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MODULAR REPRESENTATIONS OF THE HALL-JANKO GROUP

PETER SIN
University of Florida
358 Little Hall
PO Box 118105
Gainesville
FL 32611-8105 USA

Abstract. We study modular representations of the Hall-Janko group and its double cover in characteristics 2, 3 and 5. In particular, we determine all extensions of simple modules. Results on the group $G_2(2)' \cong \mathrm{PSU}(3,3)$, which is isomorphic to a maximal subgroup of the Hall-Janko group, are also included.

INTRODUCTION

In this paper we study the modular representations of the Hall-Janko group $G = J_2$, a sporadic simple group of order $604800 = 2^7.3^3.5^2.7$ and of its twofold universal covering group $\widehat{G}$. The (Brauer) characters of simple modules have been determined in all characteristics ([16], [12], [13]), as have the decomposition numbers. The blocks with cyclic defect are described in [8] (see also [10]) and the only non-principal 2-block has been treated in [15]. In these cases it is not difficult to give the Loewy structure of the projective indecomposable modules. The blocks which have not yet been studied are the principal blocks and two other blocks of $\widehat{G}$ of maximal defect, one for each of the characteristics 3 and 5. We shall determine the extensions between the simple modules in these cases. Characteristics 3 and 5 are relatively easy, and one can try to say more about the projective modules. The characteristic 2 calculations appear to be harder. In our approach, we rely heavily on the representation theory of the algebraic group $G_2(k)$, where k is an algebraically closed field of characteristic 2 which was studied in [17] and [6]. G embeds into this group in such a way that most of the irreducible representations of G over k extend to $G_2(k)$.

It can be expected that computer calculations will eventually furnish the entire Loewy structure of all of the projective indecomposable modules. In this direct approach, it is necessary first to find matrices of group generators acting on suitable projective modules and this first step is not automatic and may be quite challenging. In this paper, we calculate the extensions of simple modules by indirect methods, for which the initial data needed are essentially character-theoretic. Unfortunately, our collection of tricks does not amount to an algorithm. Nevertheless, we hope that some of them will be useful for investigations of other groups.

§1. Statement of results

Let L be a simple kG-module and P its projective cover. If $p \neq 2$, then P is also the projective cover of L when both are regarded as $k\widehat{G}$-modules. For $p = 2$, the $k\widehat{G}$-projective cover $\widehat{P}$ of L is different from P but the simple module extensions are the same; this follows from 5-term sequences arising from the group extension $1 \to Z(\widehat{G}) \to \widehat{G} \to G \to 1$, using the fact that $Z(\widehat{G}) \leq [\widehat{G}, \widehat{G}]$.

THEOREM. *In the tables below the dimension of the space of extensions between two simple modules is the entry of the row labelled by the first module and the column labelled by the second.*

$p = 2$.

There are two conjugacy classes of embeddings of G into $G_2(4)$ and they are interchanged by the Frobenius map (see [18]). We have chosen our notation (see §2 below) so that (for a fixed embedding of G in $G_2(k)$) the simple modules with subscript 1 are the restrictions of simple G-modules with *restricted* highest weights. The modules with subscripts 2 are then the restrictions to G of their twists by the Frobenius map. This turns out to be the same as twisting by the outer automorphism of G.

TABLE 1. The principal 2-block of G.

Ext^1_{kG}	k	6_1	6_2	14_1	14_2	36	84
k	0	1	1	0	0	0	1
6_1	1	0	0	0	1	1	0
6_2	1	0	0	1	0	1	0
14_1	0	0	1	0	0	0	1
14_2	0	1	0	0	0	0	1
36	0	1	1	0	0	0	0
84	1	0	0	1	1	0	0

$p = 3$. Our notation follows that of [9]. There is one choice to be made with regard to the pair of dual simple modules of dimension 36 in the non-principal block of maximal defect. Since these two are the only simple modules which are not self-dual, we can choose either to be the one labelled 36_1, since complex conjugation fixes all other simple Brauer characters. We have chosen 36_1 to be the one for which $\mathrm{Ext}^1_{k\widehat{G}}(6_1, 36_1) \neq 0$. (See below that this condition picks out one of the two 36-dimensional modules.)

TABLE 2. The principal 3-block of G.

Ext^1_{kG}	k	13_1	13_2	21_1	21_2	57_1	57_2	133
k	0	1	1	0	0	0	0	0
13_1	1	0	0	0	0	1	0	1
13_2	1	0	0	0	0	0	1	1
21_1	0	0	0	0	0	0	0	1
21_2	0	0	0	0	0	0	0	1
57_1	0	1	0	0	0	0	0	0
57_2	0	0	1	0	0	0	0	0
133	0	1	1	1	1	0	0	0

TABLE 3. The non-principal 3-block of $\widehat{G}$ of maximal defect.

$\mathrm{Ext}^1_{k\widehat{G}}$	6_1	6_2	14	36_1	36_2	50_1	50_2	236
6_1	0	0	0	1	0	0	1	0
6_2	0	0	0	0	1	1	0	0
14	0	0	0	1	1	1	1	1
36_1	0	1	1	0	0	0	0	0
36_2	1	0	1	0	0	0	0	0
50_1	0	1	1	0	0	0	0	0
50_2	1	0	1	0	0	0	0	0
236	0	0	1	0	0	0	0	0

$p = 5$.

Again our notation matches [9].

TABLE 4. The principal 5-block of G.

Ext^1_{kG}	k	14	21	41	85	189
k	0	0	1	0	0	0
14	0	1	1	0	0	1
21	1	1	1	1	0	1
41	0	0	1	0	1	0
85	0	0	0	1	0	1
189	0	1	1	0	1	1

TABLE 5. The non-principal 5-block of $\widehat{G}$ of maximal defect.

$\mathrm{Ext}^1_{k\widehat{G}}$	6	14	56	64	190	202
6	1	0	1	1	0	0
14	0	0	0	1	0	1
56	1	0	1	1	1	1
64	1	1	1	1	0	0
190	0	0	1	0	0	0
202	0	1	1	0	0	0

REMARK. Notice that there exists a bijection between the sets of simple modules in these two blocks which preserves extensions. The blocks are not Morita equivalent, since they have different Cartan matrices. However, J. Rickard has provided some good evidence to suggest that the two blocks are derived equivalent in a particularly simple way. In particular he has described a complex which ought to be a tilting complex for the above equivalence and which gives a change of basis to show that the two Cartan matrices define the same integral quadratic form. Also each block has 14 complex characters all of degrees not divisible by 5. So it is possible that the blocks are derived equivalent over a suitable 5-adic ring.

APPENDIX. EXTENSIONS FOR $U_3(3) \cong (G_2(2))'$

Here we give tables of extensions for the group $H = U_3(3)$. The subgroups of G which are isomorphic to H form a unique class of maximal subgroups. These results are obtained by similar methods, except that, as might be expected for this smaller group, the consideration of induced modules is an effective additional technique. The group order is $2^5.3^3.7$, so the characteristic 7 case is covered by the theory of blocks of defect one (see [10]). We will treat the other two primes. As far as I am aware, several of the results below appear as isolated calculations, scattered throughout the literature, and there is little doubt that most or all of the others are also known, whether published or not. The main reason for including this appendix is simply to provide a convenient reference.

$p = 2$.

H has index 2 in $G_2(2)$. The latter has three simple modules in the principal block of dimensions 1, 6 and 14. Their restrictions to H are the simple modules in the principal block of H. The other block of $G_2(2)$ is of defect zero, containing the 64-dimensional Steinberg module, whose restriction to H splits into two nonisomorphic, 32-dimensional simple projective modules.

TABLE A1. The principal 2-block of $H \cong (G_2(2))'$.

Ext^1_{kH}	k	6	14
k	0	1	0
6	1	0	1
14	0	1	0

REMARK. The extensions for the principal block of $G_2(2)$ are given by the same table except that the entries on the leading diagonal are 1 instead of 0.

$p = 3$.

$H \cong U_3(3)$ has two blocks, the principal block and a block of defect 0 containing the 27-dimesional Steinberg module. The principal block has 8 simple modules, denoted by their dimensions in the table below. The module 3 affords the natural representation and 6 is its symmetric square. The superscript "*" denotes duality. The module 7 is the nontrivial composition factor of $3 \otimes 3^*$, that is the quotient of the space of matrices of trace 0 by the scalar matrices, and the module 15 is the kernel of the natural map $3^* \otimes S^2(3) \to 3$.

TABLE A2. The principal 3-block of H.

Ext^1_{kH}	1	3	3*	6	6*	7	15	15*
1	0	0	0	1	1	1	0	0
3	0	0	1	0	0	1	0	1
3*	0	1	0	0	0	1	1	0
6	1	0	0	0	0	0	1	0
6*	1	0	0	0	0	0	0	1
7	1	1	1	0	0	0	0	0
15	0	0	1	1	0	0	0	0
15*	0	1	0	0	1	0	0	0

REMARK. In this case, a little extra work beyond the calculations needed to complete Table A2 yields the Loewy structure of the projective indecomposable modules.

Preliminary Remarks.

In what follows, we will often make use of information drawn from the ATLAS, and the tables of Brauer characters, decomposition matrices, products of characters etc. in [9], [16], [12] and [13]. We will also use the tables of weights in [5] in a few places for $p = 2$. To avoid undue length, the details of calculations which follow directly from this numerical information, such as the calculation of multiplicities of simple modules in tensor products of simple modules, or in projective modules, will be omitted, but we will try to indicate what information is used in each instance. Some care is needed to find the correct correspondences among the notations of these sources.

The following notational conventions will be used in this paper. All modules considered will be vector spaces over a field k and we will abbreviate $\mathrm{Hom}_{kG}(-,-)$ and $\mathrm{Ext}^1_{kG}(-,-)$ to $\mathrm{Hom}_G(-,-)$ and $\mathrm{Ext}^1_G(-,-)$. The symbol $[X : S]$ will denote the number of composition factors of the kG-module X which are isomorphic to the simple module S.

§2. CHARACTERISTIC 2

In §2 we assume k to be an algebraically closed field of characteristic 2.

2.1.

In [18] it is shown that G is isomorphic to a maximal subgroup of the Chevalley group $G_2(4)$ of rational points over $\mathbf{F}_4$ inside the algebraic group

$G_2(k)$. The simple $G_2(k)$-modules whose highest weights are 2-*restricted* (see [11]) are the trivial module, a 6-dimensional module, which we denote by 6_1, the 14-dimensional adjoint module 14_1 and the 64-dimensional (first) Steinberg module 64_1. These modules remain irreducible upon restriction to G; indeed, all but 64_1 remain irreducible for the subgroup $G_2(2)'$ of G. We denote the algebraic conjugates under $\mathrm{Gal}(\mathbf{F}_4/\mathbf{F}_2)$ of these modules with a subscript "2". The simple modules in the principal block are then k, 6_1, 6_2, 14_1, 14_2, $36 := 6_1 \otimes 6_2$ and $84 := 6_1 \otimes 14_2$. The module 84 is also isomorphic to $6_2 \otimes 14_1$. (Remark: Since the two tensor products are not isomorphic for $G_2(4)$ and G is maximal in $G_2(4)$,the last sentence implies that G is the full stabilizer in $G_2(4)$ of a certain hermitian form on $6_1 \otimes 14_2$.) We see then that all simple modules in the principal block are restrictions of simple modules for $G_2(k)$. There is just one other block, with defect group isomorphic to the fours group, containing 64_1, 64_2 and a simple module 160, which appears in the equation

$$14_1 \otimes 14_2 \cong 36 \oplus 160. \tag{1}$$

The structure of the non-principal block has been studied in [15] and we will make use of it later. The facts concerning simple modules can all be checked from tables of Brauer characters. Note also that all simple modules are self-dual.

2.2. This subsection contains a general lemma and some immediate applications.

In spite of the section heading, the field in the following lemma is arbitrary. This result is a variation of the main result in [1].

LEMMA 1. *Let F be a field, G a group and V a finite-dimensional FG-module, with $V^G = 0$. Let L_0, L_1 and L_2 be subgroups of G with $L_0 \leq L_1 \cap L_2$ and $G = \langle L_1, L_2 \rangle$. Set $d_i = \dim_F V^{L_i}$. Suppose*

(i) *$d_0 \leq d_1 + d_2$;*
(ii) *$H^1(L_i, V) = 0$, for $i = 1,2$.*

Then $H^1(G, V) = 0$.

PROOF: We will show that every short exact sequence of FG-modules of the form

$$0 \to V \to E \to F \to 0$$

must split, which is equivalent to $E^G \neq 0$, since $V^G = 0$. By (ii), the sequence splits upon restriction to L_1 and L_2, hence also to L_0. Thus, E^{L_i} has dimension $d_i + 1$ for all i. Since $E^{L_0} \supseteq E^{L_1} + E^{L_2}$, the fact that $E^G = E^{L_1} \cap E^{L_2}$ is not zero follows from (i) by counting dimensions.

REMARK. Under the given conditions on the subgroups L_i (excluding (i) and (ii)), the hypothesis (i) is equivalent to $d_0 = d_1 + d_2$, or to $M^{L_0} = M^{L_1} + M^{L_2}$.

We will now perform some calculations with small modules using Lemma 2.2. In order to do so, we must pick suitable subgroups L_i and study their actions on these modules. We will follow the ATLAS notation for conjugacy classes. The notation we are about to introduce will be fixed for the rest of §2. Let x be an element of class $5A$. Then its centralizer $C(x)$, is isomorphic to $A_5 \times Z_5$, the second factor being generated by x. This has index 2 in the normalizer $N(x) \cong A_5 \times D_{10}$ of $\langle x \rangle$.

Let z be an element of $C(x)$ of order 3 (in class 3A). We have $C(z) \cong 3.\mathrm{PGL}(2,9)$.

From the character table, we see that $\langle xz \rangle$ is its own centralizer and it follows then from orders that $C(x)$ and $C(z)$ generate G, in fact we may replace $C(z)$ by its derived subgroup $3.A_6$. Likewise, if x' is taken to be an element of $C(x)$ in class 5A but outside $\langle x \rangle$, then $C(x)$ and $C(x')$ generate G and have intersection $S = \langle x, x' \rangle \cong Z_5 \times Z_5$. (The statements concerning generation can of course be read off from the list of maximal subgroups, but elementary arguments suffice for our needs.) These are typical of the subgroups we will choose for L_i in applying Lemma 1. In all cases it will be routine to check that L_1 and L_2 generate G and that $L_0 \leq L_1 \cap L_2$, so we will mainly be concerned with hypotheses (i) and (ii). In verifying these we will need to make use of certain facts about representations of $A_5 = \mathrm{SL}(2,4)$, A_6, etc. These will not be difficult facts and they can be found, for instance, in the examples at the end of [2].

The nontrivial simple $k(\mathrm{SL}(2,4))$-modules are two algebraically conjugate 2-dimensional modules V_1 and V_2 and their tensor product $V_{12} := V_1 \otimes V_2$. Let ω denote both a primitive 5th root of unity in k and the character of $\langle x \rangle$ sending x to that root. As modules for the subgroup $C(x)' \cong \mathrm{SL}(2,4)$ of G, we fix V_1 to be the one on which an element of class 5C has trace $\omega + \omega^4$. For $D_{10} = \langle x \rangle.2$, we denote by D_1 and D_2 the irreducible 2-dimesional module whose restrictions to $\langle x \rangle$ are $\omega + \omega^4$ and $\omega^2 + \omega^3$ respectively.

LEMMA 2. *As modules for $C(x) \cong SL(2,4) \times D_{10}$, we have:*

(a) $6_{1|N(x)} \cong (V_1 \otimes k) \oplus (V_2 \otimes D_2)$;

(b) $14_{1|N(x)} \cong (X \otimes k) \oplus (k \otimes D_1) \oplus (V_{12} \otimes D_2)$, *where* $X \cong V_2 \otimes V_2$, *is uniserial with factors* k, V_1, k.

PROOF: This mostly follows from (Brauer) character calculations. The only point which requires further discussion is the structure of X in (b). It is clear from the characters that $14_{1|N(x)}$ has a direct summand $X \otimes k$ with the right composition factors. Since 14_1 is self-dual, so must X be. Therefore, if X were not uniserial, it would have to be semisimple. Then $14_1^{N(x)}$ would be 2-dimensional. Let $S = \langle x, x' \rangle$ be as in the discussion before the lemma. It is easy to check that $\dim_k(14_1^S) = 2$, so if X were semisimple, we would have $14_1^{N(x)} = 14_1^S$. But then the same would hold with x' in place of x since they are conjugate in G, leading to the contradiction that $14_1^G \neq 0$, because $N(x)$ and $N(x')$ generate G. This proves that X is uniserial. The isomorphism with $V_2 \otimes V_2$ is an elementary calculation in $\mathrm{SL}(2,4)$.

We next apply these facts in a simple case.

LEMMA 3.

(a) $H^1(G, 36) = 0$.
(b) $\mathrm{Ext}_G^1(14_1, 14_2) = 0$.

PROOF: Equation 2.1(1) shows that (b) follows from (a), as 160 does not belong to the principal block. Let $L_1 = C(x)$, $L_2 = C(x')$ and $L_0 = S$. Since $36 \cong 6_1 \otimes 6_2$, we find using Lemma 2(a) that as modules for $\mathrm{SL}(2,4) \cong L_1/\langle x\rangle$, we have $36^{\langle x\rangle} \cong V_1 \otimes V_2 \cong V_{12}$. Since V_{12} is an injective module, we obtain $\mathrm{H}^1(L_1, 36) \cong \mathrm{H}^1(L_1/\langle x\rangle, 36^{\langle x\rangle}) = 0$ and similarly for L_2, which is conjugate to L_1. Therefore the hypothesis (ii) of lemma 1 is verified. Hypothesis (i) of Lemma 1 holds because we see from chracters that $36^S = 0$. So Lemma 1 yields the result.

Pursuing this method further, we have the following.

LEMMA 4.

(a) $H^1(G, 6_i) \cong k$.
(b) $H^1(G, 14_i) = 0$.
(c) $H^1(G, 6_i \otimes 6_i) = 0$.

PROOF: (a) It does not matter whether we consider 6_1 or 6_2. We first prove that $\mathrm{H}^1(G, 6_i) \neq 0$ by showing that there is a nonsplit extension of k by by some 6_i. A Brauer character calculation shows that the component of $6_1 \otimes 14_1$ in the principal block has composition factors 6_1 (twice), k (twice) and 6_2. Since $\mathrm{Hom}_G(k, 6_1 \otimes 14_1) \cong \mathrm{Hom}_G(6_1, 14_1) = 0$, the existence of the claimed nonsplit extension follows. Let M be such an extension, of k by 6_1, say. To prove the lemma it will suffice to show that $\mathrm{H}^1(G, M) = 0$. In the notation of the preceding discussion and of Lemma 1, we choose $L_1 = C(x) \cong \mathrm{SL}(2,4) \times \langle x\rangle$, $L_2 = C(z)'$ and $L_0 = \langle xz\rangle$. From characters we can see that $6^{\langle z\rangle} = 0$, which implies that $\mathrm{H}^1(L_2, 6_1) = 0$, from which it follows (in the notation of Lemma 1) that $d_2 = d_0 = 1$ and also that $d_1 = 0$, as $M^G = 0$. We also obtain $\mathrm{H}^1(L_2, M) = 0$ from the long exact sequence. The only remaining hypothesis to be checked is that $\mathrm{H}^1(L_1, M) = \mathrm{H}^1(L_1/\langle x\rangle, M^{\langle x\rangle}) = 0$. By Lemma 2(a), $6_1^{\langle x\rangle} \cong V_1$, a simple 2-dimensional module for $L_1/\langle x\rangle \cong \mathrm{SL}(2,4)$. Since $d_1 = 0$ the $k(L_1/\langle x\rangle)$-module $M^{\langle x\rangle}$ must be a nonsplit extension of k by V_1. Now it is a fact about $\mathrm{SL}(2,4)$ that $\mathrm{H}^1(L_1/\langle x\rangle, V_1) \cong k$ (see [2] or [14, p. 183]) and it follows that $\mathrm{H}^1(L_1/\langle x\rangle, M^{\langle x\rangle}) = 0$.

(b) We may consider just 14_1. We take $L_1 = C(x)$, $L_2 = C(x')$ and $L_0 = S$. By Lemma 2(b), we see that $d_1 = d_2 = 1$ and $d_0 = 2$, so the condition on fixed points holds. It suffices to check hypothesis (ii) of Lemma 1 for L_1, since x and x' are conjugate in G. We have, using Lemma 2(b),

$$\mathrm{H}^1(L_1, 14_1) \cong \mathrm{H}^1(L_1/\langle x\rangle, 14_1^{\langle x\rangle}) \cong \mathrm{H}^1(\mathrm{SL}(2,4), X).$$

The last group is easily seen to be zero using $\mathrm{H}^1(\mathrm{SL}(2,4), V_1) \cong k$ and $\mathrm{H}^1(\mathrm{SL}(2,4), k) = 0$. Therfore Lemma 1 applies and (b) is proved.
(c) The composition factors of $6_1 \otimes 6_1$ are k (twice), 14_1 (twice) and 6_2. From the natural exact sequences

$$\begin{aligned} 0 \to \wedge^2(6_1) \to 6_1 \otimes 6_1 \to S^2(6_1) \to 0 \\ 0 \to 6_2 \to S^2(6_1) \to \wedge^2(6_1) \to 0, \end{aligned} \tag{1}$$

we see (for example by self-duality) that $\wedge^2(6_1) \cong k \oplus 14_1$. By (b), we thus obtain $\mathrm{H}^1(G, \wedge^2(6_1)) = 0$. It will therefore be enough to show $\mathrm{H}^1(G, S^2(6_1)) = 0$. Let E be the preimage in $S^2(6_1)$ of $k \leq \wedge^2(6_1)$ under the map in the second sequence of (1). Since $S^2(6_1)/E \cong 14_1$, we are reduced by (b) to showing $\mathrm{H}^1(G, E) = 0$. Now E is an extension of k by 6_2, so the result will follow from (a) if we can show that the extension does not split. In our situation this is equivalent to $(S^2(6_1))^G = 0$. This means we must show that G does not preserve any nonzero quadratic form on 6_1. Such a form would be nondegenerate since 6_1 is simple, so we would obtain an embedding of G into $SO(6, k)$, which is impossible as the latter is isomorphic (as an abstract group) to $SL(4, k)$ and G has no faithful 4-dimensional representation. This completes the proof.

2.3. To make further use of 2.2, Lemma 1 we need to have more detailed knowledge of certain modules.

LEMMA 1. *We have the following module structures where each row is a socle layer.*

(a)

$$6_1 \otimes 6_1 : \quad \begin{array}{ccc} k & & 14_1 \\ & 6_2 & \\ k & & 14_1 \end{array} .$$

(b)

$$6_1 \otimes 14_1 : \quad 64_1 \oplus \begin{array}{c} 6_1 \\ k \\ 6_2 \\ k \\ 6_1 \end{array} .$$

PROOF: (a) A partial description of the structure of $6_1 \otimes 6_1$ has already been given in the proof of 2.2, Lemma 4(c). To complete the proof of (a), it will be sufficient to show that the subquotient M of $6_1 \otimes 6_1$ obtained by omitting the top and bottom trivial composition factors is uniserial (with factors 14_1, 6_2, 14_1). Inside G is a subgroup $L \cong \mathrm{SL}(3,2)$ (generated by the root subgroups in $G_2(2)$ corresponding to long roots, so L lies inside a maximal subgroup of G isomorphic to $G_2(2)'$). A direct calculation shows that $6_{|L} \cong V \oplus V^*$, where V is the natural module for L. Since $\wedge^2(6_1) \cong k \oplus 14_1$, we have

$$k \oplus 14_{1|L} \cong \wedge^2(V) \oplus \wedge^2(V^*) \oplus (V \otimes V^*) \cong V^* \oplus V \oplus W \oplus k,$$

where W is the simple 8-dimensional kL-module isomorphic to the space of 3×3 matrices of trace zero. Thus, both 6_1 and 14_1 are semisimple for L and it follows that if the module M were not uniserial, then its restriction to L would have socle length 2. However, we have $(6_1 \otimes 6_1)_{|L} \cong (V \oplus V^*) \otimes (V \oplus V^*)$, and it is easily seen that the direct summand $V \otimes V$ of this module is uniserial with factors V^*, V, V^*. Thus, $M_{|L}$ does not have socle length 2 and (a) is proved.

(b) The composition factors are easily checked. Also, 64_1 splits off as a direct summand since it belongs to a different block from the other factors. Let U be the complementary summand. We have

$$\mathrm{Hom}_G(k \oplus 6_2, 6_1 \otimes 14_1) \cong \mathrm{Hom}_G(6_1 \oplus 36, 14_1) = 0,$$

so the only simple submodule of U must be 6_1. By self-duality, every simple quotient is also isomorphic to 6_1. It follows that U has a simple head and a simple socle, both isomorphic to 6_1. Then $U' := \mathrm{rad}\, U / \mathrm{soc}\, U$ is self-dual with composition factors k (twice) and 6_2. If U' were not uniserial it would have to be semisimple and then U would have $k \oplus k$ in its second socle layer, contradicting $\mathrm{H}^1(G, 6_1) \cong k$ (2.2, Lemma 4(a)).

REMARK. From the natural split surjection $6_1 \otimes \wedge^2(6_1) \to \wedge^3(6_1)$, it is immediate that the module U in (b) is isomorphic to $\wedge^3(6_1)$. Since this module will reappear often we reserve the notation U for it.

We can now establish the triviality of 1-cohomology in many cases.

LEMMA 2.

(a) $H^1(G, U) = 0$.
(b) $H^1(G, 6_1 \otimes 14_1) = 0$.
(c) *Let M be a nonsplit extension of k by 6_1 (unique up to isomorphism by 2.2, Lemma 4(a)). Then $\mathrm{Ext}^1_G(6_1, M) = 0$. In particular there exists no uniserial kG-module with factors 6_1, k, 6_1.*
(d) $\mathrm{Ext}^1_G(6_i, U) = 0$.
(e) $\mathrm{Ext}^1_G(6_1, 84) = 0$.
(f) $\mathrm{Ext}^1_G(14_1, 36) = 0$.
(g) $\mathrm{Ext}^1_G(84, 84) = 0$.
(h) $\mathrm{Ext}^1_G(36, 36) = 0$.
(i) $\mathrm{Ext}^1_G(14_1, 14_1) = 0$.

PROOF: (a) From characters, we see that $U^{\langle z \rangle} = U^{\langle xz \rangle}$ and that this is a 2-dimensional subspace of the 8-dimensional space $U^{\langle x \rangle}$. We will apply

2.2, Lemma 1 with $L_1 = C(x) \cong \mathrm{SL}(2,4) \times \langle x \rangle$, $L_2 = C(z)'$, which is a central extension of A_6 by $\langle z \rangle$, and $L_0 = \langle xz \rangle$. Since A_6 has no nontrivial 2-dimensional modules over k, $C(z)'$ must act trivially on $U^{\langle z \rangle}$, so $d_0 = d_2$ and hypothesis (i) of 2.2, Lemma 1 is satisfied. The triviality of the L_2-action on $U^{\langle z \rangle}$ also yields $\mathrm{H}^1(L_2, U) = 0$. It remains to show $\mathrm{H}^1(L_1, U) = 0$. We shall prove the stronger statement that $\mathrm{H}^1(L_1, 6_1 \otimes 14_1) \cong \mathrm{H}^1(L_1/\langle x \rangle, (6_1 \otimes 14_1)^{\langle x \rangle})$ is zero. A straightforward computation using 2.2, Lemma 2 shows that

$$(6_1 \otimes 14_1)^{\langle x \rangle} \cong (V_1 \otimes X) \oplus (V_2 \otimes V_{12}) \oplus (V_2 \otimes V_{12}).$$

Now $V_1 \otimes X \cong V_1 \otimes V_2 \otimes V_2 \cong V_{12} \otimes V_2$ and V_{12} is an injective module for $L_1/\langle x \rangle \cong \mathrm{SL}(2,4)$, so all summands are injective and the result follows.
(b) This follows from (a) and Lemma 1(b), since 64_1 is not in the principal block.
(c) We shall apply 2.2, Lemma 1 to the module $6_1 \otimes M$. We choose $L_1 = C(x)$, $L_2 = C(x')$, $L_0 = S$. First, it is easy to check that $M^S \cong k$. We claim that $M^{L_1} = 0$. If not, we would have $M^{L_1} = M^S$, but then the same would hold with L_2 in place of L_1 as they are conjugate and both contain S. This would imply $M^G \neq 0$, contrary to the definition of M. From the claim it now follows using 2.2, Lemma 2(a) that

$$M_{|L_1} \cong (Y \otimes k) \oplus (V_2 \otimes \omega^2) \oplus (V_2 \otimes \omega^2),$$

where Y is a nonsplit extension of k by V_1. Using this and 2.2 Lemma 2(a) again one calculates that $d_1 = \dim_k (6_1 \otimes M)^{L_1} = \dim_k \mathrm{Hom}_{L_1}(6_1, M) = 3$. Hence $d_2 = 3$ as well. Character computations show $d_0 = 6$, so hypothesis (i) holds. By conjugacy of L_1 and L_2 it suffices to check (ii) for L_1, namely that $\mathrm{H}^1(L_1, 6_1 \otimes M)$ vanishes. By the above description of $M_{|L_1}$ and 2.2, Lemma 2(a),

$$\begin{aligned}
\mathrm{H}^1(L_1, 6_1 \otimes M) &\cong \mathrm{H}^1(L_1/\langle x \rangle, (6_1 \otimes M)^{\langle x \rangle}) \\
&\cong \mathrm{H}^1(\mathrm{SL}(2,4), (V_1 \otimes Y) \oplus (V_2 \otimes V_2) \oplus (V_2 \otimes V_2)) \\
&\cong \mathrm{Ext}^1_{\mathrm{SL}(2,4)}(V_1, Y) \oplus \mathrm{Ext}^1_{\mathrm{SL}(2,4)}(V_2, V_2) \oplus \mathrm{Ext}^1_{\mathrm{SL}(2,4)}(V_2, V_2).
\end{aligned}$$

This reduces us to facts about $k(\mathrm{SL}(2,4))$, which are easily read off from the known structure of its projective modules (see [2] or [14, p. 183]).
(d) First we show $\mathrm{Ext}^1_G(6_1, U) = 0$. From Lemma 1(b) we see that U has an ascending filtration with factors M, 6_2, M^*, with M as in (c). By (c) and 2.2, Lemma 3(a), we are left to show that $\mathrm{Ext}^1_G(6_1, M^*) = 0$, but this follows from 2.2, Lemma 4(a), using the long exact sequence associated to $0 \to k \to M^* \to 6_1 \to 0$.

To show $\mathrm{Ext}^1_G(6_2, U) = 0$, we filter U by factors 6_1, U', 6_1, where U' is uniserial with factors k, 6_2, k. By 2.2, Lemma 3(a), it suffices to show $\mathrm{Ext}^1_G(6_2, U') = 0$. Suppose we have an extension $0 \to U' \to A \to 6_2 \to 0$. Since $\mathrm{Hom}_G(6_2, U') = 0$, the sequence splits if and only if $\mathrm{Hom}_G(6_2, A) \neq 0$, so we shall aim to show the latter. First we see that A cannot be uniserial, as this would violate the last statement of (c) (with 1 replaced by 2). Therefore there is a nonzero kG map $A \to k$. The kernel has 2 composition factors 6_1 and only one k, so we are done by 2.2, Lemma 3(a).
(e) We have

$$\begin{aligned}\mathrm{Ext}^1_G(6_1, 84) &\cong \mathrm{Ext}^1_G(6_1, 6_2 \otimes 14_1)\\ &\cong \mathrm{Ext}^1_G(6_2, 6_1 \otimes 14_1)\\ &\cong \mathrm{Ext}^1_G(6_2, 64_1 \oplus U) = 0,\end{aligned}$$

using Lemma 1(b) for the third isomorphism and (d) for the vanishing.
(f) This is immediate from (e) since $\mathrm{Ext}^1_G(6_1, 6_2 \otimes 14_1) \cong \mathrm{Ext}^1_G(14_1, 6_1 \otimes 6_2)$.
(g) We have by Lemma 1(b),

$$\begin{aligned}\mathrm{Ext}^1_G(84, 84) &\cong \mathrm{Ext}^1_G(6_1 \otimes 14_2, 6_2 \otimes 14_1)\\ &\cong \mathrm{Ext}^1_G(6_1 \otimes 14_1, 6_2 \otimes 14_2)\\ &\cong \mathrm{Ext}^1_G(64_1 \oplus U_1, 64_2 \oplus U_2),\end{aligned}$$

where $U_1 = U$ and U_2 is its Galois conjugate. By considering blocks, we see that the last group is isomorphic to $\mathrm{Ext}^1_G(64_1, 64_2) \oplus \mathrm{Ext}^1_G(U_1, U_2)$. The first summand is zero since $64_1 \otimes 64_2$, being the restriction of the Steinberg module for $G_2(4)$, is injective. and the second summand can be seen to be zero using the long exact sequence in conjunction with (a) and (d).
(h) This follows from (g) and

$$\begin{aligned}\mathrm{Ext}^1_G(84, 84) &\cong \mathrm{Ext}^1_G(6_1 \otimes 14_2, 6_2 \otimes 14_1) \cong \mathrm{Ext}^1_G(6_1 \otimes 6_2, 14_2 \otimes 14_1)\\ &\cong \mathrm{Ext}^1_G(36, 36 \oplus 160) \cong \mathrm{Ext}^1_G(36, 36).\end{aligned}$$

(i) Dualizing the exact sequences in the proof of 2.2, Lemma 4(c) and making use of the self-duality of some modules, we have

$$\begin{aligned}&0 \to S^2(6_1)^* \to 6_1 \otimes 6_1 \to \wedge^2(6_1) \to 0,\\ &0 \to \wedge^2(6_1) \to S^2(6_1)^* \to 6_2 \to 0.\end{aligned} \tag{1}$$

The structure of all the modules occurring in these sequences can be read off from Lemma 1(a). We claim that $\mathrm{Ext}^1_G(14_1, S^2(6_1)^*) \cong k$. Since $\wedge^2(6_1) \cong 14_1 \oplus k$ (proof of 2.2 Lemma 4(c)) and $\mathrm{Ext}^1_G(14_1, 6_1 \otimes 6_1) \cong \mathrm{Ext}^1_G(6_1, 6_1 \otimes$

$14_1) = 0$, by (d) and Lemma 1(b), our claim follows from long exact sequence for $\mathrm{Ext}_G(14_1, -)$ applied to the first sequence in (1). Now we apply $\mathrm{Ext}_G(14_1, -)$ to the second sequence in (1), resulting in

$$0 \to \mathrm{Ext}^1_G(14_1, \wedge^2(6_1)) \to \mathrm{Ext}^1_G(14_1, S^2(6_1)^*) \to \mathrm{Ext}^1_G(14_1, 6_2) \qquad (2)$$

Using the structure of $6_1 \otimes 6_1$ again, we see that the unique nonsplit extension of 14_1 by $S^2(6_1)^*$ established by the claim can be realized as the kernel K of any nonzero kG-map $6_1 \otimes 6_1 \to k$. One also sees from the structure that the quotient of K by $\wedge^2(6_1)$ is a nonsplit extension of 14_1 by 6_2. This shows that the last map in (2) is not zero, hence injective by the claim. Thus, $\mathrm{Ext}^1_G(14_1, \wedge^2(6_1)) \cong \mathrm{Ext}^1_G(14_1, k \oplus 14_1) = 0$.

2.4. By making use of group automorphisms and various natural identities involving tensor products, it is not difficult to check that the results of the last section leave only the following extensions of simple modules to be computed: (i) $\mathrm{Ext}^1_G(14_1, 6_2) \cong \mathrm{H}^1(G, 84)$, (ii) $\mathrm{Ext}^1_G(6_1, 36)$, (iii) $\mathrm{Ext}^1_G(14_1, 84)$ and (iv) $\mathrm{Ext}^1_G(36, 84)$.

From 2.1 we see

$$\begin{aligned}\mathrm{Ext}^1_G(14_1, 84) &= \mathrm{Ext}^1_G(14_1, 6_1 \otimes 14_2)\\ &\cong \mathrm{Ext}^1_G(6_1, 14_1 \otimes 14_2) \cong \mathrm{Ext}^1_G(6_1, 36 \oplus 160),\end{aligned}$$

so (ii) and (iii) have the same dimension e. Let d be the dimension of (i). These will turn out to be equal; we prove the easier inequality now and the reverse only after we have found e.

LEMMA 1. $1 \le d \le e$.

PROOF: The first inequality follows from 2.3, Lemma 1(a). By definition of d, there is a kG-module E with socle isomorphic to 6_2 and $E/\operatorname{soc} E$ isomorphic to the direct sum of d copies of 14_1. Tensoring with 6_1, we obtain the short exact sequence

$$0 \to 36 \to E \otimes 6_1 \to (14_1 \otimes 6_1)^d \to 0.$$

We now apply $\mathrm{Hom}_G(6_1, -)$ and consider the resulting long exact sequence. We have $\mathrm{Hom}_G(6_1, E \otimes 6) \cong \mathrm{Hom}_G(6_1 \otimes 6_1, E)$, which can be seen to be zero from the structure of $6_1 \otimes 6_1$ (2.3, Lemma 1(a)). By 2.3, Lemma 1(b), we have $\dim_k \mathrm{Hom}_G(6_1, (14_1 \otimes 6_1)^d) = d$. Then the long exact sequence yields $d \le \dim_k \mathrm{Ext}^1_G(6_1, 36) = e$.

We need to know some more about structure of some small modules.

LEMMA 2. *We have the following module structures where the rows going from top to bottom in each summand are the radical layers.*

(a)

$$6_2 \otimes 6_1 \otimes 6_1 : \quad \begin{array}{ccc} 6_2 & & 84 \\ k & & 14_2 \\ & 6_1 & \\ k & & 14_2 \\ 6_2 & & 84 \end{array} .$$

Moreover, the quotient by the unique copy of 84 *in the socle does not have a trivial submodule. Neither does the quotient by the copy of* 6_2.

(b) $6_2 \otimes 6_2 \otimes 14_1 \cong 6_2 \otimes 14_2 \otimes 6_1 \cong (64_2 \otimes 6_1) \oplus (U_2 \otimes 6_1)$:

$$\begin{array}{c} 160 \\ 64_1 \\ 160 \end{array} \quad \bigoplus \quad \begin{array}{ccc} 14_1 & & 36 \\ & 6_1 & \\ & k & \\ & 6_2 & \\ k & & 14_1 \\ & 6_1 & \\ & 36 & \end{array} .$$

In particular there is a unique quotient which is a nonsplit extension of 36 *by* 6_1.

(c) *We have*

$$(64_1 \otimes 14_2) \oplus (U \otimes 14_2) \cong 6_1 \otimes 14_1 \otimes 14_2 \cong (6_1 \otimes 160) \oplus (6_1 \otimes 6_1 \otimes 6_2).$$

Moreover, $(64_1 \otimes 14_2) \oplus 64_2 \cong 6_1 \otimes 160$.

PROOF: (a) Since we already know the structure of $6_1 \otimes 6_1$ from 2.3, Lemma 1, we see that $6_2 \otimes (6_1 \otimes 6_1)$ has a filtration with factors $6_2 \oplus 84 \cong 6_2 \otimes (k \oplus 14_1)$, $6_2 \otimes 6_2$, $6_2 \oplus 84$. The structure of the middle factor is known. Let M be the unique subquotient of $6_1 \otimes 6_1$, obtained by omitting top and bottom trivial factors, which is uniserial with series 14_1, 6_2, 14_1 and let N be the uniserial subquotient with series k, 6_2, k. Since we know the structure of $6_2 \otimes 6_2$ all statements will follow if we show that $6_2 \otimes M$ has no submodules 14_2 or k (hence also no such quotients by self-duality) and that $6_2 \otimes N$ has no trivial submodule. We have $\mathrm{Hom}_G(14_2, 6_2 \otimes M) \cong \mathrm{Hom}_G(14_2 \otimes 6_2, M) = 0$, from 2.3, Lemma 1(b) and it is clear that $\mathrm{Hom}_G(k, 6_2 \otimes M) \cong \mathrm{Hom}_G(6_2, M) = 0$ and $\mathrm{Hom}_G(k, 6_2 \otimes N) \cong \mathrm{Hom}_G(6_2, N) = 0$.

(b) The isomorphisms in the first line follow from $6_1 \otimes 14_2 \cong 84 \cong 6_2 \otimes 14_1$ and 2.3, Lemma 1(b). Using again the structure of $6_2 \otimes 6_2$, we see that $(6_2 \otimes 6_2) \otimes 14_1$ has a filtration with factors

$$14_1 \oplus (14_1 \otimes 14_2), \qquad 14_1 \otimes 6_1, \qquad 14_1 \oplus (14_1 \otimes 14_2).$$

We recall from 2.1 and 2.3, Lemma 1(b) that $14_1 \otimes 14_2 \cong 160 \oplus 36$ and $14_1 \otimes 6_1 \cong 64_1 \oplus U$.

Let us first consider the summand outside the principal block. The above shows it has a filtration with factors $160, 64_1, 160$. Since $\mathrm{Hom}_G(64_1, 6_2 \otimes 6_2 \otimes 14_1) \cong \mathrm{Hom}_G(64_1 \otimes 6_2, 84) = 0$, just from characters, we see that this summand has the claimed structure. It is easy to see from characters that this summand of $6_2 \otimes 6_2 \otimes 14_1$ corresponds under the isomorphism in the statement to $64_2 \otimes 6_1$.

The component in the principal block has a filtration with factors $14_1 \oplus 36$, U, $14_1 \oplus 36$. From the structure of U and $\mathrm{Hom}_G(6_2 \otimes 6_2 \otimes 14_1, 6_1) \cong \mathrm{Hom}_G(84, 36) = 0$, we see that the top two radical layers of $6_2 \otimes 6_2 \otimes 14_1$ are $14_1 \oplus 36$ and 6_1 respectively. The last statement of (b) now follows from this, using $\mathrm{Ext}^1_G(14_1, 6_1) = 0$ (2.3, Lemma 2(b)). With the structure of U known, the rest of the structure (the position of the second 14_1 in the radical series) can be easily deduced from 2.2, Lemma 4(b), 2.3, Lemma 2(i) and $\mathrm{Hom}_G(6_2 \otimes 6_2 \otimes 14_1, 14_1) \cong \mathrm{Hom}_G(84, 84) \cong k$. This proves (b).
(c). The main statement is immediate from the isomorphisms (2.1 and 2.3, Lemma 1(b)) $14_1 \otimes 14_2 \cong 160 \oplus 36$ and $6_1 \otimes 14_1 \cong 64_1 \oplus U$. The last statement will be established by proving the isomorphism $U \otimes 14_2 \cong (6_1 \otimes 36) \oplus 64_2$ of the complementary summands. First, these modules have the same composition factors. Next, we have by Lemma 1(b),

$$\mathrm{Hom}_G(U \otimes 14_2, 6_2) \cong \mathrm{Hom}_G(14_1 \otimes 6_1 \otimes 14_2, 6_2) \cong \mathrm{Hom}_G(84, 84) \cong k,$$

since $\mathrm{Hom}_G(64_1 \otimes 14_2, 6_2) = 0$. Thus, $6_1 \otimes 14_1 \otimes 14_2$ has a unique indecomposable summand having 6_2 as a simple quotient and, furthermore, that summand is a summand of $U \otimes 14_2$. Since $6_1 \otimes 36$ is such a summand of $6_1 \otimes 14_1 \otimes 14_2$, by (a) and the main part of (c), we are done.

The proof of the key fact $\mathrm{Ext}^1_G(36, 84) = 0$ will make use of several techniques which we have not discussed yet. First, the fact that various simple modules are restrictions of simple modules for $G_2(k)$ will allow us to to apply some fundamental results about induced modules and good filtrations from the representation theory of algebraic groups. These theorems can be found in [11]. Data such as weights of Weyl modules, composition multiplicities in Weyl modules or tensor products of simple modules are straightforward to compute. They can also be found in [5] (for weights), [17] and [6]. Secondly, we need the submodule structure of projective modules in the non-principal block. This has been given in [15].

LEMMA 3.

(a) *The radical series the projective covers of modules in the non-principal block are as follows ($P(64_2)$ omitted):*

$$P(64_1): \quad \begin{array}{c} 64_1 \\ 160 \\ 64_2 \\ 160 \\ 64_1 \end{array}; \qquad P(160): \quad \begin{array}{ccc} & 160 & \\ 64_1 & & 64_2 \\ 160 & & 160 \\ 64_2 & & 64_1 \\ & 160 & \end{array}.$$

(b) $P(64_1) \otimes 14_2 \cong P(36) \oplus P(84) \oplus (P(160))^3$.
(c) $P(64_2) \otimes 14_2 \cong P(6_2) \oplus (P(64_2))^2 \oplus (P(160))^2$.
(d) $P(160) \otimes 14_2 \cong P(14_1 \oplus (P(36))^2 \oplus (P(160))^6$.

PROOF: For (a) we refer to [15]. Parts (b), (c) and (d) are just character calculations.

LEMMA 4. $\mathrm{Ext}^1_G(36, 84) = 0$.

PROOF: From the filtration of $P := P(64_1)$:

$$\mathrm{rad}^2 P \subset \mathrm{rad}\, P \subseteq P$$

we have an induced filtration of $P \otimes 14_2$. We now consider G-homomorphisms from the factors to 36. Since $\mathrm{rad}^2 P$ is a quotient of $P(64_2)$, Lemma 3(c) shows that

$$\mathrm{Hom}_G(\mathrm{rad}^2 P \otimes 14_2, 36) = 0. \tag{1}$$

Next since $\mathrm{rad}\, P / \mathrm{rad}^2 P \cong 160$, we have

$$\begin{aligned} \mathrm{Hom}_G((\mathrm{rad}\, P / \mathrm{rad}^2 P) \otimes 14_2, 36) &\cong \mathrm{Hom}_G(160 \otimes 14_2, 36) \\ &\cong \mathrm{Hom}_G(160, 14_2 \otimes 6_2 \otimes 6_1) \cong k, \end{aligned} \tag{2}$$

by Lemma 2(b). Finally,

$$\mathrm{Hom}_G(64_1 \otimes 14_2, 36) \cong \mathrm{Hom}_G(64_1, 14_2 \otimes 6_2 \otimes 6_1) = 0, \tag{3}$$

again by Lemma 2(b). We also need

$$\begin{aligned} \mathrm{Hom}_G(64_1 \otimes 14_2, 84) &\cong \mathrm{Hom}_G(64_1 \otimes 6_2, 14_2 \otimes 14_1) \\ &\cong \mathrm{Hom}_G(64_2 \otimes 6_1, 36 \oplus 160) \cong k, \end{aligned} \tag{4}$$

using Lemma 2(b).

Now $\mathrm{Hom}_G(P \otimes 14_2, 36) \cong k$, by Lemma 3(b). Denote by N the maximal submodule of $P \otimes 14_2$ which is the kernel of a nonzero G-homomorphism to 84. Then by Lemma 3(b), we see that $\dim_k \mathrm{Ext}^1_G(84, 36) = \dim_k \mathrm{Hom}_G(N, 36) - 1$. By (4) and Lemma 3(b), we see that $\mathrm{rad}\, P \otimes 14_2 \subset N$ and that the image of N in the top factor $(P \otimes 14_2)/((\mathrm{rad}\, P) \otimes 14_2) \cong 64_1 \otimes 14_2$ is the kernel K of a nonzero G-homomorphism $64_1 \otimes 14_2 \to 84$. Now from (1), (2) and (3) it follows that

$$\dim_k \mathrm{Ext}^1_G(84, 36) \leq \dim_k \mathrm{Hom}_G(K, 36). \tag{5}$$

Now $\mathrm{Hom}_G(K/\mathrm{rad}(64_1 \otimes 14_2), 36) = 0$ by (3), so we may replace K by $\mathrm{rad}(64_1 \otimes 14_2)$ in (5). Then by Lemma 2(c), this may be replaced in turn by $\mathrm{rad}(6_1 \otimes 160)$. Since $6_1 \otimes 160$ is a quotient of $6_1 \otimes (6_2 \otimes 64_1)$, by Lemma 2(b), and $\mathrm{Hom}_G(6_1 \otimes 6_2 \otimes 64_1, 36) \cong \mathrm{Hom}_G(6_2 \otimes 64_1, 6_1 \otimes 36) = 0$, by Lemma 2, we arrive at the inequality

$$\dim_k \mathrm{Ext}^1_G(84, 36) \leq \dim_k \mathrm{Hom}_G(\mathrm{rad}(6_2 \otimes 6_1 \otimes 64_1), 36). \tag{6}$$

Our eventual aim is to show that the right hand side of (6) is zero. Before we can do this, we need to recall some ideas from the representation theory of the algebraic group $\mathbf{G} = G_2(k)$ (see [6 II,§3]). The simple rational modules $L(\lambda)$ are paramentrized by non-negative integral combinations λ of the two fundamental dominant weights ω_1 and ω_2. We denote by $V(\lambda)$ the Weyl module, with simple quotient $L(\lambda)$ and by $\mathrm{H}^0(\lambda)$ the induced module with socle $L(\lambda)$. In the notation of [6], we have that the restrictions to G of $L(\omega_1)$, $L(\omega_2)$ and $L(\omega_1 + \omega_2)$ are respectively 6_1, 14_1 and 64_1. The Weyl module $V(\omega_1)$ has radical k, while $V(\omega_2)$ and $V(\omega_1 + \omega_2)$ are both simple, hence also isomorphic to the corresponding induced modules.

The module $6_2 \otimes (6_1 \otimes 64_1)$ is the restriction of the $\mathbf{G}$-module $V = L(2\omega_1) \otimes (L(\omega_1) \otimes L(\omega_1 + \omega_2))$. Until further notice, all module structures are for $\mathbf{G}$ unless stated otherwise. A fundamental result on algebraic group representations [11, II, 4.19] states that the tensor product of two Weyl modules has an ascending filtration by Weyl modules, such that the highest weights of the factors appear in descending order with respect to the usual ordering on weights. Moreover, the multiplicities of the factors can be calculated from weight data. For $V(\omega_1) \otimes V(\omega_1 + \omega_2)$, the factors (in ascending order) are: $V(2\omega_1 + \omega_2)$, $V(2\omega_2)$, $V(3\omega_1)$, $V(\omega_1 + \omega_2)$, $V(2\omega_1)$, $V(\omega_2)$.

Since $V(\omega_1 + \omega_2) = \mathrm{H}^0(\omega_1 + \omega_2)$ and Weyl modules can extend induced modules only trivially [11, II, 4.13] we see that $V(\omega_1) \otimes V(\omega_1 + \omega_2)$ has a simple quotient $L(\omega_1 + \omega_2)$. That it also has such a simple submodule is clear from the structure of $V(\omega_1)$. But $L(\omega_1 + \omega_2)$ occurs only once as a composition factor in $V(\omega_1) \otimes V(\omega_1 + \omega_2)$, so it must be a direct summand. It now follows that $V = L(\omega_1) \otimes L(\omega_1 + \omega_2)$ has a filtration with the other factors as subquotients:

$$\begin{gathered} V_1 = V(2\omega_1 + \omega_2), \quad V_2/V_1 = V(2\omega_2), \quad V_3/V_2 = V(3\omega_1), \\ V_4/V_3 = V(2\omega_1), \quad V/V_4 = V(\omega_2). \end{gathered} \tag{7}$$

Since V is self-dual, there is also the dual filtration $A_1 \subset \ldots \subset A_5 = V$ by induced modules $\mathrm{H}^0(\lambda)$ (ascending, with the weights in the reverse order to (7)).

The composition factors of each Weyl module are easy to compute. By Considering these two filtrations and the structures of their factors, we shall show that V has a filtration $\mathcal{F}$ with the following properties:

(i) Every $\mathcal{F}$-factor with $L(\omega_1)$ as a composition factor is a nonsplit extension of either k or $L(3\omega_1)$ by $L(\omega_1)$

(ii) The unique $\mathcal{F}$-factor with $L(2\omega_1+\omega_2)$ as a composition factor is a nonsplit extension of $L(2\omega_2)$ by $L(2\omega_1+\omega_2)$.

By the calculation of $\mathbf{G}$-extensions [6], and a universal property of Weyl modules, we find that

$$\operatorname{rad}_{\mathbf{G}} V_1/\operatorname{rad}^2_{\mathbf{G}} V_1 \cong k \oplus L(\omega_2) \oplus L(2\omega_2), \tag{8}$$

so $W_1 = \operatorname{rad}_{\mathbf{G}} V_1$ has no quotients $L(\omega_1)$. There is a unique $\mathbf{G}$-homomorphism $V \to A_5/A_4 \cong \mathrm{H}^0(2\omega_1+\omega_2)$ and this factors through V/W_1, by the uniqueness of $L(2\omega_1+\omega_2)$ as a composition factor in V. Inside $\mathrm{H}^0(2\omega_1+\omega_2)$ is a nonsplit extension D of $L(2\omega_2)$ by $L(2\omega_1+\omega_2)$, from (8). Let $\overline{V}_2$ be the image of V_2 in V/W_1. Then by the uniqueness of the composition factor $L(2\omega_2)$ in V/W_1, one sees that $\overline{V}_2$ maps onto D. It follows that $\operatorname{rad}_{\mathbf{G}} \overline{V}_2 = \overline{V}_1 \oplus \overline{W}_2 \cong L(2\omega_1+\omega_2) \oplus \operatorname{rad}_{\mathbf{G}} V(2\omega_2)$. Since $\operatorname{Hom}_{\mathbf{G}}(\operatorname{rad}_{\mathbf{G}} V(2\omega_2), L(\omega_1)) \cong \operatorname{Ext}^1_{\mathbf{G}}(L(2\omega_2), L(\omega_1)) = 0$, by [6], we see that if W_2 is the preimage of $\overline{W}_2$ in V, then we have a filtration $W_1 \subset W_2 \subset V_2 \subset V_3 \subset V_4 \subset V_5 = V$ in which no factor has $L(\omega_1)$ as a homomorphic image and with the unique composition factor $L(2\omega_1+\omega_2)$ appearing in the factor V_2/W_2 as required in (ii). This filtration can now be further refined to satisfy (i), using the fact that the only composition factors of V which extend $L(\omega_1)$ as $\mathbf{G}$-modules are k and $L(3\omega_1)$. Thus, the filtration $\mathcal{F}$ can be constructed.

Then $\mathcal{F}$ is of course also a filtration of G-modules. We claim that both (i) and (ii) remain valid for G, namely, that the extensions do not split on restriction. Since the groups of $\mathbf{G}$-extensions between simple modules are at most one-dimensional, it suffices to find nonsplit $\mathbf{G}$-extensions of these modules which do not split on restriction. It is easy to see that $L(\omega_1)\otimes L(\omega_2)$ has the same structure as its restriction $6_1 \otimes 14_1$, given in 2.3, Lemma 1(b), so we have the first extension of (i). The second extension in (i) can be obtained by twisting the dual M^* of the first by Frobenius and tensoring with $L(\omega_1)$, for $\operatorname{Hom}_G(M\otimes 6_1, 6_1) \cong \operatorname{Hom}_G(M, 6_1\otimes 6_1) = 0$, by 2.3 Lemma 1(a). The extension in (ii) represents a nonzero class in $\operatorname{Ext}^1_{\mathbf{G}}(L(2\omega_2), L(2\omega_1)\otimes L(\omega_2)) \cong \operatorname{Ext}^1_{\mathbf{G}}(L(\omega_2), L(2\omega_1)\otimes L(2\omega_2))$, so it suffices to find a class in the latter which restricts to a nonzero class in $\operatorname{Ext}^1_G(L(\omega_2), L(2\omega_1)\otimes L(2\omega_2))$. The module $L(2\omega_1)\otimes L(2\omega_2)$ has a simple $\mathbf{G}$-submodule $L(2\omega_1)$ and the map $\operatorname{Ext}^1_{\mathbf{G}}(L(\omega_2), L(2\omega_1)) \to \operatorname{Ext}^1_{\mathbf{G}}(L(\omega_2), L(2\omega_1)\otimes L(2\omega_2))$ is injective by 2.3, Lemma 1 (b), so it is enough to find a class in $\operatorname{Ext}^1_{\mathbf{G}}(L(\omega_2), L(2\omega_1))$ with nonzero restriction to G. By 2.3, Lemma 1(a), such a nonsplit extension occurs as a subquotient of $L(\omega_1)\otimes L(\omega_1)$. This proves the claim.

From now on we will consider only G-modules again.

First we observe that all G-composition factors 6_1 or 84 of $6_1 \otimes 64$ are restrictions of $\mathbf{G}$-composition factors $L(\omega_1)$ and $L(2\omega_1+\omega_2)$ of $L(\omega_1)\otimes L(\omega_1+$

ω_2), and that 6_1 and 84 are the only composition factors with the property that the tensor product with 6_2 has 36 as a composition factor. Next, calculations of Hom_G-spaces shows that 84 is the only simple module in the head of the principal block summand of $6_2 \otimes 6_1 \otimes 64_1$, and we have $\mathrm{Hom}_G(6_2 \otimes 6_1 \otimes 64_1, 84) \cong \mathrm{Hom}_G(6_2 \otimes 64_1, 6_1 \otimes 6_1 \otimes 14_2) \cong k^2$, by Lemma 2(b). Thus $(6_2 \otimes 6_1 \otimes 64_1)/\operatorname{rad} 6_2 \otimes 6_1 \otimes 64_1 \cong 84 \oplus 84$. Combining this information we see that the right hand side of (6) will be shown to be zero once we show that for each of the three special $\mathcal{F}$-factors E in (i) and (ii), that $E \otimes 6_2$ does not have a homomorphic image which is 36 or a nonsplit extension of 84 by 36. First, let E be a nonsplit extension of k by 6_1 or a nonsplit extension of 14_2 by 84. Then $\mathrm{Hom}_G(E \otimes 6_2, 36) \cong \mathrm{Hom}_G(E, (6_2 \otimes 6_2) \otimes 6_1) = 0$, Lemma 2(a). Also, $\mathrm{Hom}_G(E \otimes 6_2, 84) \cong \mathrm{Hom}_G(E, 6_2 \otimes 6_2 \otimes 14_1) = 0$, by Lemma 2(b).

Finally, let E be a nonsplit extension of 36 by 6_1. As we have mentioned before, we can write E as $M \otimes 6_1$, where M is a nonsplit extension of 6_2 by k. So $E \otimes 6_2 = (M \otimes 6_2) \otimes 6_1$. Since $\mathrm{Hom}_G(k, M \otimes 6_2) \cong \mathrm{Hom}_G(M, 6_2) = 0$, we see that $M \otimes 6_2$ has a submodule Y which is a nonsplit extension of k by 6_2. Then $Y \otimes 6_1$ is a nonsplit extension of 6_1 by 36 and this submodule of $M \otimes 6_2 \otimes 6_1 \cong E \otimes 6_2$ contains the unique composition factor 36. Therefore the conditions are checked for all three special types of $\mathcal{F}$-factors and the Lemma is proved.

LEMMA 5.

(a) $(e =) \dim_k \mathrm{Ext}^1_G(36, 6_1) = 1$.
(b) $(d =) \dim_k H^1(G, 84) = 1$.

PROOF: By 2.3, Lemma 2(a), we have $0 \neq \mathrm{Ext}^1_G(14_1, 6_2) \cong \mathrm{H}^1(G, 84)$, so (b) now follows from (a).

Lemma 2(b) shows that $\mathrm{Ext}^1_G(36, 6_1) \neq 0$, so we shall aim to prove that if E is any nonsplit extension of 36 by 6_1, then it is a homomorphic image of $6_2 \otimes 6_2 \otimes 14_1$, for then Lemma 2(b) will imply the uniqueness of E up to isomorphism. We have $\mathrm{Hom}_G(6_2 \otimes 6_2 \otimes 14_1, E) \cong \mathrm{Hom}_G(84, 6_2 \otimes E)$. Now $6_2 \otimes E$ has a descending filtration with factors $6_2 \otimes 36 \cong 6_2 \otimes 6_2 \otimes 6_1$, $6_2 \otimes 6_1 = 36$. By Lemma 2(a), the first factor has a submodule isomorphic to 84, so by Lemma 4, we have the conclusion $\mathrm{Hom}_G(84, 6_2 \otimes E) \neq 0$. Finally we observe that any nonzero map in $\mathrm{Hom}_G(6_2 \otimes 6_2 \otimes 14_1, E)$ must be surjective, since $\mathrm{Hom}_G(6_2 \otimes 6_2 \otimes 14_1, 6_1) \cong \mathrm{Hom}_G(84, 36) = 0$. This completes the proof.

REMARK. In the proof of Lemma 4, we saw that the nonzero class in (a) can be realized by the tensor product of 6_2 with the unique nonsplit extension of 6_1 by k; the latter is the restriction of the corresponding Weyl module of $G_2(k)$. Thus, this nonsplit extension is the restriction of one for the algebraic group. It is not hard to show that all of the nonsplit extensions in the principal block can be obtained similarly from extensions between simple modules for the algebraic group having 4-*restricted* highest weights, through the processes of restriction and twisting by $\mathrm{Gal}(\mathbf{F}_4/\mathbf{F}_2)$.

This completes the calculation of simple module extensions in characteristic 2.

§3. CHARACTERISTIC 3

In §3 we assume k to be an algebraically closed field of characteristic 3. Our notation follows that of [9], where decomposition matrices, Brauer trees and other character-theoretic information may be found. Note that there are some differences in labelling of ordinary and 3-modular characters in the tables in [9], [12]/[13] and [4]/[16] (three distinct labellings). The principal block has eight simple modules, as does the unique non-principal block of maximal defect of $\widehat{G}$. Besides these, there are three blocks of defect zero, with simple modules 189_1, 189_2 and 216, and three blocks of 1.

We label the simple modules by their dimension. The modules with suffix "1" are conjugate to those with "2" by the outer automorphism.

The simple modules in the principal block are k, 13_1, 13_2, 21_1, 21_2, 57_1, 57_2 and 133. All these are self-dual. In the other block of maximal defect we have 6_1, 6_2, 14, 36_1, 36_2, 50_1, 50_2 and 236. All of these are self-dual except for the 36-dimensional ones.

We will give details of the calculations only for the principal block. The same method yields the extensions in the other block of maximal defect; most of them result from a straightforward application of the method and then there are a few more stubborn cases $((236, 236)$, $(236, 50_1)$ and $(50_1, 50_2))$ where more complicated arguments (but still of the same kind) are needed.

Before going into details, we sketch the method we shall use to compute the extensions. The structures of the projective indecomposable modules in the blocks of defect 1 are easily found from the Brauer trees (see [7, VII.12]). By tensoring the projective indecomposable modules in the cyclic blocks with simple modules we obtain a projective module on which we have filtrations induced from any filtration projective in the cyclic block. Moreover, the decomposition of the tensor product into indecomposable summands is a purely character-theoretic question whose solution can be read off from the character table and decomposition matrix. For example, suppose P and Q are uniserial projective indecomposable modules with P the projective cover of $\operatorname{rad} Q$, a common situation in blocks of defect 1. Then if S is any simple module, we can routinely find the decompositions $P \otimes S$ and $Q \otimes S$ into their indecomposable summands. From this, we derive information about the head of $\operatorname{rad} Q \otimes S$, which must be isomorphic to a submodule of the head of $P \otimes S$, and hence we obtain information about the indecomposable summands of $Q \otimes S$. In practice, if S is small (we shall use modules of dimensions 6 and 13 in our calculations), their may be very few summands of $Q \otimes S$ in the block of interest and many extensions of simple modules can be deduced immediately using this information, in conjunction with the Cartan matrix and knowledge of the composition factors of tensor products of simple modules.

3.1 The principal 3-block. In this subsection, we compute the extensions between simple modules in the principal block. Let us first dispose of some trivial cases:

$$\mathrm{Ext}^1_G(k,k)=0, \qquad \mathrm{Ext}^1_G(21_1,57_1)=0, \qquad \mathrm{Ext}^1_G(21_1,21_2)=0. \tag{1}$$

The first is obvious, the second is because the corresponding Cartan number is zero and the third is because $21_1\otimes 21_2$ has no component in the principal block.

The next three lemmas simply collect together some relevant character-theoretic information. The computations involved are routine. In order to avoid irrelevant information, we will usually write "$\cong_0$" to mean isomorphism of principal block components.

LEMMA 1. *We have the following uniserial projective modules in blocks of defect 1, with composition series as given:*

(a) $P(36):36,90,36;\qquad P(90):90,36,90.$
(b) $P(126_1):126_1,126_2,126_1;\qquad P(126_2):126_2,126_1,126_2.$

LEMMA 2.

(a) $P(126_1)\otimes 6_1\cong_0 P(21_2)\oplus P(57_1)$.
(b) $P(126_2)\otimes 6_1\cong_0 P(133)$.
(c) $P(90)\otimes 13_1\cong_0 P(13_1)$.
(d) $P(36)\otimes 13_1\cong_0 P(13_2)$.

LEMMA 3. *We have the following composition factors, with multiplicities written in the exponents.*

(a) $126_1\otimes 6_1\cong_0\{{13_1}^2,{21_2}^2,{57_1}^2,133\}$.
(b) $126_2\otimes 6_1\cong_0\{k^3,{13_1}^2,{13_2}^3,{21_1}^2,21_2,57_2,133^4\}$.
(c) $90\otimes 13_1\cong_0\{k^2,{13_1}^4,13_2,21_2,{57_1}^2,133^2\}$.
(d) $36\otimes 13_1\cong_0\{k^2,13_1,{13_2}^4,21_1,{57_2}^2,133^2\}$.

PROPOSITION 4. *The following hold:*

(a) $\mathrm{Ext}^1_G(21_2,k)=0$.
(b) $\mathrm{Ext}^1_G(21_2,13_2)=0$.
(c) $\mathrm{Ext}^1_G(57_1,k)=0$.
(d) $\mathrm{Ext}^1_G(57_1,13_2)=0$.
(e) $\mathrm{Ext}^1_G(57_1,57_2)=0$.
(f) $\mathrm{Ext}^1_G(13_1,13_1)=0$.

PROOF: This uses only the parts (a) and (b) of Lemmas 2 and 3. Note first that Lemma 2(b) implies that $\operatorname{rad} P(126_1) \otimes 6_1$ has a simple head isomorphic to 133. It follows that for any simple module S other than 133, we have

$$\operatorname{Ext}^1_{\hat{G}}(21_2 \oplus 57_1, S) \cong \operatorname{Hom}_{\hat{G}}(\operatorname{rad}(126_1 \otimes 6_1), S). \tag{2}$$

Parts (a)-(e) now follow from the fact that none of the corresponding simple modules S is a composition factor of $\operatorname{rad}(126_1 \otimes 6_1)$, and (f) from (a), (c) and the isomorphism

$$13_1 \otimes 13_1 \cong k \oplus 21_1 \oplus 57_1 \oplus 90, \tag{3}$$

which is easily verified (using self-duality) from character data.

To go further, we need to look at some small modules. In the following lemma the socle series are described.

LEMMA 5.

(a) $36 \cong 6_1 \otimes 6_2$.
(b) $126_1 \cong 6_1 \otimes 21_2$.
(c)

$$6_1 \otimes 6_1 \cong \begin{array}{c} k \\ 13_1 \\ k \end{array} \oplus 21_1.$$

(d)

$$21_1 \otimes 21_1 \cong_0 \begin{array}{c} k \\ 13_1 \\ k \end{array} \oplus 21_1.$$

(e) $\operatorname{soc}(21_2 \otimes 13_1) \cong 57_1$.
(f)

$$6_1 \otimes 13_1 \cong \begin{array}{c} 14 \\ 50_1 \\ 14 \end{array}.$$

(g)

$$6_2 \otimes 14 \cong \begin{array}{ccc} & 13_2 & \\ k & & 57_2 \\ & 13_2 & \end{array}.$$

(h)

$$6_2 \otimes 50_1 \cong \begin{array}{ccc} & 133 & \\ 13_1 & & 21_1 \\ & 133 & \end{array}.$$

(i)

$$6_1 \otimes 21_1 \cong \begin{array}{ccc} & 50_1 & \\ 14 & & 6_2 \\ & 50_1 & \end{array} \oplus 6_1 .$$

(j)

$$13_1 \otimes 21_1 \cong_0 \begin{array}{ccc} & 13_1 & \\ k & & 57_1 \\ & 13_1 & \end{array} .$$

PROOF: The composition factors are obtained from the character data, so only the module structure needs proof. There is nothing to prove in (a) and (b). Parts (c) and (d) follow from the decomposition of the tensor square into the symmetric and alternating squares and the fact that the tensor square has one-dimensional fixed points. For (e), we calculate first that $21_2 \otimes 13_1 \cong_0 \{13_1{}^2, 57_1{}^2, 133\}$. Since $[13_1 \otimes 13_1 : 21_2] = 0$ and since 133 cannot be a direct summand as its dimension is not divisible by 3 [3, Prop. 2.2], (e) is proved. That the module in (f) must have simple head and socle is clear from the fact that it is a tensor factor of $36 \otimes 13_1$, which in turn is a quotient of $P(13_2)$, By Lemma 2(d), so (f) holds by self-duality. It then follows from (f) and Lemma 2(d) that $6_2 \otimes 14$ has simple head and socle 13_2 so, by self-duality, (g) holds. It is easily checked that $[6_2 \otimes 13_1 : 50_1] = 0 = [6_2 \otimes 21_1 : 50_1]$ and (h) follows from this. Similar considerations of spaces of $k\widehat{G}$-homomorphisms establish (i).

PROPOSITION 6.

(a) $\mathrm{Ext}^1_{\widehat{G}}(57_1, 57_1) = 0$.
(b) $\mathrm{Ext}^1_{\widehat{G}}(21_2, 57_1) = 0$.
(c) $\mathrm{Ext}^1_{\widehat{G}}(13_2, k) \cong k$.
(d) $\mathrm{Ext}^1_{\widehat{G}}(13_2, 57_2) \cong k$.
(e) $\mathrm{Ext}^1_{\widehat{G}}(13_2, 21_1) = 0$.
(f) $\mathrm{Ext}^1_{\widehat{G}}(13_2, 133) \cong k$.
(g) $\mathrm{Ext}^1_{\widehat{G}}(133, 21_1) \cong k$.
(h) $\mathrm{Ext}^1_{\widehat{G}}(133, 57_2) = 0$.
(i) $\mathrm{Ext}^1_{\widehat{G}}(13_1, 13_2) = 0$.
(j) $\mathrm{Ext}^1_{\widehat{G}}(133, k) = 0$.
(k) $\mathrm{Ext}^1_{\widehat{G}}(133, 133) = 0$.
(l) $\mathrm{Ext}^1_{\widehat{G}}(21_1, 21_1) = 0$.

PROOF: To prove (a) and (b), we return to (2) in the proof of Proposition 4. By self-duality we have $\mathrm{soc}(126_1 \otimes 6_1) \cong 21_2 \oplus 57_1$. We claim that every map in the right hand side of (2) has the simple submodule 57_1 in its kernel.

By Lemma 5 (b) we have $126_1 \otimes 6_1 \cong 21_2 \otimes (6_1 \otimes 6_1)$. The only composition factor of $6_1 \otimes 6_1$ whose tensor product with 21_2 has 57_1 as a composition factor is 13_1. The claim now follows from Lemma 5(e). Now (a) and (b) follow from the claim and (2), using Lemma 3.

Next we consider the module $P(13_2)$, which by Lemma 2 (d) has a filtration with factors equal to the principal block parts of $36 \otimes 13_1$ and $\operatorname{rad} P(36) \otimes 13_1$ respectively. Since the latter has simple head 13_1, by Lemma 2 (c), we see that for any simple module $S \not\cong 13_1$, we have

$$\operatorname{Ext}^1_{\hat{G}}(13_2, S) \cong \operatorname{Hom}_{\hat{G}}(\operatorname{rad}(36 \otimes 13_1), S). \tag{3}$$

Now $36 \otimes 13_1 \cong 6_2 \otimes (6_1 \otimes 13_1)$ has, by Lemma 5 (f), a filtration with factors $6_2 \otimes 14$, $6_2 \otimes 50_1$, $6_2 \otimes 14$. The structures of these factors are given in (g) and (h) of that lemma and (c), (d) (e) and (f) now follow from this information, using (3). We can now obtain (l) from Lemma 5(d) and existing results (Proposition 4 (a) and part (c) of this proposition). Now we examine $P(133) \cong_0 P(126_2 \otimes 6_1)$. From its filtration induced by that of $P(126_2)$, we obtain for $S \not\cong 21_2$, 57_1,

$$\operatorname{Ext}^1_{\hat{G}}(133, S) \cong \operatorname{Hom}_{\hat{G}}(\operatorname{rad}(126_2 \otimes 6_1), S). \tag{4}$$

By Lemma 5(b), we have $126_2 \otimes 6_1 \cong 6_2 \otimes (21_1 \otimes 6_1)$ and by Lemma 5(i), this has a filtration with factors $6_2 \otimes 50_1$, $6_2 \otimes (14 \oplus 6_2)$, $6_2 \otimes 50_1$. The structures of the factors are given in Lemma 5 (g) and (h). From this information, we can deduce (g) and (h) from (4). Next, we note that the isomorphism (3) in the proof of Proposition 4 implies that $\operatorname{Hom}_{\hat{G}}(13_1, 13_1 \otimes 57_1) \cong \operatorname{Hom}_{\hat{G}}(13_1 \otimes 13_1, 57_1) \cong k$, so $13_1 \otimes 57_1$ has a unique submodule isomorphic to 13_1. Since $[13_1 \otimes 57_1 : 13_2] = 0$, we obtain an embedding $\operatorname{Ext}^1_{\hat{G}}(13_2, 13_1)$ into $\operatorname{Ext}^1_{\hat{G}}(13_2, 13_1 \otimes 57_1) \cong \operatorname{Ext}^1_{\hat{G}}(13_2 \otimes 13_1, 57_1)$. Since $13_1 \otimes 13_2 \cong_0 133$, (i) follows from (h) and then (j) follows from (i). To prove (k) using (4), we need to examine $126_2 \otimes 6_1 \cong 6_2 \otimes (6_1 \otimes 21_1)$ more closely. By Lemma 5 (i), the module $6_1 \otimes 21_1$ has a uniserial submodule U with composition factors 14, 50_1. and the quotient of $\operatorname{rad}(6_2 \otimes 6_1 \otimes 21_1)$ by $6_2 \otimes U$ has no nonzero homomorphisms to 133, so it is enough to prove

$$\operatorname{Hom}_{\hat{G}}(6_2 \otimes U, 133) = 0. \tag{5}$$

We claim that U is a quotient of $6_1 \otimes 13_1$. Assuming this, we obtain

$$\operatorname{Hom}_{\hat{G}}(6_2 \otimes U, 133) \leq \operatorname{Hom}_{\hat{G}}(6_2 \otimes 6_1 \otimes 13_1, 133) \cong \operatorname{Hom}_{\hat{G}}(36 \otimes 13_1, 133) = 0,$$

by Lemma 2(d), which proves (5). It remains to verify the claim. By Lemma 5(f) and (i), any nonzero homomorphism from $6_1 \otimes 13_1$ to $6_1 \otimes 21$ maps onto U. We compute:

$$\mathrm{Hom}_{\widehat{G}}(6_1 \otimes 13_1, 6_1 \otimes 21_1) \cong \mathrm{Hom}_{\widehat{G}}(6_1 \otimes 6_1, 13_1 \otimes 21_1).$$

The fact that this space is not zero follows from Lemma 5 (c) and (j), using $\mathrm{Ext}^1_{\widehat{G}}(k, 13_1) \cong k$ (part (c) of this proposition).

Up to automorphisms and duality we have now computed all entries in Table 2.

§4. Characteristic 5

Our notation matches [9], which is our main source for data concerning Brauer characters, decomposition numbers and products of characters.

In §4 we assume k to be an algebraically closed field of characteristic 5.

4.1. The simple $k\widehat{G}$-modules will be denoted by their dimensions. There are two simple modules of dimension 14 of which one is faithful; this we denote by $\hat{14}$. There are also two mutually dual simple modules of dimension 50, but these play no part in our discussion.

The group G has five 5-blocks. There are three blocks of defect zero containing the simple modules 175, 225 and 300, a block of defect 1 with two simple modules 70 and 90, and the principal block, with Cartan matrix

4.1a) The principal 5-block of G.

	k	14	21	41	85	189
k	3	1	2	1	0	0
14	1	6	3	0	1	3
21	2	3	7	2	1	3
41	1	0	2	3	2	1
85	0	1	1	2	3	2
189	0	3	3	1	2	6

The projective indecomposable modules in the block of defect 1 have the following structures:

$$P(70): \quad \begin{matrix} & 70 & \\ 70 & & 90 \\ & 70 & \end{matrix}, \qquad P(90): \quad \begin{matrix} 90 \\ 70 \\ 90 \end{matrix} \ . \tag{1}$$

The group $\widehat{G}$ has four further blocks containing faithful modules. Three, containing 50_1, 50_2 and 350 respectively, are of defect zero, while the fourth whose Cartan matrix is given below has maximal defect.

4.1b) The non-principal 5-block of $\widehat{G}$ of maximal defect.

	6	$\hat{14}$	56	64	190	202
6	6	1	3	3	1	0
$\hat{14}$	1	4	1	2	0	2
56	3	1	7	3	2	2
64	3	2	3	6	0	1
190	1	0	2	0	2	1
202	0	2	2	1	1	3

We now describe some tensor products of simple modules. An expression in braces indicates a module having the enclosed composition factors and multiplicities are indicated by an exponent.

LEMMA 1.

(a) $14 \otimes 14 \cong k \oplus 14 \oplus 21 \oplus 70 \oplus 90$.
(b) $14 \otimes 21 \cong 14 \oplus 21 \oplus 189 \oplus 70$.
(c) $14 \otimes 6 \cong 6 \oplus \hat{14} \oplus 64$.
(d) $14 \otimes 90 \cong \{14^3, 21, 85, 189^3\} \oplus \{70^2, 90^2\} \oplus 225$.
(e) $14 \otimes 70 \cong \{k, 14^3, 21^4, 41, 85, 189^3\} \oplus 70 \oplus 90$.
(f) $14 \otimes 189 \cong \{14^3, 21^4, 41^2, 85^3, 189^7\} \oplus 70 \oplus 90 \oplus 175 \oplus 225 \oplus 300$.
(g) $14 \otimes 41 \cong 85 \oplus 189 \oplus 300$.
(h) $21 \otimes 21 \cong k \oplus 14 \oplus 21 \oplus 41 \oplus 85 \oplus 189 \oplus 90$.
(i) $21 \otimes 41 \cong \{14, 21^2, 41^2, 85^2, 189^2\} \oplus 175$.
(j) $6 \otimes 6 \cong k \oplus 14 \oplus 21$.
(k) $6 \otimes \hat{14} \cong 14 \oplus 70$.
(l) $6 \otimes 56 \cong 21 \oplus 41 \oplus 85 \oplus 189$.
(m) $6 \otimes 64 \cong 14 \oplus 21 \oplus 70 \oplus 90$.
(n) $6 \otimes 41 \cong 56 \oplus 190$.
(o) $6 \otimes 85 \cong \{6, 56^2, 190, 202\}$.
(p) $6 \otimes 70 \cong \{6, \hat{14}^2, 56, 64^2, 202\}$.
(q) $6 \otimes 90 \cong \{6, 56, 64^2\} \oplus 350$.
(r) $\hat{14} \otimes 70 \cong \{6^4, 56^4, 64^3, 190\} \oplus 350$.
(s) $\hat{14} \otimes 90 \cong \{\hat{14}^4, 56, 64^3, 202^3\} \oplus 350$.

PROOF: These results all follow from character computations, block decomposition and self-duality.

Next, we consider tensor products of small projective modules with simple modules. Again, the proof is pure calculation with characters.

LEMMA 2.

(a) $P(90) \otimes 14 \cong P(14) \oplus P(189) \oplus P(70) \oplus P(90)^2 \oplus 225$.
(b) $P(70) \otimes 14 \cong P(14) \oplus P(21) \oplus P(189) \oplus P(70) \oplus P(90)^2 \oplus 225$.
(c) $P(90) \otimes 6 \cong P(64) \oplus 350^2$.
(d) $P(70) \otimes 6 \cong P(\hat{14}) \oplus P(64) \oplus 350$.
(e) $P(90) \otimes \hat{14} \cong P(\hat{14}) \oplus P(64) \oplus P(202) \oplus 350^3$.
(f) $P(70) \otimes \hat{14} \cong P(6) \oplus P(56) \oplus P(64) \oplus 350^4$.
(g) $P(k) \otimes 14 \cong P(14) \oplus P(70) \oplus 300$.
(h) $P(k) \otimes 21 \cong P(21) \oplus P(85) \oplus P(90) \oplus 175$.
(i) $P(k) \otimes 6 \cong P(6)$.

From the above two lemmas, we can obtain information about the heads (maximal semisimple quotients) of certain tensor products which will appear as subquotients in filtrations of projective modules in subsequent Ext^1 calculations. By self-duality, the socles are the same.

LEMMA 3.

(a) $\mathrm{head}(90 \otimes 14) \cong 14 \oplus 189 \oplus 70 \oplus 90 \oplus 225$.
(b) $\mathrm{head}(70 \otimes 14) \cong 14 \oplus 21 \oplus 189 \oplus 70 \oplus 90$.
(c) $\mathrm{head}(90 \otimes 6) \cong 64 \oplus 350$.
(d) $\mathrm{head}(70 \otimes 6) \cong \hat{14} \oplus 64$.
(e) $\mathrm{head}(90 \otimes \hat{14}) \cong \hat{14} \oplus 64 \oplus 202 \oplus 350$.
(f) $\mathrm{head}(70 \otimes \hat{14}) \cong 6 \oplus 56 \oplus 64$.

PROOF: Upper bounds on the multiplicities are given by the previous lemma since, for example, $90 \otimes 14$ is a quotient of $P(90) \otimes 14$. The exact multiplicities are then easily found by elementary considerations of composition factors, self-duality, etc, using Lemma 1. We give the details for (a) only. The occurence of 225 is clear from Lemma 1(d), that of 14 follows from Lemma 1(a) and that of 189 from Lemma 1(f). By Lemma 1(e), 70 appears, so we need only determine the multiplicity of 90. If it appeared twice then $90 \oplus 90$ would be a direct summand of $90 \otimes 14$. It would then follow from the structure of $P(90)$ that $P(90) \otimes 14$ had a subquotient $\{90^5, 70\}$, which contradicts what we know about $P(90)$.

REMARK. A general result [3, Prop. 2.2] states that if the field characteristic divides the dimension of an indecomposable module then it divides the dimension of every direct summand of the tensor product of that module with another. We shall use this here and on one later occasion. It has the following consequence: Let M be the component in a block of maximal defect of one of the modules whose heads are given in Lemma 3. Then $\operatorname{soc} M \subseteq \operatorname{rad} M$. Here we could also have argued using relative projectivity, but not in our later application.

4.2 Extensions. In several instances, where the Cartan number is zero or is a 2 on the leading diagonal, it is clear that there are only split extensions between the corresponding simple modules, so these entries in the table of extensions can be filled in immediately, and of course, $\operatorname{Ext}^1_{\hat{G}}(k,k) = 0$.

In the first result the filtrations of $P(90) \otimes 14$ and $P(70) \otimes 14$ induced by certain filtrations of $P(90)$ and $P(70)$ will be used to calculate a number of Ext^1 groups between simple modules. Further extensions can then be computed from formal manipulations with the equations of Lemma 1. In the succeeding propositions we apply the same method to other tensor products of projective modules with simple modules. Nearly all entries in the tables of extensions for characteristic 5 will be found with little difficulty by this process. The rest require some extra arguments, but still based on the same ideas. Lemmas referred to by number are from 4.1.

PROPOSITION 1.

(a) $\operatorname{Ext}^1_{\hat{G}}(14,k) = 0$.
(b) $\operatorname{Ext}^1_{\hat{G}}(189,41) = 0$.
(c) $\operatorname{Ext}^1_{\hat{G}}(21,k) = k$.
(d) $\operatorname{Ext}^1_{\hat{G}}(21,41) = k$.
(e) $\operatorname{Ext}^1_{\hat{G}}(14,14) = k$.
(f) $\operatorname{Ext}^1_{\hat{G}}(14,85) = 0$.
(g) $\operatorname{Ext}^1_{\hat{G}}(189,85) = k$.
(h) $\operatorname{Ext}^1_{\hat{G}}(21,85) = 0$.
(i) $\operatorname{Ext}^1_{\hat{G}}(14,21) = k$.
(j) $\operatorname{Ext}^1_{\hat{G}}(6,6) = k$.
(k) $\operatorname{Ext}^1_{\hat{G}}(6,\hat{14}) = 0$.
(l) $\operatorname{Ext}^1_{\hat{G}}(6,56) = k$.
(m) $\operatorname{Ext}^1_{\hat{G}}(6,64) = k$.

PROOF: We consider the filtration $\operatorname{rad} P(90) \otimes 14 \subset P(90) \otimes 14$. Since $\operatorname{rad} P(90)$ is a quotient of $P(70)$, it follows that $\operatorname{rad} P(90) \otimes 14$ is a quotient of $P(70) \otimes 14$, whose head is given in Lemma 3. In particular, neither k nor 41 appears. By Lemma 2, we have

$$P(90) \otimes 14 \cong_0 P(14) \oplus P(189), \tag{1}$$

where "$\cong_0$" denotes isomorphism of parts in the principal block. Since we have composition multiplicities $[90 \otimes 14 : k] = 0$ and $[90 \otimes 14 : 41] = 0$, we see that k and 41 are not quotients of $\operatorname{rad}(P(90) \otimes 14)$, proving (a) and (b).

Now consider the filtration $\operatorname{rad} P(70) \otimes 14 \subset P(70) \otimes 14$. Since $\operatorname{rad} P(70)$ is a quotient of $P(70) \oplus P(90)$, it follows that $\operatorname{rad} P(90) \otimes 14$ is a quotient of $(P(70) \otimes 14) \oplus (P(90) \otimes 14)$, whose head is given in Lemma 3. In particular k and 41 do not appear. By Lemma 2,

$$P(70) \otimes 14 \cong_0 P(14) \oplus P(21) \oplus P(189). \tag{2}$$

Since we already know $\operatorname{Ext}^1_{\hat{G}}(14, k \oplus 41) = 0$ and $\operatorname{Ext}^1_{\hat{G}}(189, k \oplus 41) = 0$, from (a), (b) and the Cartan matrix, it follows that

$$\operatorname{Ext}^1_{\hat{G}}(21, k) \cong \operatorname{Hom}_{\hat{G}}(\operatorname{rad}/\operatorname{soc}(70 \otimes 14), k), \tag{3}$$

and similarly for 41. Now,

$$\operatorname{rad}/\operatorname{soc}(70 \otimes 14) \cong_0 \{21^2, k, 41, 14, 189, 85\}.$$

We claim (c) and (d) hold. If not, then $\operatorname{Ext}^1_{\hat{G}}(21, k) = 0$ and $\operatorname{Ext}^1_{\hat{G}}(21, 41) = 0$, but then using self-duality, we see that $\operatorname{rad}/\operatorname{soc}(70 \otimes 14)$ has a quotient $k \oplus 41$, contrary to (3).

Now (e) is immediate from Lemma 1(a).

Let us return to the filtration of $P(90) \otimes 14$ at the beginning. We know by Lemma 2(b) that 85 is not a quotient of $\operatorname{rad} P(90) \otimes 14$ so we have by (1),

$$\operatorname{Ext}^1_{\hat{G}}(14 \oplus 189, 85) \cong \operatorname{Hom}_{\hat{G}}(\operatorname{rad}/\operatorname{soc}(90 \otimes 14), 85). \tag{4}$$

It is clear from self-duality that

$$\operatorname{rad}/\operatorname{soc}(90 \otimes 14) \cong_0 14 \oplus 21 \oplus 85 \oplus 189. \tag{5}$$

Thus, $\operatorname{Ext}^1_{\hat{G}}(14 \oplus 189, 85) \cong k$. From (5), (1) and self-duality it follows that $\operatorname{rad}/\operatorname{soc}(P(14) \oplus P(189))$ has a unique submodule 85 and unique quotient 85, with the submodule being in the kernel of a quotient map. Therefore by self-duality, the image by projection onto exactly one of $\operatorname{rad}/\operatorname{soc} P(14)$ or $\operatorname{rad}/\operatorname{soc} P(189)$ has the same property. Since $[P(14) : 85] = 1$, it must be the latter. This proves (f) and (g).

Next, returning to the filtration of $P(70) \otimes 14$, we see from (2) and Lemma 2 (a), (b) that $\operatorname{Ext}^1_{\hat{G}}(14 \oplus 21 \oplus 189, 85) \cong \operatorname{Hom}_{\hat{G}}(\operatorname{rad}/\operatorname{soc}(70 \otimes 14), 85)$, so (g) and $[\operatorname{rad}/\operatorname{soc}(70 \otimes 14) : 85] = 1$ imply (h).

Parts (i)-(m) are easily derived from the identities in Lemma 1 and the parts already proved.

Next we consider $P(k) \otimes 14$ and $P(k) \otimes 21$.

PROPOSITION 2.

(a) $\operatorname{Ext}^1_{\hat{G}}(k, 41) = 0$.

(b) $\operatorname{Ext}^1_{\hat{G}}(6, 190) = 0$.

(c) $\operatorname{Ext}^1_{\hat{G}}(21, 21) = k$.

(d) $\mathrm{Ext}^1_{\hat{G}}(14,189)=k$.
(e) $\mathrm{Ext}^1_{\hat{G}}(21,189)=k$.
(f) $\mathrm{Ext}^1_{\hat{G}}(189,189)=k$.

PROOF: If $\mathrm{Ext}^1_{\hat{G}}(k,41)\neq 0$, then by Lemma 2(g), 14 extends nontrivially any simple quotient of $41\otimes 14$ in the principal block. Therefore (a) follows from Lemma 1(g) and Proposition 1(f). For (b), we use parts (n) and (j) of Lemma 1 to obtain

$$\begin{aligned}\mathrm{Ext}^1_{\hat{G}}(6,56\oplus 190)&\cong \mathrm{Ext}^1_{\hat{G}}(6,6\otimes 41)\cong \mathrm{Ext}^1_{\hat{G}}(6\otimes 6,41)\\ &\cong \mathrm{Ext}^1_{\hat{G}}(k\oplus 14\oplus 21,41),\end{aligned}$$

which gives a reduction to Proposition 1 and (a).

(c) follows from Lemma 1(h) using (a) and Proposition 1.

Lemma 1(a) and (b) yield

$$\begin{aligned}\mathrm{Ext}^1_{\hat{G}}(14,14\oplus 21\oplus 189)&\cong \mathrm{Ext}^1_{\hat{G}}(14,14\otimes 21)\cong \mathrm{Ext}^1_{\hat{G}}(14\otimes 14,21)\\ &\cong \mathrm{Ext}^1_{\hat{G}}(k\oplus 14\oplus 21,21),\end{aligned}$$

and now (d) follows from existing results.

From Lemma 2(h), Proposition 1(c) and Lemma 1(h), it follows that $\mathrm{Ext}^1_{\hat{G}}(21,189)\neq 0$. Equation (1) of Proposition 1 shows that $\dim \mathrm{Ext}^1_{\hat{G}}(14\oplus 189,21)=\dim \mathrm{Hom}_{\hat{G}}(\mathrm{rad}(P(90)\otimes 14),21)$ and from the filtration of $P(90)\otimes 14$ and the structure of $90\otimes 14$ and $70\otimes 14$ described there, one sees that this dimension is at most 2. Now (e) follows from Proposition 1(i).

To prove (f), we use Lemma 1(b), (a) and (h), to obtain

$$\begin{aligned}&\mathrm{Ext}^1_{\hat{G}}(14\oplus 21\oplus 189\oplus 70,14\oplus 21\oplus 189\oplus 70)\\ &\cong \mathrm{Ext}^1_{\hat{G}}(14\otimes 21,14\otimes 21)\cong \mathrm{Ext}^1_{\hat{G}}(14\otimes 14,21\otimes 21)\\ &\cong \mathrm{Ext}^1_{\hat{G}}(k\oplus 14\oplus 21\oplus 70\oplus 90,k\oplus 14\oplus 21\oplus 41\oplus 85\oplus 189\oplus 90),\end{aligned}$$

and all extensions in the first and last terms other than the one in question are known.

REMARK. It is now easy to see that $P(k)$ has radical (and socle) series

$$\begin{array}{ccc} & k & \\ & 21 & \\ k & 14 & 41\,. \\ & 21 & \\ & k & \end{array}$$

Next we consider the tensor products of $P(70)$ and $P(90)$ with 6 and $\hat{14}$.

PROPOSITION 3.

(a) $\mathrm{Ext}^1_{\hat{G}}(64, 202) = 0$.
(b) $\mathrm{Ext}^1_{\hat{G}}(64, 56) = k$.
(c) $\mathrm{Ext}^1_{\hat{G}}(\hat{14}, 202) = k$.
(d) $\mathrm{Ext}^1_{\hat{G}}(\hat{14}, 56) = 0$.
(e) $\mathrm{Ext}^1_{\hat{G}}(\hat{14}, 64) = k$.
(f) $\mathrm{Ext}^1_{\hat{G}}(202, 190) = 0$.
(g) $\mathrm{Ext}^1_{\hat{G}}(202, 202) = 0$.
(h) $\mathrm{Ext}^1_{\hat{G}}(202, 56) = k$.
(i) $\mathrm{Ext}^1_{\hat{G}}(56, 190) = k$.
(j) $\mathrm{Ext}^1_{\hat{G}}(\hat{14}, \hat{14}) = 0$.
(k) $\mathrm{Ext}^1_{\hat{G}}(64, 64) = k$.
(l) $\mathrm{Ext}^1_{\hat{G}}(56, 56) = k$.
(m) $\mathrm{Ext}^1_{\hat{G}}(85, 41) = k$.
(n) $\mathrm{Ext}^1_{\hat{G}}(41, 41) = 0$.

PROOF: From Lemma 3 (c), (d) and Lemma 1 (p), (q) we see that $70 \otimes 6$ and $90 \otimes 6$ have the following structures:

$$70 \otimes 6: \quad \begin{matrix} \hat{14} & & 64 \\ 6 & 56 & 202 \\ \hat{14} & & 64 \end{matrix}, \qquad 90 \otimes 6: \quad 6 \quad \begin{matrix} 64 \\ \\ 64 \end{matrix} \quad 56 \oplus 350. \tag{6}$$

Since these modules filter $P(90) \otimes 6 \cong P(64) \oplus 350^2$ we can deduce (a) and (b). Since they also filter $P(70) \otimes 6 \cong P(\hat{14}) \oplus P(64) \oplus 350$, we also obtain (c) (using (a)) and (d) (using (b)). The module $\operatorname{rad} P(90) \otimes 6$ has a simple quotient $\hat{14}$ and we have seen that the composition factors 6 and 56 of $90 \otimes 6$ do not extend $\hat{14}$, which yields (e). Next we consider $90 \otimes \hat{14}$ and $70 \otimes \hat{14}$, which filter $P(90) \otimes \hat{14} \cong P(\hat{14}) \oplus P(64) \oplus P(202) \oplus 350^3$. Since $\operatorname{rad} P(90) \otimes \hat{14}$ is a quotient of $P(70) \otimes 14$ and since $[90 \otimes \hat{14} : 190] = 0$, we deduce (f). Since by Lemma 1(s) and Lemma 3(e), $[\operatorname{rad}/\operatorname{soc}(90 \otimes \hat{14}) : 202] = 1$, (g) follows from (c). Also since $[\operatorname{rad}/\operatorname{soc}(90 \otimes \hat{14}) : 56] = 1$ and 56 appears only once in the head of $\operatorname{rad} P(90) \otimes \hat{14}$, we see that $\dim \mathrm{Ext}^1_{\hat{G}}(\hat{14} \oplus 64 \oplus 202, 56) \le 2$ from which it follows by (b) that $\dim \mathrm{Ext}^1_{\hat{G}}(202, 56) \le 1$. To prove (h), we need to find a nonsplit extension of 56 by 202. By Lemma 1(o), $6 \otimes 85 \cong \{6, 56^2, 202, 190\}$. We have $\mathrm{Hom}_{\hat{G}}(6 \otimes 85, 56) \cong \mathrm{Hom}_{\hat{G}}(85, 6 \otimes 56) \cong k$ by Lemma 1(l). The subquotient of $6 \otimes 85$ obtained by omitting the top and bottom factors 56 must be semisimple by self-duality. Since any direct summand of $6 \otimes 85$ must have dimension divisible by 5 (Remark after Lemma 3), we see that 202 is not one and so we have found the desired extension.

To prove (i), we note that the part of $P(70) \otimes \hat{14}$ in the relevant block is $P(6) \oplus P(56) \oplus P(64)$. Neither $90 \otimes \hat{14}$ nor $70 \otimes \hat{14}$ has 190 as a quotient, so since $\mathrm{Ext}^1_{\hat{G}}(6 \oplus 64, 190) = 0$ (by Proposition 2(b) and the Cartan matrix),

$$\mathrm{Ext}^1_{\hat{G}}(56, 190) \cong \mathrm{Ext}^1_{\hat{G}}(6 \oplus 56 \oplus 64, 190) \cong \mathrm{Hom}_{\hat{G}}(\mathrm{rad}\,/\,\mathrm{soc}(70 \otimes \hat{14}), 190). \quad (7)$$

Since $\mathrm{rad}\,/\,\mathrm{soc}(70 \otimes \hat{14}) \cong \{6^2, 56^2, 64, 190\}$, the assumption $\mathrm{Ext}^1_{\hat{G}}(56, 190) = 0$ would contradict (7), so (i) holds.

From the filtration of $P(90) \otimes \hat{14}$ we obtain

$$\mathrm{Ext}^1_{\hat{G}}(\hat{14} \oplus 64 \oplus 202, \hat{14}) \cong \mathrm{Hom}_{\hat{G}}(\mathrm{rad}\,/\,\mathrm{soc}(90 \otimes \hat{14}), \hat{14}),$$

and this has dimension at most 2. Then (c) and (e) imply (j).

By Lemma 1(c), (a) and (j), we have

$$\begin{aligned} &\mathrm{Ext}^1_{\hat{G}}(6 \oplus \hat{14} \oplus 64, 6 \oplus \hat{14} \oplus 64) \cong \mathrm{Ext}^1_{\hat{G}}(14 \otimes 6, 14 \otimes 6) \\ &\cong \mathrm{Ext}^1_{\hat{G}}(14 \otimes 14, 6 \otimes 6) \cong \mathrm{Ext}^1_{\hat{G}}(k \oplus 14 \oplus 21, k \oplus 14 \oplus 21), \end{aligned}$$

and all extensions except $\mathrm{Ext}^1_{\hat{G}}(64, 64)$ are already known, whence (k).

We prove (l), (m) and (n) together. First we claim $\dim \mathrm{Ext}^1_{\hat{G}}(56, 56) \leq 1$. From the structure of $P(70)$, we see that $P = P(70) \otimes \hat{14}$ has a filtration $P_1 \subset P_2 \subset P$ where P_1 is a quotient of $P(90) \otimes \hat{14}$ while P_2/P_1 and P/P_2 are both isomorphic to $70 \otimes \hat{14}$. Consideration of the heads of these factors and the composition factors of $\mathrm{rad}\,/\,\mathrm{soc}(70 \otimes \hat{14})$ shows that the $\dim \mathrm{Hom}_{\hat{G}}(\mathrm{rad}(P(70) \otimes \hat{14}), 56) \leq 3$. Since the part of $P(70) \otimes \hat{14}$ in the block of maximal defect is $P(6) \oplus P(56) \oplus P(64)$, our claim follows from (b) and Proposition 1 (l). Now using Lemma 1 (l) and (n) and known extensions we have

$$\begin{aligned} &k \oplus \mathrm{Ext}^1_{\hat{G}}(41, 41) \oplus \mathrm{Ext}^1_{\hat{G}}(85, 41) \cong \mathrm{Ext}^1_{\hat{G}}(21 \oplus 41 \oplus 85 \oplus 189, 41) \\ &\cong \mathrm{Ext}^1_{\hat{G}}(6 \otimes 56, 41) \cong \mathrm{Ext}^1_{\hat{G}}(56, 6 \otimes 41) \\ &\cong \mathrm{Ext}^1_{\hat{G}}(56, 56 \oplus 190) \cong k \oplus \mathrm{Ext}^1_{\hat{G}}(56, 56). \end{aligned} \quad (8)$$

By our claim, the dimension of the last member is at most 2. On the other hand we have $\mathrm{Ext}^1_{\hat{G}}(85, 41) \neq 0$, for example from the fact that the reduction mod 5 of the complex character of degree 126 has 41 and 85 as composition factors. Now (8) implies (l), (m) and (n).

Finally, we turn to the last missing entry.

PROPOSITION 4. $\mathrm{Ext}^1_{\hat{G}}(85, 85) = 0$.

PROOF: Ignoring components outside the principal block, we have $P(k) \otimes 21 \cong P(21) \oplus P(85)$. We define the filtration by setting F_i to be the principal block part of $\operatorname{soc}^i P(k) \otimes 21$, $i = 1, \ldots 5$. Thus $F_5/F_4 \cong 21$ and $F_4/F_3 \cong k \oplus 14 \oplus 12 \oplus 41 \oplus 85 \oplus 189$. From existing results on extensions, it now follows that

$$F_3 \cong \operatorname{rad}^2 P(21) \oplus \operatorname{rad} P(85). \tag{9}$$

We will prove that $\dim \operatorname{Hom}_{\widehat{G}}(F_3, 85) \leq 1$, while $\operatorname{Hom}_{\widehat{G}}(\operatorname{rad}^2(P(21), 85)) \neq 0$. The lemma will then follow from (9). Dual to (9), we have $F_2 \cong \operatorname{soc}^2 P(21) \oplus \operatorname{soc} P(85)$, so $\operatorname{soc}^2 P(21) \subseteq \operatorname{rad}^2 P(21)$. Since

$$\operatorname{rad}^2 / \operatorname{soc}^2(P(21)) \cong \{14, 21^3, 85, 189\}$$

is self-dual and 21 does not extend 85, we have $\operatorname{Hom}_{\widehat{G}}(\operatorname{rad}^2(P(21), 85) \neq 0$. Now consider any nonzero $\phi \in \operatorname{Hom}_{\widehat{G}}(F_3, 85)$. We use the identification (9). Since $\operatorname{soc} P(85)$ is the unique minimum submodule of $\operatorname{rad} P(85)$ and the latter is not simple, we see that $\operatorname{soc} P(85)$ is in the kernel of ϕ. Let $\overline{F}_i$ denote the images of F_i in $F_3/\operatorname{soc} P(85)$ (i=1, 2,3) then $\overline{F}_3/\overline{F}_2 \cong F_3/F_2$ is the only factor of this filtration of $\overline{F}_3$ which has 85 as a composition factor. Also, from the structure of $P(k)$, we have $\overline{F}_3/\overline{F}_2 \cong (41 \otimes 21) \oplus (k \otimes 21) \oplus (14 \otimes 21)$ and all the composition factors 85 belong to $41 \otimes 21$. Thus, we have shown $\operatorname{Hom}_{\widehat{G}}(F_3, 85) \cong \operatorname{Hom}_{\widehat{G}}(21 \otimes 41, 85)$ and it remains to bound the dimension of the second space by 1. Using Lemma 1(i),(j) and (n), we have

$$\begin{aligned}
k \oplus \operatorname{Hom}_{\widehat{G}}(21 \otimes 41, 85) &\cong \operatorname{Hom}_{\widehat{G}}((k \oplus 14 \oplus 21) \otimes 41, 85) \\
&\cong \operatorname{Hom}_{\widehat{G}}((6 \otimes 6) \otimes 41, 85) \cong \operatorname{Hom}_{\widehat{G}}(6 \otimes 41, 6 \otimes 85) \\
&\cong \operatorname{Hom}_{\widehat{G}}(56, 6 \otimes 85) \oplus \operatorname{Hom}_{\widehat{G}}(190, 6 \otimes 85) \\
&\cong k \oplus \operatorname{Hom}_{\widehat{G}}(190, 6 \otimes 85).
\end{aligned} \tag{10}$$

Since $[6 \otimes 85 : 190] = 1$, we obtain our estimate and the proposition is proved.

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