Graphs With Large Sets of Strongly Cospectral Vertices

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Overview

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Both are equivalence relations. The concept of strongly cospectral vertices, introduced by Godsil around 2012, is important in the theory of quantum walks on graphs.

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acting on $\mathbb{C}V(X)$.
X has **perfect state transfer** from a to $b \in V(X)$ if, for some τ, we have $|U(\tau)_{b,a}| = 1$.

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- SC classes can have more than 2 elements. 4 in $P_2 \square P_3$, 8 in $P_2 \square P_3 \square P_4$.
- Cartesian products of k paths of suitable lengths where the 2^k "corners" are SC.
- What about vertex-transitive graphs?

Restrictions on SC classes

- If a and b are SC then any automorphism fixing a must fix b.
- No tree can have 3 SC vertices (Coutinho-Juliano-Spier 2022) .
- Arnadottir-Godsil (2021): In a normal Cayley graph the SC class of identity element forms an central elementary abelian 2-group G_{sc}.

  $|G_{sc}| \leq |G|/m$, where m is the maximum eigenvalue multiplicity. For cubelike graphs, this bound is roughly $\sqrt{|G|}$. Examples of size 4 in cubelike graphs.

SC classes in Cayley graphs are unbounded

We shall describe two families of examples which show that an SC class in certain abelian Cayley graphs can be arbitraily large.
In the first family, all involutions belong to G_{sc}.
The graphs in the second family are cubelike, hence integral.

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Theorem

Distinct elements g and h of G are strongly cospectral iff there is a central involution z such that the following hold.

(a) $h = zg$.

(b) $(\forall \chi, \psi \in \mathrm{Irr}(G))$, $\chi(S) = \psi(S)$ *implies* $\chi(z) = \psi(z)$.

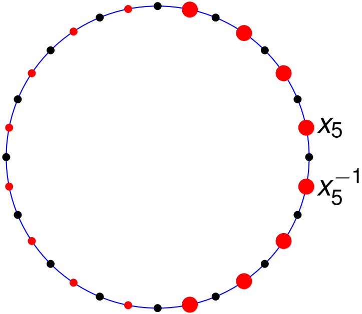
Overview

Let Z_d denote a cyclic group of order d, written mutiplicatively. We use the notation $C_m = \langle x_m \rangle$ for Z_{2^m}. We shall assume that $m \geq 3$. Let $\omega_m = \exp(\frac{2\pi i}{2^m})$, a primitive 2^m-th root of unity in $\mathbb{C}$. We identify the group of irreducible complex characters with $\mathbb{Z}/2^m\mathbb{Z}$, where $a \in \mathbb{Z}/2^m\mathbb{Z}$ corresponds the the character $[a] : x_m \mapsto \omega_m^a$.
If $u = x_m^{2^{m-1}}$ is the involution in C_m then $[a](u) = (-1)^a$.

The sets $T_m \subset C_m$

In the group C_m, we consider the generating set

$$T_m = \{x_m^{2i+1} \mid 0 \leq i \leq 2^{m-3} - 1\} \cup \{x_m^{-(2i+1)} \mid 0 \leq i \leq 2^{m-3} - 1\}.$$



The generating set $T_5 \subset C_5$.

The Graphs $\mathrm{Cay}(G_J, S_J)$

Let J be a finite set of positive integers $j \geq 3$. Let $G_J = \bigoplus_{j \in J} C_j$. $S_J = \cup_{j \in J} T_j$ is a generating set of G_J The characters of G_J are given by tuples $a_J = ([a_j])_{j \in J}$ with $a_j \in \mathbb{Z}/2^j\mathbb{Z}$. The eigenvalues of $\mathrm{Cay}(G_J, S_J) = \square_{j \in J} \mathrm{Cay}(C_j, T_j)$ are given by

$$a_J(S_J) = \sum_{j \in J} [a_j](T_j).$$

Some Galois theory

Consider the fields $\mathbb{Q}(\omega_m)$ and $F_m = \mathbb{Q}(\omega_m + \omega_m^{-1})$. The following lemma summarizes some well known facts from Galois theory that we shall need.

Lemma

(a) $\mathrm{Gal}(\mathbb{Q}(\omega_m)/\mathbb{Q}) = \langle\beta_m\rangle \times \langle\gamma_m\rangle \cong Z_2 \times Z_{2^{m-2}}$, *where* $\beta(\omega_m) = (\omega_m^{-1})$ *and* $\gamma_m(\omega_m) = \omega_m^5$.

(b) *Let* τ_m *be the unique involution of* $\langle\gamma_m\rangle$. *Then* $\tau_m(\omega_m) = -\omega_m$.

(c) *The restriction map* $\mathrm{Gal}(\mathbb{Q}(\omega_m)/\mathbb{Q}) \to \mathrm{Gal}(F_m/\mathbb{Q})$ *defines an isomorphism of* $\langle\gamma_m\rangle$ *with* $\mathrm{Gal}(F_m/\mathbb{Q})$.

(d) *The field* F_{m-1} *is the subfield of* F_m *fixed by* τ_m.

□

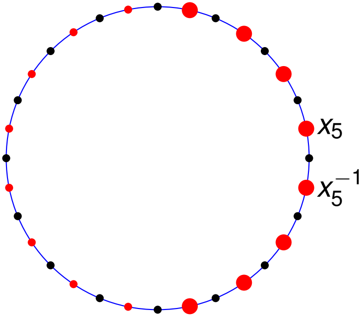
Corollary

Let $[a]$ be a character of C_m.

(a) $[a](T_m) \neq 0$ *if a is odd.*

(b) $\tau_m([a](T_m)) = -[a](T_m)$ *if a is odd.*

(c) $[a](T_m) \in \mathbb{Q}$ *if a is even.*



$\square$

All involutions are SC in $\mathrm{Cay}(G_J, S_J)$

Recall $G_J = \bigoplus_{j \in J} C_j$, $S_J = \cup_{j \in J} T_j$, ($J \subset \{3, 4, \ldots\}$).

Lemma

Let $a_J = ([a_j])_{j \in J}$ and $b_J = ([b_j])_{j \in J}$ be characters of G_J. Suppose that $a_J(S_J) = b_J(S_J)$. Assume that not all a_j and b_j are even, and let $m \in J$ be the largest element for which either a_m or b_m is odd. Then both a_m and b_m are odd and $[a_m](T_m) = [b_m](T_m)$.

Proof.

$$\sum_{j \in J} [a_j](T_j) = \sum_{j \in J} [b_j](T_j) \in F_m.$$

By the Galois theory lemma, all terms except $[a_m](T_m)$ and $[b_m](T_m)$ are in the 1-eigenspace of τ_m, and $[a_m](T_m)$ is in the (-1)-eigenspace. $\square$

Corollary

Let $a_J = ([a_j])_{j\in J}$ *and* $b_J = ([b_j])_{j\in J}$ *be characters of* G_J*. and suppose that* $a_J(S_J) = b_J(S_J)$*. Then for every* $j \in J$*,* a_j *and* b_j *are either both odd or both even. In particular* $a_J(t) = b_J(t)$ *for every involution* $t \in G_J$*. Thus, all involutions belong to* G_{sc}*.*

Overview

cubelike graphs

A cubelike graph is $\mathrm{Cay}(G, S)$ where G is an elementary abelian 2-group and the connecting set S is any subset of G that does not contain the identity. Let $|G| = 2^n$. We will identify G with the additive group of the vector space $\mathbb{F}_2^n$ over $\mathbb{F}_2$, so we may speak of hyperplanes instead of subgroups of index 2, make use of the dot product and, later on, quadratic forms.

Characters

For $w \in G$, $\chi_w : G \to \{\pm 1\}$ is the character defined by $\chi_w(x) = (-1)^{w \cdot x}$. Then the eigenvalues of $\mathrm{Cay}(G, S)$, counted with multiplicity, are the 2^n values

$$\chi_w(S) := \sum_{s \in S} \chi_w(S) = \sum_{s \in S} (-1)^{w \cdot s} = |S| - 2n_w,$$

where $n_w = |\{s \in S \mid w \cdot s = 1\}|$.
We have $n_w = |S| - |S \cap H_w|$, where H_w is the hyperplane orthogonal to w with respect to the dot product, if $w \neq 0$, and $H_0 = G$. Let $\sigma = \sum_{s \in S} s$, where we mean the sum in the group G. Then

$$\chi_w(\sigma) = \prod_{s \in S} (-1)^{w \cdot s} = (-1)^{n_w}.$$

Thus for any w, knowing $|S \cap H_w|$ is equivalent to knowing the eigenvalue $\chi_w(S)$, and this information completely determines $\chi_w(\sigma)$.

Idea behind construction

Suppose $S = S_1 \cup S_2 \cup S_3$ is the disjoint union of three subsets. Let σ_i be the sum in G of the elements of S_i.
Suppose S has the property that for all w, $|H_w \cap S|$ determines $|H_w \cap S_i|$ for all i.
Then $\chi_w(S)$ determines $|H_w \cap S|$, which determines $|H_w \cap S_i|$ for all i, which determines $\chi_w(\sigma_i)$. Thus, two characters that give the same eigenvalue of $\mathrm{Cay}(G, S)$ agree on each element σ_i, and so the elements σ_i are cospectral to 0 (although they may be equal to zero in some cases). If we can find S and S_i as above such that the σ_i generate a group of order 8, we will have an example of a strongly cospectral subgroup of order 8.
Of course, we can generalize this idea to subsets S_i, $i = 1, \ldots,$ k, to construct a strongly cospectral subgroup of order 2^k.

First try

We consider a prototype for this idea (in which unfortunately the $\sigma_i = 0$).

Let $n = n_1 + n_2 + n_3$, with $n_1 \ll n_2 \ll n_3$.

$G = V_1 \oplus V_2 \oplus V_3$, with $\dim_{\mathbb{F}_2} V_i = n_i$. (View V_i as a subspace of G in the usual way.)

Let $S_i = V_i \setminus \{0\}$ and $S = \cup_{i=1}^{3} S_i$.

For any hyperplane H of G, $|H \cap S_i| = 2^{n_i} - 1$ or $2^{n_i - 1} - 1$.

If the n_i are chosen properly, say with n_3 very large, n_2 moderate and n_1 small, we can deduce $|H \cap S_i|$, $i = 1, 2, 3$ from $|H \cap S|$. In this prototype, we can see that $\sigma_i = 0$, so we do not have a working construction yet, but we can now try to fix things up.

Quadratic forms over $\mathbb{F}_2$

We will use quadratic forms over $\mathbb{F}_2$. Let $V = \mathbb{F}_2^d$, where $d = 2e + 1$, $e \geq 1$. On V we take coordinates x_1,. . . , x_d quadratic form $q(x_1, \ldots, x_d) = x_d^2 + \sum_{i=1}^{e} x_i x_{e+i}$. Let Q be the set of zeros of q in $V \setminus \{0\}$. The bilinear form $b(v, v') = q(v + v') - q(v) - q(v')$ associated with q has a 1-dimensional radical $\langle p \rangle$, where $p = (0, \ldots, 0, 1)$, called the nucleus of q. Note that $q(p) = 1$.

First we shall consider $\sigma_Q = \sum_{v \in Q} v$.

Lemma

If $d = 3$, then $\sigma_Q = (0, 0, 1)$ and for $d \geq 5$ we have $\sigma_Q = 0$.

$\square$

By the Lemma, if $d \geq 5$ and we set $S' = Q \cup \{p\}$, then $\sigma_{S'} := \sigma_Q + p = p \neq 0$.

Lemma

Suppose $G = V_1 \oplus V_2 \oplus \cdots \oplus V_k$, *with* $\dim V_i = n_i$. *Suppose* $S_i \subset V_i \setminus \{0\}$ *and* $S = \cup_{i=1}^{k} S_i$. *Let* $N_i = \{|H \cap S_i| \mid H \text{ a hyperplane of } G\}$ *and* $\epsilon_i = \min\{|a - b| \mid a, b \in N_i, a \neq b\}$. *Suppose*

$$\epsilon_i \geq 2^{n_{i-1}+2} \quad \text{for all } i = 2, \ldots, k.$$

Then $|H \cap S|$ *determines* $|H \cap S_i|$ *for all* $i = 1, \ldots, k$.

We can choose odd n_i, $S_i = Q_i \cup \{p_i\} \subset V_i$ as above. We can check that $\epsilon_i \geq 2^{\frac{n_i-3}{2}}$, so n_i can be chosen so that the hypotheses of the lemma hold. Assuming such a choice, it follows from our discussion that all the σ_i are strongly cospectral in $\mathrm{Cay}(G, S)$, hence also all elements of the subgroup they generate. Since the elements $\sigma_{S_i} = p_i$ are linearly independent, this group has order 2^k.

Overview

Pretty good state transfer, open problems

A quantum walk exhibits pretty good state transfer from vertex a to vertex b if for every $\epsilon > 0$ there exists $t > 0$ with $|U(t)_{b,a}| > 1 - \epsilon$.

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- SC classes can be arbitrarily large. Is the same true of PGST classes? Is it true for PGST classes in vertex-transitive graphs?
- If the graph is periodic then PGST is equivalent to PST (H. Pal).

PGST in Cayley graphs

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- What can we say about G_{pg}?
- What is G_{pg} for the examples in this talk? Cubelike graphs are integral, hence periodic, so PGST=PST.