Quantum walks on Cayley graphs

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Overview

Quantum walks on graphs

Strong cospectrality

Strong Cospectrality in Normal Cayley Graphs

Perfect state transfer

Pauli spin matrices

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These act on a 2-diml. complex inner product space (one qubit), orthonormal basis v_0, v_1.

N qubits with nearest-neighbour interaction

Let Γ be a graph with N vertices, and adjacency matrix A. The state space of N qubits is the tensor power $(\mathbb{C}^2)^{\otimes N}$ with orthonormal basis vectors $v_I = v_{i_1} \otimes \cdots \otimes v_{i_N}$, $I = \{k \mid i_k = 1\}$. If we assign each qubit to a vertex of Γ, then one possible Hamiltonian for the qubits to interact with adjacent qubits is

$$H = \frac{1}{4} \sum_{i,j} A_{ij}(X_i X_j + Y_i Y_j),$$

where, for example, X_i is the operator acting as the Pauli X matrix on the i-th qubit and as the identity on all other qubits. (This Hamiltonian is variously referred to as the XX or XY Hamiltonian.)

H commutes with the operator $\sum_{i=1}^{N} Z_i$, so leaves invariant its eigenspaces, such as the 1-excitation subspace $W = \langle v_{\{i\}}, 1 \leq i \leq N \rangle$. We have $H_{|W} = A$. On W, the Schrödinger equation takes the form

$$i \frac{d}{dt} v(t) = A v(t),$$

whose solutions gives the time evolution $v(t) = e^{-itA} v(0)$ of any state in W.

Quantum walks

A quantum walk on Γ consists of an initial state $v(0) \in W$ and the family $U(t) = e^{-itA}$ of unitary matrices. Change notation slightly and write the orthonormal basis elements of W as e_a, for vertices a of Γ. Often the initial state is taken as e_a for some a.

Perfect state transfer

We say that the quantum walk on Γ admits perfect state transfer from vertex a to vertex b at time τ if

$$U(\tau)e_a = \gamma e_b,$$

for some $\gamma \in C$ of norm 1. (The γ enters because states are not vectors but rather points of the projective space.)

First Examples

P_2, $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $U(t) = \begin{bmatrix} \cos t & -i \sin t \\ -i \sin t & \cos t \end{bmatrix}$, so we have PST at $t = \pi/2$.

Similarly, P_3 admits PST between its end vertices, but it can be shown that there can be no PST for P_n, $n \geq 4$.

In general, PST is a rare phenomenon. Godsil showed that if the maximum degree of a graph is fixed, then there are only finitely many connected graphs that admit PST.

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Let the spectral decomposition of A be

$$A = \sum_{r=1}^{d} \theta_r E_r$$

where the E_r are the idempotent projections onto eigenspaces.

$$U(t) = e^{-itA} = \sum_{r=1}^{d} e^{-it\theta_r} E_r$$

Suppose we have PST fro m a to b at time τ. Then there exist τ and γ with

$$\sum_{r=1}^{d} e^{-i\tau\theta_r} E_r e_a = \gamma e_b,$$

so for every r we have

$$e^{-i\tau\theta_r} E_r e_a = \gamma E_r e_b. \tag{1}$$

Since the entries of E_r, e_a and e_b are real, it follows that $\gamma^{-1} e^{-i\tau\theta_r} = \pm 1$. Thus we have

$$E_r e_b = \epsilon_r E_r e_a, \qquad \epsilon_r = \pm 1 \qquad \text{for all } r \text{ with } E_r e_a \neq 0. \tag{2}$$

We say that a and b are strongly cospectral iff (2) holds.

Theorem

(Coutinho-Godsil) Vertices a and b of a graph are s.c. iff there is a matrix Q such that

1. *Q is a polynomial in A;*
2. *$Q^2 = I$; and*
3. *$Qe_a = e_b$*

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Strong Cospectrality in normal Cayley Graphs

$X = \mathrm{Cay}(G, S)$ simple, normal Cayley graph, (S closed under inversion, conjugation, $1 \notin S$, connected if S generates G)
Eigenvalues come from irreducible characters. $\chi \in \mathrm{Irr}(G)$ gives the eigenvalue

$$\theta_\chi = \frac{\chi(S)}{\chi(1)}.$$

Theorem

Distinct elements g and h of G are strongly cospectral iff there is a central involution z such that the following hold.

(a) $h = zg$.

(b) $(\forall \chi, \psi \in \mathrm{Irr}(G))$, $\frac{\chi(S)}{\chi(1)} = \frac{\psi(S)}{\psi(1)}$ *implies* $\frac{\chi(z)}{\chi(1)} = \frac{\psi(z)}{\psi(1)}$.

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Back to PST

For a vertex a, let $\Phi_a = \{\theta_r \mid E_r e_a \neq 0\}$. Suppose a and b are strongly cospectral. Then $\Phi_b = \Phi_a$ and we define $\Phi^+_{a,b} = \{\theta_r \in \Phi_a \mid \epsilon_r = 1\}$ and $\Phi^-_{a,b} = \{\theta_r \in \Phi_a \mid \epsilon_r = -1\}$.

Theorem

A k-regular graph Γ admits PST between vertices a and b at some time if and only if the following hold.

1. *a and b are strongly cospectral.*
2. $\Phi_a \subseteq \mathbb{Z}$.
3. *There exists a nonnegative integer M such that $v_2(k-\mu) = N$ for all $\mu \in \Phi^-_{a,b}$ and $v_2(k-\mu) > N$ for all $\mu \in \Phi^+_{a,b}$.*

Remark

Assume a and b are strongly cospectral. We'll sketch how above theorem is related to the equation (2) *for PST*

$$e^{-i\tau\theta_r} E_r e_a = \gamma E_r e_b = \gamma \epsilon_r E_r e_a.$$

which must hold for all r with $\theta_r \in \Phi_a$. If we set $\theta_1 = k$, then we know that $\theta_1 \in \Phi_a$ and $\epsilon_1 = 1$, since the entries of E_1 are positive. Thus,

$$e^{-i\tau\theta_1} E_1 e_a = \gamma E_1 e_a$$

so $\gamma = e^{-i\tau k} = \gamma$. Thus we have for all $\theta_r \in \Phi_a$,

$$e^{-i\tau(k-\theta_r)} = \epsilon_r.$$

This means $\tau(k - \theta_r)$ must be an integral multiple of π, and this multiple must be odd or even according to ϵ_r. This is the meaning of part (3) of the Theorem.

PST in extraspecial 2-groups (joint work with Julien Sorci)

Let G be an extraspecial 2-group (of either type) of order 2^{2n+1}. Let z be the central involution. The the noncentral classes have the form $\{x, zx\}$. We consider $\mathrm{Cay}(G, S)$ with $S \setminus \{z\} = \cup_{i=1}^{\ell} \{x_i, zx_i\}$ containing ℓ noncentral classes. Let $\overline{x}_i$ be the image of x_i in $G/Z(G) \cong (\mathbb{Z}/2\mathbb{Z})^{2n}$. For $y \in (\mathbb{Z}/2\mathbb{Z})^{2n}$, let $n_y = |\{i \mid \overline{x}_i \cdot y = 0\}|$.

Theorem

Let $g \in G$. Then $\mathrm{Cay}(G, S)$ has PST from g to zg at some time if and only if one of the following holds.

1. *$z \in S$ and for all $y \in (\mathbb{Z}/2\mathbb{Z})^{2n}$ we have $v_2(\ell - n_y) \geq v_2(\ell + 1)$; or*
2. *$z \notin S$ and for all $y \in (\mathbb{Z}/2\mathbb{Z})^{2n}$ we have $v_2(\ell - n_y) \geq v_2(\ell)$.*

In particular (1) holds whenever ℓ is even and (2) holds whenever ℓ is odd.