X_0^p, \dots, X_n^p)$ gives us a resolution

$$\dots \longrightarrow K_1(X_0^p, \dots, X_n^p) \longrightarrow K_0(X_0^p, \dots, X_n^p) \longrightarrow \bar{S} \longrightarrow 0. \tag{16}$$

We may identify $K_i(X_0^p, \dots, X_n^p)$ with $\wedge^i(V^{*(p)}) \otimes_k S(V^*)$ so that (16) becomes an exact sequence of $\mathrm{GL}(n+1,k)$-modules. The scalar matrices act on homogeneous elements according to their total degrees, so for each $\lambda$, the components $K_i^\lambda$ of degree $\lambda$ in each term of the complex form a resolution of $\bar{S}^\lambda$. We have

$$K_i^\lambda = \wedge^i(V^{*(p)}) \otimes S^{\lambda - ip}(V^*). \tag{17}$$

Therefore, the character of $\bar{S}^\lambda$ is the same as the Euler (alternating sum) character of $K_*^\lambda$. Clearly, the resolution $K_*^\lambda$ runs out when $i$ exceeds $\lfloor \lambda/p \rfloor$, so the lemma is proved. □

COROLLARY 2.1.

$$\dim S(\lambda_0, \dots, \lambda_{t-1}) = \prod_{j=0}^{t-1} \left[ \sum_{i=0}^{\lfloor \lambda_j/p \rfloor} (-1)^i \binom{n+1}{i} \binom{n+\lambda_j - ip}{n} \right].$$

2.5.

We record here some standard statements about duality in $\bar{S}$.

LEMMA 2.5. (a) $\dim \bar{S}^{(n+1)(p-1)} = 1$ *and multiplication defines nonzero* $\mathrm{GL}(n+1,k)$-*bilinear maps*

$$\bar{S}^{\lambda} \times \bar{S}^{(n+1)(p-1)-\lambda} \longrightarrow \bar{S}^{(n+1)(p-1)}.$$

(b) $\bar{S}^{\lambda}$ *and* $\bar{S}^{(n+1)(p-1)-\lambda}$ *are mutually dual as* $\mathrm{SL}(n+1,k)$-*modules. In particular they have the same dimension.*

(c) $S(\lambda_0,\dots,\lambda_{t-1})$ *and* $S((n+1)(p-1)-\lambda_0,\dots,(n+1)(p-1)-\lambda_{t-1})$ *are dual as* $kG$-*modules.*

*Proof.* Part (a) is immediate and (b) follows from the simplicity of the modules $\bar{S}^{\lambda}$. Part (c) follows from (a), the tensor product theorem and the fact that $G$ acts trivially on $S((n+1)(p-1),\dots,(n+1)(p-1))$. □

## 3. *Twisted degrees, filtrations and Hamada's parametrization*

The purpose of this section is to give a second parametrization of the composition factors of $k^P$, by a set $\mathscr{H}$, with a natural partial order which will be seen later to describe the submodule structure.

### 3.1. *Types of monomials*

Let $X = \prod_{i=0}^{n} X_i^{b_i} \in k[X_0,\dots,X_n]$, with $0 \leqslant b_i \leqslant q-1$. Write $b_i = \sum_{j=0}^{t-1} a_{ij} p^j$, $0 \leqslant a_{ij} \leqslant p-1$, and set $\lambda_j = \sum_{i=0}^{n} a_{ij}$, so that

$$X = \prod_{j=1}^{t-1} \left( \prod_{i=0}^{n} X_i^{a_{ij}} \right)^{p^j}. \tag{18}$$

We say that $X$ is of type $(\lambda_0,\dots,\lambda_{t-1})$. The basis monomials in $B$ and $A$ and of their graded modules are each the image of exactly one of the above monomials, so we may define their types in the same way. For $k^P$, in order to make the preimage of a basis monomial unique, we exclude the monomial $X_0^{q-1}\dots X_n^{q-1}$, that is, the type of 1 is $(0,\dots,0)$.

The types of basis monomials of $k^P$ (and respectively $Y_P$) consist of all those $(\lambda_0,\dots,\lambda_{t-1})$ with $0 \leqslant \lambda_j \leqslant (n+1)(p-1)$ which satisfy the following conditions:

(1) $\sum_{j=0}^{t-1} \lambda_j p^j \equiv 0 \pmod{q-1}$.

(2) $(\lambda_0,\dots,\lambda_{t-1}) \neq ((n+1)(p-1),\dots,(n+1)(p-1))$ (respectively nor $(0,\dots,0)$).

### 3.2. *Twisted degrees and the set* $\mathscr{H}$

The filtration $\{F_s\}$ of $k[X_0,\dots,X_n]$ by degree induces a filtration $\{\mathscr{F}_r\}$ on $k^P$ given by

$$\mathscr{F}_r = \text{span of basis monomials } x = \prod_i x_i^{b_i} \text{ with } \sum_i b_i = s(q-1) \text{ for some } s \leqslant r. \tag{19}$$

This filtration is not stable under the action of the Galois group $\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ so we can define conjugate filtrations $\{\mathscr{F}_r^{(e)}\}$ for $0 \leqslant e \leqslant t-1$ by setting

$$\mathscr{F}_r^{(e)} = \sigma^e(\mathscr{F}_r), \tag{20}$$

where $\sigma$ is the generator of $\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ whose action on $A$ sends $f$ to $f^p$. To see what this means in terms of basis monomials, let

$$x = \prod_{j=1}^{t-1} \left( \prod_{i=0}^{n} x_i^{a_{ij}} \right)^{p^j} \tag{21}$$

be a basis monomial of $k^P$ and let its type be $(\lambda_0, \dots, \lambda_{t-1})$. Then

$$\sigma^{-e}(x) = \prod_{j=1}^{t-1} \left( \prod_{i=0}^{n} x_i^{a_{ij}} \right)^{p^{[j-e]}}, \tag{22}$$

where '[ ]' indicates that the exponents of $p$ are taken mod $t$. Thus $\sigma^{-e}(x)$ is of type $(\lambda_e, \dots, \lambda_{[t-1+e]})$.

We define the twisted degrees of basis monomials of $k^P$ by

$$\deg^e(x) = \deg(\sigma^{-e}(x)) = \sum_{j=0}^{t-1} \lambda_j p^{[j-e]} \tag{23}$$

for $e = 0, \dots, t-1$. Then $\deg^e(x) \equiv 0 \mod q-1$ and $\mathscr{F}_r^{(e)}$ is spanned by all basis monomials $x$ with $(1/(q-1)) \deg^e(x) \leqslant r$. It is clear that the twisted degrees of a monomial depend only on its type. Thus, to each type $(\lambda_0, \dots, \lambda_{t-1})$ of a basis monomial of $k^P$, we may assign the $t$-tuple $(s_0, \dots, s_{t-1})$, where

$$(q-1)s_e = \sum_{j=0}^{t-1} \lambda_j p^{[j-e]}. \tag{24}$$

We define $\mathscr{H}$ to be the set of $t$-tuples of integers $(s_0, \dots, s_{t-1})$ such that the following conditions hold:

(1) $1 \leqslant s_j \leqslant n$.

(2) $0 \leqslant ps_{j+1} - s_j \leqslant (n+1)(p-1)$ (set $s_t = s_0$).

The following follows easily from the definition of $\mathscr{H}$ and the description of the types of basis monomials of $k^P$ at the end of §3.1.

LEMMA 3.1. *Formula* (24) *defines a bijection from the set of types of basis monomials of* $Y_P$ (*respectively* $k^P$) *and the set* $\mathscr{H}$ (*respectively* $\mathscr{H} \cup \{(0, \dots, 0)\}$). *The inverse map is given by the formula* $\lambda_j = ps_{j+1} - s_j$, $j = 0, \dots, t-1$.

Thus we may also speak of the $\mathscr{H}$-tuple of a basis monomial of $Y_P$.

3.3.

By Lemma 3.1, the set $\mathscr{H} \cup \{(0, \dots, 0)\}$ also parametrizes the set of composition factors of $k^P$. We set

$$L(s_0, \dots, s_{t-1}) = S(\lambda_0, \dots, \lambda_{t-1}), \tag{25}$$

for $(s_0, \dots, s_{t-1})$ corresponding to $(\lambda_0, \dots, \lambda_{t-1})$.

There is a natural partial order on $\mathscr{H}$ given by

$$(s'_0, \dots, s'_{t-1}) \leqslant (s_0, \dots, s_{t-1}) \text{ if and only if } s'_j \leqslant s_j \text{ for all } j. \tag{26}$$

We now consider the submodules

$$\mathscr{Y}_{s_0, \ldots, s_{t-1}} = Y_P \cap \mathscr{F}_{s_0}^{(0)} \cap \ldots \cap \mathscr{F}_{s_{t-1}}^{(t-1)} \tag{27}$$

of $Y_P$. Then $\mathscr{Y}_{s'_0, \ldots, s'_{t-1}} \subseteq \mathscr{Y}_{s_0, \ldots, s_{t-1}}$ if and only if $(s'_0, \ldots, s'_{t-1}) \leqslant (s_0, \ldots, s_{t-1})$.

LEMMA 3.2. (a) *The quotient of* $\mathscr{Y}_{s_0, \ldots, s_{t-1}}$ *by the sum of all the* $\mathscr{Y}_{s'_0, \ldots, s'_{t-1}}$ *with* $(s'_0, \ldots, s'_{t-1}) \lneqq (s_0, \ldots, s_{t-1})$ *is isomorphic to* $L(s_0, \ldots, s_{t-1})$.

(b) *The composition factors of* $\mathscr{Y}_{s_0, \ldots, s_{t-1}}$ *are the simple modules* $L(s'_0, \ldots, s'_{t-1})$ *for* $(s'_0, \ldots, s'_{t-1}) \leqslant (s_0, \ldots, s_{t-1})$.

*Proof.* Let $\bar{\mathscr{Y}}$ denote the quotient in part (a). This module has a basis consisting of the images of all monomials in $A$ of type $(\lambda_0, \ldots, \lambda_{t-1})$ corresponding to $(s_0, \ldots, s_{t-1})$. Let $M$ be the $kG$-submodule of $A$ generated by these monomials. Then $M \subseteq A^{\mathbb{F}_q^\times}$ and we have $kG$-maps

$$\bar{\mathscr{Y}} \longleftarrow M \longrightarrow \mathrm{gr}^{(q-1)s_0} A \cong \mathrm{gr}^{(q-1)s_0} B. \tag{28}$$

The monomials of $A$ of type $(\lambda_0, \ldots, \lambda_{t-1})$ map to the monomials of the same type in $\mathrm{gr}\, B$. The latter are weight vectors in $\mathrm{gr}\, B$ under the action of the maximal torus $T$, and map to a basis of the simple $\mathrm{GL}(n+1, k)$-module $S(\lambda_0, \ldots, \lambda_{t-1})$ under the $T$-isomorphism of Lemma 2.3. Therefore, the $\mathrm{GL}(n+1, k)$-submodule of $\mathrm{gr}\, B$ generated by monomials of type $(\lambda_0, \ldots, \lambda_{t-1})$ has a simple quotient $L$ isomorphic to $S(\lambda_0, \ldots, \lambda_{t-1})$ such that the images of the monomials in question form a basis. It is now clear that the map $M \longrightarrow L$ induces the isomorphism in part (a). Part (b) now follows easily. □

## 4. *Combinatorics of* $\mathscr{H}$

In this section, we study the partially ordered set $\mathscr{H}$. Similar combinatorics has been considered in earlier papers (cf. [**5**, 2.4 Proposition, 2.5 Lemma C; **11**, pp. 209–210; **12**, Definitions 1.4, 2.2, Proposition 2.3; **16**, §3]).

### 4.1.

For $(s_0, \ldots, s_{t-1}) \in \mathscr{H}$ we set $\lambda_j = ps_{j+1} - s_j$ as in Lemma 3.1.

LEMMA 4.1. *Let* $(s_0, \ldots, s_{t-1}) \in \mathscr{H}$. *Then* $(s_0, \ldots, s_j - 1, \ldots, s_{t-1}) \in \mathscr{H}$ *if and only if the following conditions hold:*

(1) $\lambda_j < (n+1)(p-1)$.

(2) $\lambda_{j-1} \geqslant p$.

*Moreover if* $(s_0, \ldots, s_{t-1}) \neq (1, \ldots, 1)$, *there exists some* $j$ *with* $(s_0, \ldots, s_j - 1, \ldots, s_{t-1}) \in \mathscr{H}$.

*Proof.* The necessary and sufficient conditions are immediate from the definition of $\mathscr{H}$. To prove the last statement, suppose that $s_e > 1$. Then $\sum_j \lambda_j p^{[j-e]} = (q-1)s_e > q-1$, which forces $\lambda_{j_1} \geqslant p$ for some $j_1$. Also, since $s_j \leqslant n$ for all $j$, there is some $j_2$ with $\lambda_{j_2} < (n+1)(p-1)$. Therefore either there exists $j$ with $\lambda_{j-1} = (n+1)(p-1) \geqslant p$ and $\lambda_j < (n+1)(p-1)$, in which case, this $j$ satisfies conditions (1) and (2), or else for all $j$ we have $\lambda_j < (n+1)(p-1)$, in which case $j_1 + 1$ satisfies conditions (1) and (2). □

4.2.

LEMMA 4.2. *Let* $(s_0,\dots,s_{t-1}),(s'_0,\dots,s'_{t-1})\in\mathscr{H}$ *with* $(s'_0,\dots,s'_{t-1})\lneqq(s_0,\dots,s_{t-1})$. *Then for some* $j$ *we have* $(s_0,\dots,s_j-1,\dots,s_{t-1})\in\mathscr{H}$ *and*

$$(s'_0,\dots,s'_{t-1})\leqslant(s_0,\dots,s_j-1,\dots,s_{t-1}).$$

*Proof.* Let $s_j=s'_j+a_j$. We must show that the conditions of Lemma 4.1 hold for some $j$ with $a_j\neq 0$. Let $I=\{j\,|\,a_j\neq 0\}$. By assumption $I\neq\emptyset$. Suppose for a contradiction that for every $j\in I$, either condition (1) or condition (2) in Lemma 4.1 fails. Assume first that there exists $j\in I$ such that $\lambda_j=(n+1)(p-1)$. Then

$$\begin{aligned}(n+1)(p-1)=\lambda_j&=ps_{j+1}-s_j\\&=ps'_{j+1}+pa_{j+1}-s'_j-a_j\\&=\lambda'_j+pa_{j+1}-a_j.\end{aligned}\tag{29}$$

Since $\lambda'_j\leqslant(n+1)(p-1)$ and $a_j\neq 0$, it follows that $a_{j+1}\neq 0$, so $j+1\in I$. Hence either (1) or (2) fails with $j+1$ in place of $j$. However, $\lambda_{(j+1)-1}=\lambda_j=(n+1)(p-1)\geqslant p$ so it must be (1) which fails with $j+1$ in place of $j$, namely $\lambda_{j+1}=(n+1)(p-1)$. Repeating the argument, we eventually deduce that $I=\{0,\dots,t-1\}$, contradicting the last sentence of Lemma 4.1. We may therefore assume, for every $j\in I$, that $\lambda_j<(n+1)(p-1)$ and hence that condition (2) fails for all $j$, which is to say that $\lambda_j<p$ for every $j\in I$. Then

$$0\leqslant\lambda'_{j-1}=\lambda_{j-1}-pa_j+a_{j-1}.\tag{30}$$

Since $a_j\neq 0$, we must have $a_{j-1}\neq 0$, whence $j-1\in I$. This leads us eventually to the same contradiction as before, that $I=\{0,\dots,t-1\}$. $\square$

By induction, we obtain the following corollary.

COROLLARY 4.1. *Under the hypotheses of the lemma, there is a descending chain in* $\mathscr{H}$ *from* $(s_0,\dots,s_{t-1})$ *to* $(s'_0,\dots,s'_{t-1})$ *in which successive terms are obtained by subtracting* 1 *from a suitable entry of the* $t$*-tuple.*

## 5. *Submodules generated by monomials*

The main aim of this section is to show that any basis monomial with $\mathscr{H}$-tuple $(s_0,\dots,s_{t-1})$ generates $\mathscr{Y}_{s_0,\dots,s_{t-1}}$ as a $kG$-module.

5.1.

For $0\leqslant r\neq s\leqslant n$ let $g_{rs}$ denote the linear substitution in $k[X_0,\dots,X_n]$ given by

$$g_{rs}:X_r\longmapsto X_r+X_s,\quad X_l\longmapsto X_l\quad(l\neq r).\tag{31}$$

To simplify the notation, we take $r=0$ and $s=1$. Let

$$X=\prod_{i=0}^{n}X_i^{b_i}=\prod_{j=1}^{t-1}\left(\prod_{i=0}^{n}X_i^{a_{ij}}\right)^{p^j}\tag{32}$$

be a monomial with $b_i \leqslant q-1$. We consider the following two expressions for $g_{01}X$:

$$g_{01}X = \left(\sum_{u=0}^{b_0}\binom{b_0}{u}X_0^{(b_0-u)}X_1^{b_1+u}\right)\prod_{i=2}^{n}X_i^{b_i} \tag{33}$$

$$g_{01}X = \prod_{j=0}^{t-1}\left(\left(\sum_{u_j=0}^{a_{0j}}\binom{a_{0j}}{u_j}X_0^{a_{0j}-u_j}X_1^{a_{1j}+u_j}\right)\prod_{i=2}^{n}X_i^{a_{ij}}\right)^{p^j}. \tag{34}$$

From equation (34) we see that there are $\prod_{j=0}^{t-1}(a_{0j}+1)$ distinct monomials appearing with nonzero coefficients in $g_{01}X$.

From equation (33) we see that all monomials in $g_{01}X$ afford distinct characters of $T(q)$ (the subgroup of diagonal matrices in $G$), except when $b_0 = q-1$, in which case the monomials $X$ and $X_1^{b_1+q-1}X_2^{b_2}\ldots X_n^{b_n}$ are the only two which afford the same character, for this is the only case where exponents of $X_0$ in the monomials in (33) are congruent $\bmod\, q-1$.

Now suppose that

$$x = \prod_{i=0}^{n}x_i^{b_i} = \prod_{j=0}^{t-1}\left(\prod_{i=0}^{n}x_i^{a_{ij}}\right)^{p^j} \tag{35}$$

is a basis monomial of $Y_P$. From equation (33), we obtain

$$g_{01}x = \left(\sum_{u=0}^{b_0}\binom{b_0}{u}x_0^{(b_0-u)}x_1^{\{b_1+u\}}\right)\prod_{i=2}^{n}x_i^{b_i}, \tag{36}$$

where '{c}' indicates that $q-1$ should be subtracted if $c \geqslant q$.

Since the map $k[X_0,\ldots,X_n]^{\mathbb{F}_q^{\times}} \longrightarrow k^P$ is a $kG$-map, the $\prod_{j=0}^{t-1}(a_{0j}+1)$ distinct basis monomials occurring in equation (36) afford distinct characters of $T(q)$, except when $b_0 = q-1$, in which case $x$ and $x' = x_1^{b_1+q-1}x_2^{b_2}\ldots x_n^{b_n}$ afford the same character. Since any $kG$-module is the direct sum of its $T(q)$-isotypic components, it follows that the submodule $kGx$ generated by $x$ contains all of the basis monomials which appear in (36) with nonzero coefficients (including $x'$ when $b_0 = q-1$, since then $x+x'$ is a $T(q)$-isotypic component of $g_{01}x$, so both $x$ and $x+x'$ belong to $kGx$). We have proved the following result.

LEMMA 5.1. *Let $x$ be a basis monomial of $Y_P$. Then when $g_{rs}x$ is expressed as a linear combination of basis monomials, each basis monomial which appears with a nonzero coefficient lies in the $kG$-submodule generated by $x$. If $x$ is given by equation* (35) *then these monomials are precisely the images of the* $\prod_{j=0}^{t-1}(a_{rj}+1)$ *monomials which occur in the product*

$$\prod_{j=0}^{t-1}\left(\left(\sum_{u_j=0}^{a_{rj}}\binom{a_{rj}}{u_j}X_r^{a_{rj}-u_j}X_s^{a_{sj}+u_j}\right)\prod_{i\neq r,s}X_i^{a_{ij}}\right)^{p^j}.$$

5.2.

LEMMA 5.2. *If $x$ is a basis monomial of $Y_P$ of type $(\lambda_0,\ldots,\lambda_{t-1})$, then the submodule $kGx$ contains every basis monomial of type $(\lambda_0,\ldots,\lambda_{t-1})$.*

*Proof.* Write

$$x = \prod_{j=1}^{t-1}\left(\prod_{i=0}^{n}x_i^{a_{ij}}\right)^{p^j} = \left(\prod_{i=0}^{n}x_i^{a_{i0}}\right)x^*. \tag{37}$$

Suppose that $a_{r0} \neq 0$ and $a_{s0} < p-1$. Then $g_{rs}x$ will involve the basis monomial

$$x_r^{a_{r0}-1} x_s^{a_{s0}+1} \Big( \prod_{i \neq r,s} x_i^{a_{i0}} \Big) x^* \tag{38}$$

with coefficient $a_{r0} \neq 0$, so by Lemma 5.2, this monomial belongs to the submodule $kGx$. It is clear that monomial (38) is also of type $(\lambda_0, \ldots, \lambda_{t-1})$. By repeating this procedure for different choices of $r$ and $s$, we see that $kGx$ contains all basis monomials of the form $(\prod_{i=0^n} x_i^{c_i})x*$, for any values of $c_i$ with $0 \leqslant c_i \leqslant p-1$ and $\sum_i c_i = \lambda_0$. Then we can carry out the whole process with a different value of $j$ fixed instead of 0 in (37). It follows that all monomials of type $(\lambda_0, \ldots, \lambda_{t-1})$ belong to $kGx$. □

5.3

LEMMA 5.3. *Suppose that* $(s_0, \ldots, s_{t-1}), (s_0, \ldots, s_j - 1, \ldots, s_{t-1}) \in \mathscr{H}$. *Then there exists a basis monomial x with* $\mathscr{H}$*-tuple* $(s_0, \ldots, s_{t-1})$ *such that the submodule kGx contains a basis monomial with* $\mathscr{H}$*-tuple* $(s_0, \ldots, s_j - 1, \ldots, s_{t-1})$.

*Proof.* Let $(\lambda_0, \ldots, \lambda_{t-1})$ correspond to $(s_0, \ldots, s_{t-1})$. We set out the argument only for $t > 0$ and $j = 0$; the case of $j \neq 0$ is similar, while if $t = 1$ a simpler argument will suffice. Since $(s_0 - 1, \ldots, s_{t-1}) \in \mathscr{H}$, we have $\lambda_{t-1} \geqslant p$ and $\lambda_0 < (n+1)(p-1)$. Therefore there exists a monomial of type $(\lambda_0, \ldots, \lambda_{t-1})$

$$x = \prod_{j=1}^{t-1} \left( \prod_{i=0}^{n} x_i^{a_{ij}} \right)^{p^j} \tag{39}$$

such that $a_{0,t-1} = p-1$, $a_{1,t-1} \neq 0$ and $a_{0,0} < p-1$. Write $x = \prod_{j=0}^{t-1} m_j^{p^j}$ and $X = \prod_{j=0}^{t-1} M_j^{p^j}$ with $M_j = \prod_{i=0}^{n} X_i^{a_{ij}}$. By equation (34), $g_{10}X$ involves all of the monomials in

$$\prod_{j \neq t-1} M_j^{p^j} (g_{10} M_{t-1})^{p^{t-1}} = \left( \prod_{j=0}^{t-2} M_j^{p^j} \right) \left( \sum_{l=0}^{a_{1,t-1}} \binom{a_{1,t-1}}{l} X_0^{p-1+l} X_1^{a_{1,t-1}-l} \prod_{i=2}^{n} X_i^{a_{i,t-1}} \right)^{p^{t-1}}. \tag{40}$$

The term for $l = 1$ is

$$a_{1,t-1} \left( \prod_{j=0}^{t-2} M_j^{p^j} \right) \left( X_0^p X_1^{a_{1,t-1}-1} \prod_{i=2}^{n} X_i^{a_{i,t-1}} \right)^{p^{t-1}}. \tag{41}$$

When this is mapped into $Y_P$, the $X_0^q$ from the last factor maps to $x_0^q = x_0$, so the image of (41) is

$$a_{1,t-1} \left( x_0^{a_{00}+1} \prod_{i=1}^{n} x_i^{a_{i0}} \right) \left( \prod_{j=1}^{t-2} m_j^{p^j} \right) \left( x_1^{a_{1,t-1}-1} \prod_{i=2}^{n} x_i^{a_{i,t-1}} \right)^{p^{t-1}}. \tag{42}$$

By Lemma 5.2, the monomial in (42) lies in $kGx$, since $a_{1,t-1} \neq 0$. Its type is $(\lambda_0 + 1, \ldots, \lambda_{t-1} - p)$, which corresponds to the $\mathscr{H}$-tuple $(s_0 - 1, \ldots, s_{t-1})$, by Lemma 3.1. □

Combining Lemmas 5.2 and 5.3 yields the main result of this section.

THEOREM 5.1. *Any basis monomial with* $\mathscr{H}$*-tuple* $(s_0, \ldots, s_{t-1})$ *generates* $\mathscr{Y}_{s_0,\ldots,s_{t-1}}$ *as a kG-module.*

## 6. *Non-split module extensions*

6.1.

Suppose that $(s_0, \dots, s_{t-1})$, $(s_0, \dots, s_j - 1, \dots, s_{t-1}) \in \mathscr{H}$. Assume for simplicity that $j = 0$. Let $E$ be the quotient of $\mathscr{Y}_{s_0,\dots,s_{t-1}}$ by the sum of the submodules $\mathscr{Y}_{s'_0,\dots,s'_{t-1}}$ with both $(s'_0, \dots, s'_{t-1}) \lneqq (s_0, \dots, s_{t-1})$ and $(s'_0, \dots, s'_{t-1}) \neq (s_0 - 1, \dots, s_{t-1})$.

Then we have a short exact sequence

$$0 \longrightarrow L(s_0 - 1, \dots, s_{t-1}) \longrightarrow E \longrightarrow L(s_0, \dots, s_{t-1}) \longrightarrow 0. \tag{43}$$

THEOREM 6.1. *The sequence* (43) *does not split.*

*Proof.* We shall adopt the following notational conventions. For a basis monomial $x$ of $\mathscr{Y}_{s_0,\dots,s_{t-1}}$ we shall denote its image in $E$ by the same symbol and use $\bar{x}$ for its image in the quotient $L(s_0, \dots, s_{t-1})$. Also, for $I \subseteq \{0, \dots, n\}$, we will write $X_I^{q-1}$ for the monomial $\prod_{i \in I} X_i^{q-1}$ in $k[X_0, \dots, X_n]$, with corresponding notations for monomials in $Y_P$ or $E$ or $L(s_0, \dots, s_{t-1})$.

Let $(\lambda_0, \dots, \lambda_{t-1})$ be the type corresponding to $(s_0, \dots, s_{t-1})$. For each $j$ write $\lambda_j = (p-1)k_j + r_j$, with $0 \leqslant r_j < p-1$, set

$$M_j = X_0^{p-1} \dots X_{k_j-1}^{p-1} X_{k_j}^{r_j}, \tag{44}$$

and let $m_j$ and $\bar{m}_j$ denote its images in $E$ and $L(s_0, \dots, s_{t-1})$ respectively.

Let $M = \prod_j M_j^{p^j}$, with images $m$ and $\bar{m}$. We can write $m$ in the forms

$$m = \prod_{i=0}^{n} x_i^{b_i} = x_I^{q-1} x', \tag{45}$$

where $I = \{0, \dots, \min\{k_j\} - 1\}$ and $x'$ is the remaining factor.

In order to prove that (45) does not split, it will be sufficient to show that every submodule $E'$ of $E$ which contains a preimage $\hat{m}$ of $\bar{m}$ actually contains $m$, since it follows from Theorem 5.1 that $m$ generates $E$. Suppose such an $E'$ and $\hat{m}$ are given. Since every $kG$-module is the direct sum of its $T(q)$-isotypic components, we may assume that $k\hat{m}$ is $T(q)$-stable (and therefore affords the same character of $T(q)$ as $m$). Therefore, we know that $\hat{m}$ has the form

$$\hat{m} = x_I^{q-1} x' + \sum_J a_J x_J^{q-1} x', \tag{46}$$

where $0 \neq a_J \in k$ and the monomials $x_J^{q-1} x'$ have $\mathscr{H}$-tuples $(s_0 - 1, \dots, s_{t-1})$ and no variable $x_i$ with $i \in J$ occurs in $x'$. We may suppose that $\hat{m}$ has been chosen in $E'$ so that the number of error terms $a_J x_J^{q-1} x'$ in (46) is as small as possible. If there are none, we have nothing to prove, so we assume that the number of error terms is positive and show that this leads to a contradiction. This will be achieved by defining a linear endomorphism of $E$ which preserves $E'$ and reduces the number of error terms. This definition will depend on the special choice of two indices $r$ and $s$ in $\{0, \dots, n\}$, which we describe next. Under our assumption, we observe that $I \neq \varnothing$ and that each subset $J$ has size $|I| - 1$, so we may pick $r \in I$ which lies outside at least one $J$. To choose $s$, recall that in our usual notation $b_i = \sum_{j=0}^{t-1} a_{ij} p^j$ and $\lambda_j = \sum_{i=0}^{n} a_{ij}$. Then by Lemma

4.1(1) we have $\lambda_0 < (n+1)(p-1)$, so we may choose $s$ such that $a_{s0} < p-1$. We note that $b_s+1 \not\equiv 0 \bmod p$ and $s \notin I$.

Let $T_i(q) \cong \mathbb{F}_q^{\times}$ denote the subgroup of $T(q)$ consisting of matrices having all diagonal entries equal to 1 except in the $i$th diagonal position. In the group algebra $k(T_r(q) \times T_s(q))$ of the subgroup $T_r(q) \times T_s(q)$ of $T(q)$, let $\mu$ be the primitive idempotent corresponding to the character whose value at the diagonal matrix with $r$th diagonal entry $\gamma^{-1}$ and $s$th diagonal entry $\theta^{-1}$ is $\gamma^{q-2}\theta^{b_s+1}$; in the group algebra $kT_r(q)$ let $v$ be the primitive idempotent corresponding to the trivial character of $T_r(q)$.

Of course, when $q = 2$, both $\mu$ and $v$ act as the identity map on $E$, so we first finish the argument under the assumption that $q > 2$. Now a monomial $\prod_i x_i^{c_i}$ is fixed by $\mu$ if $c_r = q-2$ and $c_s = b_s+1$, and it is otherwise annihilated by $\mu$. It is fixed by $v$ if $c_r \equiv 0 \bmod q-1$, and otherwise annihilated by $v$. Recalling from 5.1 the elements $g_{rs}$ and $g_{sr}$ of $G$, we shall show that the element $vg_{sr}\mu g_{rs}$ annihilates at least one error term in (46), while multiplying all other terms by the nonzero scalar $-(b_s+1)$. This will give the desired contradiction to our minimal choice of $\hat{m}$.

Let us examine the effect of $vg_{sr}\mu g_{rs}$ on the monomials in (46). Consider first $m = x_I^{q-1}x'$ itself. Writing $x' = x_s^{b_s}x''$ we see that $\mu g_{rs}$ maps $m$ to $-x_{I\setminus\{r\}}^{q-1}x_r^{q-2}x_s^{b_s+1}x''$ and that this is then mapped by $vg_{sr}$ to $-(b_s+1)m$. Next consider the monomials $x_J^{q-1}x'$ with $r \in J$ and $s \notin J$. In these monomials the variable $x_s$ has the same exponent $b_s$ as it has in $m$ and a similar calculation shows that $vg_{sr}\mu g_{rs}$ also multiplies them by $-(b_s+1)$. The monomials $x_J^{q-1}x'$ with $r \notin J$ do not involve $x_r$ so they are fixed by $g_{rs}$ and then annihilated by $\mu$. There remain the monomials $x_J^{q-1}x'$ with $r$, $s \in J$. These arise only if $b_s = 0$. Now

$$
\begin{aligned}
g_{rs}(x_J^{q-1}x') &= x_J^{q-1}x' + x_{J\setminus\{r,s\}}^{q-1}\sum_{l=1}^{q-1}\binom{q-1}{l}x_r^{q-1-l}x_s^{l}x' \\
&= x_J^{q-1}x'
\end{aligned}
\tag{47}
$$

because the other summands have degrees $2(q-1)$ lower than the degree of $m$, and so are equal to zero in $E$. Then $\mu$ annihilates $x_J^{q-1}x'$. These computations show that $vg_{sr}\mu g_{rs}\hat{m}$ is $-(b_s+1)$ times the element

$$
x_I^{q-1} + {\sum_J}' a_J x_J^{q-1}x'
\tag{48}
$$

where the summation extends over only those $J$ for which $r \in J$ and $s \notin J$. By our choice of $r$ the number of error terms here is strictly smaller than in (46). This contradiction of our minimal choice of $\hat{m}$ completes the proof when $q > 2$.

For $q = 2$ the same proof works if we use $(g_{sr}-1)(g_{rs}-1)$ instead of $vg_{sr}\mu g_{rs}$.

□

COROLLARY 6.1. *$\mathcal{Y}_{s_0,\dots,s_{t-1}}$ is the smallest submodule of $Y_P$ which has $L(s_0,\dots,s_{t-1})$ as a composition factor.*

*Proof.* Let $Z$ be the smallest submodule of $Y_P$ which has $L(s_0,\dots,s_{t-1})$ as a composition factor and let $P(s_0,\dots,s_{t-1})$ denote the projective $kG$-module which has $L(s_0,\dots,s_{t-1})$ as its unique simple quotient (see [**8**, I.13.6]). Since $Y_P$ is multiplicity-free, the space $\mathrm{Hom}_{kG}(P(s_0,\dots,s_{t-1}), Y_P)$ is one-dimensional and any nonzero map in this space has $Z$ as its image (cf. [**8**, I.13.9]). We proceed by induction on $\Sigma_j s_j$. Consider the maps

$$\begin{tikzcd} \mathscr{Y}_{s_0,\ldots,s_{t-1}} \arrow[r, "\gamma"] & E \arrow[r, "\tau"] & L(s_0,\ldots,s_{t-1}) \\ & & P(s_0,\ldots,s_{t-1}). \arrow[u, "\pi"] \end{tikzcd} \tag{49}$$

By projectivity, the map $\pi$ of $P(s_0, \ldots, s_{t-1})$ onto its simple head may be lifted to a map

$$P(s_0, \ldots, s_{t-1}) \xrightarrow{\tilde{\pi}} E,$$

so that $\tau\tilde{\pi} = \pi$. Theorem 6.1 shows that $E$ has a simple head, so $\tilde{\pi}$ must be surjective. Again, projectivity yields a lifting

$$P(s_0, \ldots, s_{t-1}) \xrightarrow{\pi^*} \mathscr{Y}_{s_0,\ldots,s_{t-1}}, \quad \text{with } \gamma\pi^* = \tilde{\pi}.$$

Therefore, the image of $\pi^*$, which must be equal to $Z$, has $L(s_0 - 1, \ldots, s_{t-1})$ as a composition factor. The same is true with 0 replaced by any $j$ such that $(s_0, \ldots, s_j - 1, \ldots, s_{t-1}) \in \mathscr{H}$. The inductive hypothesis implies that $Z$ contains all the modules $\mathscr{Y}_{s'_0,\ldots,s'_{t-1}}$ for which $(s'_0, \ldots, s'_{t-1}) \leqslant (s_0, \ldots, s_{t-1})$ and $\sum_j s'_j = (\sum_j s_j) - 1$. By Lemma 4.2, it now follows that $Z$ contains all $\mathscr{Y}_{s'_0,\ldots,s'_{t-1}}$ with $(s'_0, \ldots, s'_{t-1}) \lneqq (s_0, \ldots, s_{t-1})$. Therefore, $Z = \mathscr{Y}_{s_0,\ldots,s_{t-1}}$, by Lemma 3.2(a). $\square$

## 7. *Proof of Theorems* A *and* B

### 7.1. *Proof of Theorem* A

Parts (a) and (b) have been proved in Lemma 2.1 and Theorem 2.1. (See also §3.3.) To prove part (c), note that Corollary 6.1 implies that each submodule of $Y_P$ is a sum of the submodules $\mathscr{Y}_{s_0,\ldots,s_{t-1}}$. Since the $\mathscr{H}$-tuples of the composition factors of $\mathscr{Y}_{s_0,\ldots,s_{t-1}}$ form the ideal in $\mathscr{H}$ of the elements dominated by $(s_0, \ldots, s_{t-1})$ and since taking the sum of submodules corresponds to taking the union of their sets of composition factors, part (c) is proved. The mapping in part (d) is injective, by (a). Surjectivity is also immediate; any ideal $\mathscr{I}$ of $\mathscr{H}$ corresponds to the submodule

$$\sum_{(s_0,\ldots,s_{t-1}) \in \mathscr{I}} \mathscr{Y}_{s_0,\ldots,s_{t-1}}. \tag{50}$$

This completes the proof of Theorem A. $\square$

### 7.2. *Proof of Theorem* B

Write $f \in k^P$ as

$$f = \sum_{(s_0,\ldots,s_{t-1}) \in \mathscr{H}_f} f_{(s_0,\ldots,s_{t-1})}, \tag{51}$$

where $f_{(s_0,\ldots,s_{t-1})}$ is a linear combination of basis monomials of $k^P$ with tuple $(s_0, \ldots, s_{t-1}) \in \mathscr{H} \cup \{0\}$.

If $(0,\dots,0)\in\mathscr{H}_f$, then $f$ has a nonzero projection onto the space of constant functions $k1$. Since $k^P$ is multiplicity-free, this implies that $k1\subseteq kGf$. Therefore, from now on, we may assume that $f\in Y_P$ and $\mathscr{H}_f\subseteq\mathscr{H}$. Then

$$kGf\subseteq\sum_{(s_0,\dots,s_{t-1})\in\mathscr{H}_f}\mathscr{Y}_{s_0,\dots,s_{t-1}} \tag{52}$$

and in the sum we may even replace the set $\mathscr{H}_f$ by its set $\mathscr{H}_f^*$ of maximal elements. For each $(s'_0,\dots,s'_{t-1})\in\mathscr{H}_f^*$, we have a nonzero $kG$-map

$$\sum_{(s_0,\dots,s_{t-1})\in\mathscr{H}_f^*}\mathscr{Y}_{s_0,\dots,s_{t-1}}\longrightarrow L(s'_0,\dots,s'_{t-1}). \tag{53}$$

Therefore, $kGf$ has $L(s'_0,\dots,s'_{t-1})$ as a quotient. By the theorem, we then have $\mathscr{Y}_{s'_0,\dots,s'_{t-1}}\subset kGf$, so the inclusion (52) is actually equality. Since the right-hand side of (52) is the smallest submodule of $Y_P$ which has all $L(s_0,\dots,s_{t-1})$ with $(s_0,\dots,s_{t-1})\in\mathscr{H}_f$ as composition factors, Theorem B is proved. □

### 7.3. *Socle and radical series*

Recall that the *radical* of a module is the smallest submodule with semisimple quotient (called the *head*). Iterating, we obtain the *radical series*. Dually, the maximal semisimple submodule is called the *socle* and we have the *socle series*.

Define $\mathscr{G}_r\subset Y_P$ to be the span of the basis monomials $x$ whose $\mathscr{H}$-tuples in $H$ satisfy $\sum_j s_j\leqslant r$. In other words,

$$\mathscr{G}_r=\sum_{\substack{(s_0,\dots,s_{t-1})\\ \sum_j s_j=r}}\mathscr{Y}_{s_0,\dots,s_{t-1}}. \tag{54}$$

Then, since elements of $\mathscr{H}$ with the same value for $\sum_j s_j$ are incomparable, we have semisimple layers

$$\mathscr{G}_r/\mathscr{G}_{r-1}\cong\bigoplus_{\substack{(s_0,\dots,s_{t-1})\\ \sum_j s_j=r}}L(s_0,\dots,s_{t-1}). \tag{55}$$

The following result now follows easily from Theorem A.

COROLLARY 7.1. *The filtration $\mathscr{G}_r$ is equal (after suitable re-indexing) to the socle filtration and to the radical filtration in reverse.*

## 8. *Hamada's formula*

### 8.1.

Let $\mathscr{C}_r\subseteq k^P$ be the subspace spanned by the characteristic functions of $r$-dimensional linear subspaces of $P$. Since $G$ acts transitively on the set of these subspaces, $\mathscr{C}_r$ is equal to $kG\chi_L$, where $L$ is defined by the equations $X_i=0$, $i=r+1,\dots,n$. Its characteristic function can be written as

$$\chi_L=\prod_{i=r+1}^{n}(1-x_i^{q-1})=\sum_{I\subseteq\{r+1,\dots,n\}}(-1)^{|I|}x_I^{q-1}. \tag{56}$$

For $I \neq \varnothing$ the monomial $x_I^{q-1}$ has $\mathscr{H}$-tuple $(|I|,\dots,|I|)$, which lies below the $\mathscr{H}$-tuple $(n-r,\dots,n-r)$ of $\prod_{i=r+1}^{n} x_i^{q-1}$. Therefore, Theorems A and B yield

$$\mathscr{C}_r = k1 \oplus \mathscr{Y}_{n-r,\dots,n-r}. \tag{57}$$

Therefore by Lemma 3.2(b) and Corollary 2.1 we obtain

$$\dim \mathscr{C}_r = 1 + \sum_{(s_0,\dots,s_{t-1})} \prod_{j=0}^{t-1} \sum_{i=0}^{\lfloor ps_{j+1}-s_j/p \rfloor} (-1)^i \binom{n+1}{i} \binom{n+ps_{j+1}-s_j-ip}{n}, \tag{58}$$

where the sum extends over $(s_0,\dots,s_{t-1}) \in \mathscr{H}$ with $1 \leqslant s_j \leqslant n-r$. Next, we note that in the definition of $\mathscr{H}$ (§3.2), if we change condition 1 to condition 1′: $0 \leqslant s_j \leqslant n$, the only extra tuple will be $(0,\dots,0)$, so we could rewrite equation (58) by summing over $\mathscr{H} \cup \{(0,\dots,0)\}$ and omitting the term 1 from (58). Then, applying Lemma 2.5(c), we obtain Hamada's rank formula [**10**, Theorem 1].

$$\dim \mathscr{C}_r = \sum_{\substack{(s_0,\dots,s_{t-1}) \\ r+1 \leqslant s_j \leqslant n+1 \\ 0 \leqslant ps_{j+1}-s_j \leqslant (n+1)(p-1)}} \prod_{j=0}^{t-1} \sum_{i=0}^{\lfloor ps_{j+1}-s_j/p \rfloor} (-1)^i \binom{n+1}{i} \binom{n+ps_{j+1}-s_j-ip}{n}. \tag{59}$$

REMARKS. (1) Note that this formula could have been proved without the full strength of Theorems A and B, for equation (57) can be deduced from (56) using only Theorem 5.1 and induction on $n-r$.

(2) One can also ask about the ranks of the incidence matrices with entries over fields of characteristic prime to $q$, or for the elementary divisors over the integers. The first question is answered in [**9**], using results of G. D. James. The second question has been settled so far only in the case of point-hyperplane incidences and $q = p$ [**2**].

## 9. *Structure of A*

As we mentioned in the introduction, we can extend our results on $Y_P$ to obtain the submodule structure of $A = k[X_0,\dots,X_n]/(X_i^q - X_i)_{i=0}^{n}$. Under the action of $Z(G) \cong \mathbb{F}_q^{\times}$, we have the decomposition

$$A = \bigoplus_{[d] \in \mathbb{Z}/(q-1)\mathbb{Z}} A[d] \tag{60}$$

into isotypic components, where $[d]$ is the character $\epsilon \longmapsto \epsilon^{-d}$. The summand $A[0]$ is isomorphic to $k \oplus k^P$ by equation (8), so we may restrict our attention to the other summands.

From now on let $d$ be a fixed integer with $0 < d < q-1$. Our aim is to prove Theorem C. The proof consists of generalizing the definitions we have made to study $Y_P$, and then checking that the arguments used in proving Theorem A carry over with suitable modifications. We note that $A[d]$ is multiplicity-free, by Lemma 2.1.

### 9.1.

The purpose of this subsection is to set up the notation for Theorem C. The discussion runs parallel to that of §3. Recall that $z_i$ denotes the image of $X_i$ in $A$. Then $A[d]$ has a basis consisting of the monomials $\prod_{i=0}^{n} z_i^{b_i} : 0 \leqslant b_i \leqslant q-1$, whose degrees are congruent to $d \bmod q-1$. We shall call this the monomial basis of $A[d]$. The types (§3.1) of the basis monomials are all the possible tuples $(\lambda_0,\dots,\lambda_{t-1})$ of integers satisfying $0 \leqslant \lambda_j \leqslant (n+1)(p-1)$ and $\sum_j \lambda_j p^j \equiv d \bmod q-1$. The degree filtration of $k[X_0, \dots, X_n]$ induces the filtration $\{\mathscr{F}_r[d]\}$ on $A[d]$ given by

$$\mathscr{F}_r[d] = \text{span of basis monomials } z = \prod_i z_i^{b_i} \text{ with } \sum_i b_i = d + s(q-1) \text{ for some } s \leqslant r. \tag{61}$$

In its action on $A$, the generator $\sigma$ of $\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ maps $A[m]$ to $A[pm]$. Let $d^{(e)}$ be the unique integer with $0 < d^{(e)} < q-1$ such that $p^e d^{(e)} \equiv d \bmod q-1$. Then we can define for each $e = 0, \dots, t-1$ a filtration $\{\mathscr{F}_r^{(e)}[d]\}$ by

$$\mathscr{F}_r^{(e)}[d] = \sigma^e(\mathscr{F}_r[d^{(e)}]). \tag{62}$$

We define the twisted degree of a basis monomial $z$ of type $(\lambda_0, \dots, \lambda_{t-1})$ to be

$$\deg^e(z) = \sum_j \lambda_j p^{[j-e]}. \tag{63}$$

Then $\deg^e(z) \equiv d^{(e)} \bmod q-1$ and $\mathscr{F}_r^{(e)}[d]$ is spanned by basis monomials $z$ with $(1/(q-1))(\deg^e(z) - d^{(e)}) \leqslant r$. Thus we may associate with each type $(\lambda_0, \dots, \lambda_{t-1})$ the $t$-tuple $(r_0, \dots, r_{t-1})$ of nonnegative integers by the formula

$$\sum_j \lambda_j p^{[j-e]} = d^{(e)} + (q-1)r_e, \quad (e = 0, \dots, t-1). \tag{64}$$

Let $d = \sum d_j p^j$, $0 \leqslant d_j \leqslant p-1$, be the $p$-adic expression for $d$. The type $\lambda$ can be recovered from its tuple by the formula

$$\lambda_j = d_j + p r_{j+1} - r_j, \quad (r_t = r_0). \tag{65}$$

Define the set

$$\mathscr{H}[d] = \{(r_0, \dots, r_{t-1}) \mid 0 \leqslant r_j \leqslant n, 0 \leqslant d_j + p r_{j+1} - r_j \leqslant (n+1)(p-1)\}. \tag{66}$$

Then (cf. Lemma 3.1) formulae (64) and (65) define inverse bijections between the set of types of basis monomials for $A[d]$ and the set $\mathscr{H}[d]$. It follows from Lemma 2.3, Lemma 2.2 and Theorem 2.1 that the set of types parametrizes the set of composition factors of $A[d]$, so we can also parametrize them with $\mathscr{H}[d]$, setting

$$L[d](r_0, \dots, r_{t-1}) = S(\lambda_0, \dots, \lambda_{t-1}). \tag{67}$$

The set $\mathscr{H}[d]$ is partially ordered by the rule $(r'_0, \dots, r'_{t-1}) \leqslant (r_0, \dots, r_{t-1})$ if and only if $r'_j \leqslant r_j$ for each $j$.

We consider now the submodules

$$\mathscr{F}[d]_{r_0, \dots, r_{t-1}} = \mathscr{F}_{r_0}^{(0)}[d] \cap \dots \cap \mathscr{F}_{r_{t-1}}^{(t-1)}[d] \tag{68}$$

for $(r_0, \dots, r_{t-1}) \in \mathscr{H}[d]$. These will play the same role as the modules $\mathscr{Y}_{s_0, \dots, s_{t-1}}$ in the discussion of $Y_P$. We have $\mathscr{F}[d]_{r'_0, \dots, r'_{t-1}} \subseteq \mathscr{F}[d]_{r_0, \dots, r_{t-1}}$ if and only if $(r'_0, \dots, r'_{t-1}) \leqslant (r_0, \dots, r_{t-1})$. Also, the analogue of Lemma 3.2 holds; the quotient of $\mathscr{F}[d]_{r_0, \dots, r_{t-1}}$ by the sum of all the submodules $\mathscr{F}[d]_{r'_0, \dots, r'_{t-1}}$, where $(r'_0, \dots, r'_{t-1}) \lneqq (r_0, \dots, r_{t-1})$, is isomorphic to $L[d](r_0, \dots, r_{t-1})$ and the composition factors of $\mathscr{F}[d]_{r_0, \dots, r_{t-1}}$ are the simple modules $L[d](r'_0, \dots, r'_{t-1})$ for $(r'_0, \dots, r'_{t-1}) \leqslant (r_0, \dots, r_{t-1})$.

### 9.2. *Proof of Theorem* C

We carry on with the notation of the previous section.

In order to prove Theorem C it remains to point out the necessary changes to the statements and proofs in §§4–7 used in proving Theorem A.

§4: If we replace $\mathscr{H}$ by $\mathscr{H}[d]$, the results remain valid. In Lemma 4.1, the minimal element $(1, \ldots, 1)$ must of course be replaced by $(0, \ldots, 0)$. The inequality in the second line becomes $\sum_j \lambda_j p^{[j-e]} = d^{(e)} + (q-1)r_e > q-1$, which holds because $d^{(e)} > 0$. In Lemma 4.2, the relation $\lambda_j = ps_{j+1} - s_j$ must be replaced by equation (65), but the equality between the first and last members of (29) still holds, as does equation (30) of that lemma.

§5: The very general discussion here makes no use of the congruence class of the degrees of the basis monomials, so all results remain valid with $Y_P$, $\mathscr{Y}_{s_0, \ldots, s_{t-1}}$ and $\mathscr{H}$ replaced by $A[d], \mathscr{F}[d]_{r_0, \ldots, r_{t-1}}$ and $\mathscr{H}[d]$.

§6: Again, the arguments do not depend on the congruence class of $d \bmod q-1$, and all that is required is to substitute the relevant notations into both the statements and proofs. In fact, since we are assuming that $0 < d < q-1$, the proof of the analogue of Theorem 6.1 is simplified by not having to consider the case $q = 2$.

§7: Parts (a) and (b) of Theorem C have been proved in the discussion leading up to equation (67). The other parts are proved just as for Theorem A, using the analogue of Corollary 6.1.

This completes our outline of the proof of Theorem C. □

## 10. *Structure of symmetric powers*

Let $S^d \subseteq k[X_0, \ldots, X_n]$ denote the space of homogeneous polynomials of degree $d$. It has a natural structure as a rational module for the algebraic group $\mathrm{GL}(n+1, k)$. Since $d < q-1$ the map $S^d \longrightarrow A[d]$ is an embedding of $kG$-modules with image $\mathscr{F}_0[d]$. Thus $S^d$ corresponds to the ideal

$$\mathscr{H}[d]_{S^d} = \{(r_0, \ldots, r_{t-1}) \in \mathscr{H}[d] \mid r_0 = 0\}. \tag{69}$$

This gives the submodule structure of $S^d$ as a module for $G = \mathrm{GL}(n+1, p^t)$.

We now examine what happens if we fix $d$ and replace $p^t$ by a higher power $p^N$. Let $A[d](N)$, $\mathscr{H}[d](N)$, etc. denote the corresponding objects for $G(N) = \mathrm{GL}(n+1, p^N)$. Then

$$\mathscr{H}[d](N)_{S^d} = \{(r_0, \ldots, r_{N-1}) \in \mathscr{H}[d](N) \mid r_0 = 0\}. \tag{70}$$

The $p$-adic expression for $d$ is unchanged, so that we have $d_j = 0$ for $t \leqslant j \leqslant N-1$. Let $(r_0, \ldots, r_{N-1}) \in \mathscr{H}[d](N)_{S^d}$. Then from the definitions we have $0 \leqslant d_{N-1} + pr_0 - r_{N-1} = -r_{N-1}$, which forces $r_{N-1} = 0$. Repeating this, we obtain $r_j = 0$ for $t \leqslant j \leqslant N-1$. Moreover, the conditions on the entries $r_j$ for $0 \leqslant j \leqslant t-1$ are exactly the conditions for the $t$-tuple $(r_0, \ldots, r_{t-1})$ to belong to $\mathscr{H}[d]_{S^d}$.

We have proved the following theorem.

THEOREM D. *The submodule lattice of $S^d$ is the same for all of the groups $\mathrm{GL}(n+1, p^t)$ for $p^t - 1 > d$. Consequently, it is also the same for the algebraic group $\mathrm{GL}(n+1, k)$. This lattice is isomorphic to the lattice of ideals in the partially ordered set $\mathscr{H}[d]_{S^d}$.*

The $\mathrm{GL}(n+1, k)$ submodule structure of $S^d$ was first given in [**5**] and [**12**, **13**], the authors of which were working independently.

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