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# On the 1-Cohomology of the Groups $G_2(2^n)$

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Abstract. We compute the first cohomology group with coefficients in a simple module for the algebraic group $G_2(\overline{\mathbf{F}}_2)$ and related finite groups.

## INTRODUCTION

The principal result of this paper is the calculation of the cohomology group $H^1(G, M)$, where $G$ is either the finite group $G_2(2^n)$ or the algebraic group $G_2(\overline{\mathbf{F}}_2)$, and $M$ is a simple module. We shall consider only rational modules for the algebraic group. This cohomology group turns out to be trivial for most simple modules. As in [6] and [2] we prove this by exploiting an interaction of $\mathrm{Ext}^1$ with tensor products, cf. Lemma 3.1 below.

## §1. PRELIMINARIES AND STATEMENT OF THE MAIN RESULT

Let $F = \overline{\mathbf{F}}_2$ be an algebraic closure of the field $\mathbf{F}_2$ of two elements. We shall identify a finite extension of $\mathbf{F}_2$ with its isomorphic image in $F$. Let $\Phi$ be a root system of type $G_2$, with fundamental roots $\alpha_1$ (long) and $\alpha_2$ (short) and corresponding fundamental dominant weights $\nu_1$ and $\nu_2$. Let $\mathbf{G}$ denote the algebraic group $G_2(F)$ and let $G$ be the finite subgroup $G_2(2^n)$ of $\mathbf{F}_{2^n}$-rational points. The simple $\mathbf{G}$-modules are parametrized by dominant weights $\nu = a\nu_1 + b\nu_2$, where $a$ and $b$ are non-negative integers, the simple module being denoted by $L(\nu)$. The corresponding Weyl module, having $L(\nu)$ is its unique simple quotient, is denoted by $V(\nu)$. We shall now describe some of these modules more explicitly. The group $\mathbf{G}$ is the automorphism group of the (split) Cayley algebra $\mathfrak{C}$ over $F$, an 8-dimensional alternative algebra carrying an invariant linear form (trace) and a nonsingular quadratic form (norm). The Weyl module $V(\nu_2)$ is then the 7-dimensional space of elements of trace zero and $L(\nu_2)$ is the quotient of this by the scalar multiples of $1_{\mathfrak{C}}$. The module $V(\nu_1)$ is the adjoint module for $\mathbf{G}$, namely, the 14-dimensional simple Lie algebra of derivations of $\mathfrak{C}$, with $\mathbf{G} = \mathrm{Aut}\,\mathfrak{C}$ acting by conjugation (in $\mathrm{End}_F\,\mathfrak{C}$). Being a simple module it is equal to $L(\nu_1)$. The module $V(\nu_1 + \nu_2)$ is the first Steinberg module. It is also simple and has dimension $2^{|\Phi^+|} = 64$. It will be convenient to denote the simple modules $L(\nu_2)$, $L(\nu_1)$ and $L(\nu_1 + \nu_2)$ by $C$, $L$ and $S$ respectively. They remain simple upon restriction to the finite group $G$.

For any $F$-vector space $V$ and natural number $i$, we denote by $V_i$ the $i$-th Frobenius twist of $V$, which is the same abelian group but with $\gamma \in F$ acting on $V_i$ as $\gamma^{2^{-i}}$ acts on $V$. For

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a finite set $I$ of natural numbers we shall write $V_I$ for the tensor product $\bigotimes_{i\in I} V_i$. Then by Steinberg's Tensor Product Theorem, the simple $\mathbf{G}$-modules are the modules

$$C_I \otimes L_J \otimes S_K,$$

where $I$, $J$ and $K$ are pairwise disjoint finite subsets (possibly empty) of natural numbers. For any $FG$-module $V$, we have $V_{i+n} \cong V_i$ and the simple $FG$-modules are the $2^{2n}$ modules $C_I \otimes L_J \otimes S_K$, where now $(I, J, K)$ is an ordered triple of disjoint subsets of $N = \{0, 1, \ldots, n-1\}$, which we shall simply call a *triple* from now on. The module $S_N$ is the Steinberg module for $G$; it is projective. Each simple module is isomorphic to its dual and it will be unnecessary to make any distinction. It is clear that $\mathrm{Gal}(\mathbf{F}_{2^n}/\mathbf{F}_2)$ acts on the set of triples; the automorphism $\gamma \mapsto \gamma^{2^i}$ acts by adding $i$ to each element of $N$ and taking the remainder modulo $n$.

THEOREM. *Suppose*[1] $n > 6$, *and let* $(I, J, K)$ *be a triple as above. Then*

$$H^1(G, C_I \otimes L_J \otimes S_K) \cong F$$

*if* $(I, J, K)$ *is Galois conjugate to* $(\{0\}, \emptyset, \emptyset)$ *or* $(\{0\}, \{1\}, \emptyset)$ *and is zero otherwise. The same result holds for* $\mathbf{G}$ *if* $I$, $J$ *and* $K$ *are allowed to be any three disjoint finite subsets of the natural numbers and conjugation is by* $\mathrm{Gal}(F/\mathbf{F}_2) \cong \mathbf{Z}$.

The result for $\mathbf{G}$ follows from the result for $G$ by means of Theorem 7.1 of [3] (See also [1] Proposition 2.7), which states that the restriction map

$$\mathrm{Ext}^1_{\mathbf{G}}(L(\nu), L(\mu)) \to \mathrm{Ext}^1_{FG}(L(\nu), L(\mu)) \tag{1.1}$$

is injective if the coefficients in $\nu$ and $\mu$ of $\nu_1$ and $\nu_2$ are less than $2^n$, and that it is an isomorphism if $n$ is sufficiently large with respect to these coefficients. We shall also make use of this result in the opposite direction to apply results about $\mathbf{G}$ in the course of the proof for $G$.

Until further notice, we shall assume $n > 2$.

## §2. TENSOR PRODUCTS OF SIMPLE MODULES

Let us denote the (Brauer) character of the simple $FG$-module $C_I \otimes L_J \otimes S_K$ by $\chi_I \lambda_J \sigma_K$.

LEMMA 2.1.

(a) $\chi^2 = 2\lambda + \chi_1 + 2$.
(b) $\chi\lambda = \sigma + \chi_1 + 2\chi + 2$.
(c) $\lambda^2 = 2\chi_{\{0,1\}} + \lambda_1 + 4\lambda + 4\chi_1 + 4\chi + 6$.
(d) $\chi\sigma = \chi_1\lambda + 3\chi_{\{0,1\}} + 2\lambda_1 + 6\lambda + 6\chi_1 + 6\chi + 8$.
(e) $\lambda\sigma = \chi\lambda_1 + 3\chi_1\lambda + 2\sigma + 4\chi_{\{0,1\}} + 4\lambda_1 + 6\lambda + 2\chi_2 + 10\chi_1 + 10\chi + 16$.
(f) $\sigma^2 = 2\chi_1\sigma + 2\lambda_{\{0,1\}} + 4\chi\lambda_1 + 8\chi_1\lambda + \sigma_1 + 4\sigma + 2\chi_{\{0,2\}} + 16\chi_{\{0,1\}} + 16\lambda_1 + 20\lambda + 8\chi_2 + 32\chi_1 + 28\chi + 48$.

PROOF: Parts (a), (b) and (c) are calculated by decomposing tensor products of Weyl modules. The other parts are then readily derived from these.

For use in inductive arguments we define the *mass* of the simple $FG$-module $C_I \otimes L_J \otimes S_K$ (or of its character or of the triple) to be $2|I|+3|J|+5|K|$. The mass of an arbitrary module is then defined to be the maximum of the masses of its composition factors. The mass of a module is clearly preserved under taking duals and Galois conjugates. The usefulness of this invariant lies in the following property, which follows at once from Lemma 2.1.

[1] The hypothesis $n > 6$ is used only at the beginning of §4; everywhere else in this paper $n > 2$ is strong enough.

LEMMA 2.2. *Let $(I, J, K)$ and $(P, Q, R)$ be triples. Then*

$$\mathit{mass}(\chi_I\lambda_J\sigma_K . \chi_P\lambda_Q\sigma_R) \le 2(|I| + |P|) + 3(|J| + |Q|) + 5(|K| + |R|),$$

*with equality if and only if $(I \cup K) \cap (P \cup R) = \emptyset = (J \cup K) \cap (Q \cup R)$.*

LEMMA 2.3. *Let $(I, J, K)$ be a triple and $i \in N$.*

(a) *$\chi_i(\chi_I\lambda_J\sigma_K)$ has no constituent of the form $\sigma_T$ with $|T| > |K| + 1$ if $i \in I \cup J$ and none with $|T| > |K|$ if $i \notin I \cup J$*
(b) *$\lambda_i(\chi_I\lambda_J\sigma_K)$ has no constituent of the form $\sigma_T$ with $|T| > |K| + 1$ if $i \in I \cup J$ and none with $|T| > |K|$ if $i \notin I \cup J$.*
(c) *$\sigma_i(\chi_I\lambda_J\sigma_K)$ has no constituent of the form $\sigma_T$ with $|T| > |K| + 1$.*

PROOF: Parts (a) and (b) will be proved together by induction on the mass of $(I, J, K)$. The statements are obvious if $i \notin I \cup J \cup K$, in particular for mass zero. We assume (a) and (b) hold for triples of smaller mass. First we prove (a). If $i \in I$, then by Lemma 2.1(a),

$$\begin{aligned}\chi_i\chi_I\lambda_J\sigma_K &= (2 + 2\lambda + \chi_{i+1})\chi_{I\setminus\{i\}}\lambda_J\sigma_K \\ &= 2\chi_{I\setminus\{i\}}\lambda_J\sigma_K + 2\chi_{I\setminus\{i\}}\lambda_{J\cup\{i\}}\sigma_K + \chi_{i+1}(\chi_{I\setminus\{i\}}\lambda_J\sigma_K).\end{aligned}$$

The first two terms are multiples of irreducible characters not of the form $\sigma_T$ for $|T| > |K|+1$, and induction applies to the last term. A similar argument using Lemma 2.1(b) works if $i \in J$. If $i \in K$, then by Lemma 2.1(d) we have

$$\begin{aligned}\chi_i(\chi_I\lambda_J\sigma_K) &= (\chi_{i+1}\lambda_i + 3\chi_{\{i,i+1\}} + 2\lambda_{i+1} + 6\lambda_i + 6\chi_{i+1} + 6\chi_i + 8)\chi_I\lambda_J\sigma_{K\setminus\{i\}} \\ &= \chi_{i+1}(\chi_I\lambda_{J\cup\{i\}}\sigma_{K\setminus\{i\}}) + 3\chi_{i+1}(\chi_{I\cup\{i\}}\lambda_J\sigma_{K\setminus\{i\}}) + 2\lambda_{i+1}(\chi_I\lambda_J\sigma_{K\setminus\{i\}}) + 6\chi_I\lambda_{J\cup\{i\}}\sigma_{K\setminus\{i\}} \\ &\quad + 6\chi_{I\cup\{i\}}\lambda_J\sigma_{K\setminus\{i\}} + 6\chi_{i+1}(\chi_I\lambda_J\sigma_{K\setminus\{i\}}) + 8\chi_I\lambda_J\sigma_{K\setminus\{i\}},\end{aligned}$$

and each term is either irreducible (and not of the form $\sigma_T$, $|T| > |K| + 1$) or else one to which we can apply part (a) or part (b) of the inductive hypothesis, with $K$ replaced by $K \setminus \{i\}$.

The argument to prove (b) is similar, using Lemma 2.1(b), (c) and (e). This proves (a) and (b). We omit the details of the proof of (c), which proceeds similarly by induction on mass, making use of parts (a) and (b).

The next lemma concerns tensoring with the Steinberg module. Here, $P(M)$ denotes the projective cover of the module $M$.

LEMMA 2.4.

(a) $C_i \otimes S_N \cong P(L_i \otimes S_{N\setminus\{i\}})$.
(b) $L_i \otimes S_N \cong P(C_i \otimes S_{N\setminus\{i\}}) \oplus S_N \oplus S_N$.

PROOF: We shall prove (b) and leave the simpler case (a). Since $S_N$ is both projective and simple, the multiplicity of $P(C_I \otimes L_J \otimes S_K)$ as a direct summand of $L_i \otimes S_N$ is simply the multiplicity of $\sigma_N$ as a constituent of $\lambda_i(\chi_I\lambda_J\sigma_K)$. If $i \in J$, then $\sigma_N$ is not a constituent, for by Lemma 2.1 (c) and Lemma 2.2,

$$\mathrm{mass}(\lambda_i^2(\chi_I\lambda_{J\setminus\{i\}}\sigma_K)) = 4 + 2|I| + 3(|J| - 1) + 5|K| < 5|N|.$$

If $i \in I$, then by Lemma 2.1(b) and Lemma 2.2,

$$\mathrm{mass}(\chi_i\lambda_i(\chi_{I\setminus\{i\}}\lambda_J\sigma_K)) = 5 + 2(|I| - 1) + 3|J| + 5|K|,$$

and this will equal $5|N|$ if and only if $I = \{i\}$, $J = \emptyset$ and $K = N \setminus \{i\}$. In this case the multiplicity in question will be 1. If $i \in K$, then by Lemma 2.1(e) and Lemma 2.2,

$$\text{mass}(\lambda_i\sigma_i(\chi_I\lambda_J\sigma_{K\setminus\{i\}})) = 5 + 2|I| + 3|J| + 5(|K| - 1).$$

The mass will be $5|N|$ if and only if $I = J = \emptyset$ and $K = N$. The terms of mass 5 in $\lambda_i\sigma_i$ are $2\sigma_i$, $\chi_i\lambda_{i+1}$ and $3\chi_{i+1}\lambda_i$. It is easily seen that $\chi_i\lambda_{i+1}\sigma_{N\setminus\{i\}}$ and $\chi_{i+1}\lambda_i\sigma_{N\setminus\{i\}}$ have no constituent $\sigma_N$, so it follows that the multiplicity of $\sigma_N$ in $\lambda_i\sigma_N$ is 2. The lemma is proved.

LEMMA 2.5.

(a) *$C \otimes S$ has simple head and socle, isomorphic to $L$.*
(b) $\mathrm{Hom}_{FG}(L \otimes S, C) \cong F$
(c) *$C \otimes L \cong S \oplus M$, where $M$ has simple head and socle, isomorphic to $C$.*

PROOF: The first two parts are corollaries of Lemma 2.4. To prove (c), we note that by (b), $C \otimes L$ has both a submodule and a quotient isomorphic to $S$. But since $S$ appears only once as a composition factor of $C \otimes L$, by Lemma 2.1, it must be a direct summand. A complementary summand $M$ will then be a self-dual module with composition factors $C$ (twice), $F$ (twice) and $C_1$. Since $\mathrm{Hom}_{FG}(C \otimes L, F) = 0$ and $\mathrm{Hom}_{FG}(C \otimes L, C_1) = 0$, the result follows.

LEMMA 2.6. *Let $i \in I \subseteq N$. Then $C_i \otimes C_I$ has at most two simple modules in each Loewy layer. Furthermore, if $I = \{i, i+1, \dots, i+r\}$, then the Loewy layers are*

$$C_{I\setminus\{i\}} \oplus (C_{I\setminus\{i\}} \otimes L_i), C_{I\setminus\{i,i+1\}} \oplus (C_{I\setminus\{i,i+1\}} \otimes L_{i+1}), \dots,$$
$$F \oplus L_{i+r}, C_{i+r+1}, F \oplus L_{i+r}, \dots,$$
$$C_{I\setminus\{i\}} \oplus (C_{I\setminus\{i\}} \otimes L_i).$$

PROOF: It is enough to prove the result when $I$ has the form $\{i, i+1, \dots, i+r\}$ because for any $I$, the module $C_i \otimes C_I$ is a tensor factor of $C_i \otimes C_N$. We argue by induction on $r$. For the case $r = 0$ we observe that $\mathrm{Hom}_{FG}(C \otimes C, F) \cong F$, that $\mathrm{Hom}_{FG}(C \otimes C, C_1) = 0$ and that by Lemma 2.5(c) $\mathrm{Hom}_{FG}(C \otimes C, L) \cong F$. The result then follows from Lemma 2.1(a) and the fact that $C \otimes C$ is self-dual. Let $I' = \{i+1, \dots i+r\}$. Then $C_i \otimes C_I = (C_i \otimes C_i) \otimes C_{I'}$ has a (descending) filtration with factors in the order

$$C_{I'} \oplus (C_{I'} \otimes L_i), \qquad C_{i+1} \otimes C_{I'}, \qquad C_{I'} \oplus (C_{I'} \otimes L_i).$$

Induction applies to the middle factor which therefore has head isomorphic to $C_{I'\setminus\{i+1\}} \oplus (C_{I'\setminus\{i+1\}} \otimes L_{i+1})$, and so the proof is completed by the fact that $C_i \otimes C_I$ is self dual and the equation

$$\mathrm{Hom}_{FG}(C_i \otimes C_I, C_{I'\setminus\{i+1\}} \oplus (C_{I'\setminus\{i+1\}} \otimes L_{i+1}))$$
$$\cong \mathrm{Hom}_{FG}(C_I, C_{I\setminus\{i+1\}} \oplus (C_{I\setminus\{i+1\}} \otimes L_{i+1})) = 0.$$

## §3. THE MAIN CASE

The aim of this section is to show that $H^1(G, C_I \otimes L_J \otimes S_K) = 0$ unless $I \cup J$ and $K$ are both very small, cutting the proof of the main theorem down to a manageable size.

The first lemma appears in [6] and will provide the basis of our inductive arguments. For two $FG$-modules $A$ and $B$, we let $d(A, B) = \dim_F \mathrm{Ext}^1_{FG}(A, B)$ and if they are simple, we let $X(A, B)$ be an $FG$-module with a unique maximal submodule isomorphic to a direct sum of $d(A, B)$ copies of $B$ and the quotient by this submodule isomorphic to $A$. It is determined up to isomorphism by these properties.

LEMMA 3.1. *Let $D$ be any $FG$-module and $E$ a simple quotient of $B \otimes D$. Then*

$$\mathrm{Hom}_{FG}(X(A, B) \otimes D, E) = 0 \quad \textit{implies} \quad d(A, B) \leq d(A \otimes D, E).$$

We shall apply this where $A \otimes D$ is simple. Most frequently, we shall check the premise $\mathrm{Hom}_{FG}(X(A,B) \otimes D, E) = 0$ simply by checking that $A$ is not a composition factor of $D^* \otimes E$.

LEMMA 3.2. *Let $(I, J, K)$ be a triple and $T \subseteq N$. If $|T| > |K| + 2$, then $\mathrm{Ext}^1_{FG}(S_T, C_I \otimes L_J \otimes S_K) = 0$.*

PROOF: Since the result is true when $T = N$, we shall argue by downward induction on $|T|$. We may also assume that $K \subseteq T$, since $S$ is self-dual so

$$\mathrm{Ext}^1_{FG}(S_T, C_I \otimes L_J \otimes S_K) \cong \mathrm{Ext}^1_{FG}(S_{T\cup K}, C_I \otimes L_J \otimes S_{T\cap K}).$$

Suppose first that $I \cup J \not\subseteq T$. Pick $r \in (I \cup J) \setminus T$. If $r \in I$, we shall prove

$$d(S_T, C_I \otimes L_J \otimes S_K) \leq d(S_{T\cup\{r\}}, C_{I\setminus\{r\}} \otimes L_{J\cup\{r\}} \otimes S_K) \tag{3.2.1}$$

and if $r \in J$ we shall prove

$$d(S_T, C_I \otimes L_J \otimes S_K) \leq d(S_{T\cup\{r\}}, C_{I\cup\{r\}} \otimes L_{J\setminus\{r\}} \otimes S_K). \tag{3.2.2}$$

The result will then follow by induction. Suppose $r \in I$. then by Lemma 2.5(a), $(C_I \otimes L_J \otimes S_K) \otimes S_r$ has a quotient isomorphic to $C_{I\setminus\{r\}} \otimes L_{J\cup\{r\}} \otimes S_K$. Therefore (3.2.1) will follow from Lemma 3.1 if we prove

$$\mathrm{Hom}_{FG}(X(S_T, C_I \otimes L_J \otimes S_K), S_r \otimes (C_{I\setminus\{r\}} \otimes L_{J\cup\{r\}} \otimes S_K)) = 0.$$

This is immediate from Lemma 2.3 and the hypothesis $|T| > |K| + 2$. The proof of (3.2.2) if $r \in J$ is similar. We may therefore assume that $I \cup J \cup K \subseteq T$. For $r \in N \setminus T$ we shall prove

$$d(S_T, C_I \otimes L_J \otimes S_K) \leq d(S_{T\cup\{r\}}, C_I \otimes L_J \otimes S_{K\cup\{r\}}), \tag{3.2.3}$$

and apply induction. To use Lemma 3.1, we must show

$$\mathrm{Hom}_{FG}(X(S_T, C_I \otimes L_J \otimes S_K), S_r \otimes (C_I \otimes L_J \otimes S_{K\cup\{r\}})) = 0,$$

which is true because by Lemma 2.3, the right hand module has no composition factor $S_T$. The lemma is proved.

LEMMA 3.3. *Let $I$ and $J$ be disjoint subsets of $N$ such that $|I \cup J| > 2$. Then*

$$\mathrm{Ext}^1_{FG}(S_T, C_J \otimes L_J) = 0.$$

PROOF: If $|T| > 2$ then the preceding lemma applies. This is so if $T \supseteq I \cup J$. Therefore, we may assume there exists $r \in N \setminus T$. We shall find disjoint subsets $I'$ and $J'$ such that $I' \cup J' = I \cup J$ and

$$d(S_T, C_I \otimes L_J) \leq d(S_{T\cup\{r\}}, C_{I'} \otimes L_{J'}). \tag{3.3.1}$$

Iteration of this process will bring us back to the case $|T| > 2$.

Case **A.** $r \in I$. Here, $S_r \otimes (C_I \otimes L_J)$ has a simple quotient $C_{I\setminus\{r\}} \otimes L_{J\cup\{r\}}$, and by Lemma 3.1, (3.3.1) will follow from

$$\mathrm{Hom}_{FG}(X(S_T, C_I \otimes L_J), S_r \otimes (C_{I\setminus\{r\}} \otimes L_{J\cup\{r\}})) = 0. \tag{3.3.2}$$

In order to establish (3.3.2), we consider the factors (ignoring multiplicities) in a filtration of $S_r \otimes (C_{I\setminus\{r\}} \otimes L_{J\cup\{r\}}) = (S_r \otimes L_r) \otimes C_{I\setminus\{r\}} \otimes L_J$ induced by a composition series of $S_r \otimes L_r$. By Lemma 2.1, these are

(1) $C_{r+1} \otimes (C_{I\setminus\{r\}} \otimes L_{J\cup\{r\}})$,
(2) $C_{r+1} \otimes (C_I \otimes L_J)$,

(3) $C_{r+1} \otimes (C_{I\setminus\{r\}} \otimes L_J)$,
(4) $C_{r+2} \otimes (C_{I\setminus\{r\}} \otimes L_J)$,
(5) $L_{r+1} \otimes (C_{I\setminus\{r\}} \otimes L_J)$,
(6) $L_{r+1} \otimes (C_I \otimes L_J)$,

together with simple factors $C_{I\setminus\{r\}} \otimes L_{J\cup\{r\}}$, $C_I \otimes L_J$ and $C_{I\setminus\{r\}} \otimes L_J$. It will suffice to check that there are no (nonzero) homomorphisms from $X(S_T, C_I \otimes L_J)$ into any of these filtration factors. This is clear for the simple ones.

(1). We have

$$\begin{aligned}
&\dim_F \mathrm{Hom}_{FG}(X(S_T, C_I \otimes L_J), C_{r+1} \otimes (C_{I\setminus\{r\}} \otimes L_{J\cup\{r\}}))\\
&\quad = \dim_F \mathrm{Hom}_{FG}(X(S_T, C_I \otimes L_J) \otimes C_{r+1}, C_{I\setminus\{r\}} \otimes L_{J\cup\{r\}})\\
&\qquad \leq \dim_F \mathrm{Hom}_{FG}(S_T \otimes C_{r+1}, C_{I\setminus\{r\}} \otimes L_{J\cup\{r\}})\\
&\qquad\qquad + \dim_F \mathrm{Hom}_{FG}((C_I \otimes L_J) \otimes C_{r+1}, C_{I\setminus\{r\}} \otimes L_{J\cup\{r\}}).
\end{aligned}$$

Now $S_T \otimes C_{r+1}$ has a simple head by Lemma 2.4, and it is not isomorphic to $C_{I\setminus\{r\}} \otimes L_{J\cup\{r\}}$, so the first term on the right of the inequality is zero. If $r+1 \notin I \cup J$ then it is obvious that the second term is also zero. If $r+1 \in I$, then without any difficulty, it can be seen using Lemma 2.2 that

$$\chi_I \lambda_J \chi_{r+1} = \chi_{r+1}^2 \chi_{I\setminus\{r+1\}} \lambda_J = (2\lambda_{r+1} + \chi_{r+2} + 2)\chi_{I\setminus\{r+1\}} \lambda_J$$

has no constituent $\chi_{I\setminus\{r\}} \lambda_{J\cup\{r\}}$. The same kind of argument works if instead $r+1 \in J$.

(2) We have

$$\begin{aligned}
&\dim_F \mathrm{Hom}_{FG}(X(S_T, C_I \otimes L_J), C_{r+1} \otimes (C_I \otimes L_J))\\
&\quad = \dim_F \mathrm{Hom}_{FG}(X(S_T, C_I \otimes L_J) \otimes C_{r+1}, C_I \otimes L_J)\\
&\qquad \leq \dim_F \mathrm{Hom}_{FG}(S_T \otimes C_{r+1}, C_I \otimes L_J)\\
&\qquad\qquad + \dim_F \mathrm{Hom}_{FG}((C_I \otimes L_J) \otimes C_{r+1}, C_I \otimes L_J).
\end{aligned}$$

Now $\mathrm{Hom}_{FG}(S_T \otimes C_{r+1}, C_I \otimes L_J) = 0$, since by Lemma 2.4 $S_T \otimes C_{r+1}$ is either simple or has head isomorphic to $L_{r+1} \otimes S_{T\setminus\{r+1\}}$, which, since $|I \cup J| > 1$, is different from $C_I \otimes L_J$. The last term of the inequality above is clearly zero if $r+1 \notin I \cup J$. If $r+1 \in I$, then using Lemma 2.2, it is straightforward to verify that $\chi_I \lambda_J$ is not a constituent of

$$\chi_I \lambda_J \chi_{r+1} = \chi_{r+1}^2 \chi_{I\setminus\{r+1\}} \lambda_J = (2\lambda_{r+1} + \chi_{r+2} + 2)\chi_{I\setminus\{r+1\}} \lambda_J.$$

Likewise, if $r+1 \in J$, then $\chi_{r+1}\chi_I\lambda_J$ has no constituent $\chi_I\lambda_J$.

The proof that there are no nonzero homomorphisms from $X(S_T, C_I \otimes L_J)$ into the factors (3) and (4) involves similar character calculations; it should be pointed out perhaps that the standing hypothesis $n > 2$ is used in the proof for these two parts.

(5). It is safe to assume $r+1 \in I \cup J$. If $r+1 \in I$, then $(L_{r+1} \otimes C_{r+1}) \otimes C_{I\setminus\{r,r+1\}} \otimes L_J$ has a filtration with factors (ignoring multiplicities) $S_{r+1} \otimes C_{I\setminus\{r,r+1\}} \otimes L_J$, $C_{I\setminus\{r,r+1\}} \otimes L_J$, $C_{I\setminus\{r,\}} \otimes L_J$ and $C_{r+2} \otimes (C_{I\setminus\{r,r+1\}} \otimes L_J)$. Only the last may have a composition factor $S_T$ (since $|I \cup J| > 2$) and from Lemma 2.3 it follows that this can happen only if $T = \{r+2\}$. Moreover, by Lemma 2.3, $\mathrm{Hom}_{FG}(S_{r+2}, C_{r+2} \otimes (C_{I\setminus\{r,r+1\}} \otimes L_J))$. is zero unless $I = \{r, r+1\}$ and $J = \{r+2\}$. However, in this case we claim that the original space of maps, $\mathrm{Hom}_{FG}(X(S_T, C_I \otimes L_J), L_{r+1} \otimes (C_{I\setminus\{r\}} \otimes L_J))$, is zero. Indeed, by direct computation, we see that $C_{\{r,r+1\}} \otimes L_{r+2}$ is not a composition factor of $L_{r+1} \otimes C_{r+1} \otimes L_{r+2}$ and that $\mathrm{Hom}_{FG}(S_{r+2}, L_{r+1} \otimes C_{r+1} \otimes L_{r+2}) = 0$.

If $r+1 \in J$, then $L_{r+1} \otimes (C_{I\setminus\{r\}} \otimes L_J) = (L_{r+1} \otimes L_{r+1}) \otimes (C_{I\setminus\{r\}} \otimes L_{J\setminus\{r+1\}})$ has a filtration with factors $C_{I\setminus\{r\}} \otimes L_{J\setminus\{r+1\}}$, $C_{I\cup\{r+1\}\setminus\{r\}} \otimes L_{J\setminus\{r+1\}}$, $C_{I\setminus\{r\}} \otimes L_J$, (i) $C_{r+2} \otimes (C_{I\cup\{r\}\setminus\{r\}} \otimes L_{J\setminus\{r+1\}})$, (ii) $C_{r+2} \otimes (C_{I\setminus\{r\}} \otimes L_{J\setminus\{r+1\}})$ and (iii) $L_{r+2} \otimes (C_{I\setminus\{r\}} \otimes L_{J\setminus\{r+1\}})$.

The first three are simple and not of the form $S_T$ since $|I \cup J| > 2$. As for the others, it follows from Lemma 2.3 that they cannot have $S_T$ as a composition factor unless $T \subseteq \{r+2\}$, so we assume this. Moreover, by considering masses, we see that $C_I \otimes L_J$ is not a composition factor, so maps of $X(S_T, C_I \otimes L_J)$ into (i),(ii) and (iii) must annihilate $C_I \otimes L_J$. Now by Lemma 2.5, $S_T \otimes C_{r+2}$ will have a simple head isomorphic to either $C_{r+2}$ or $L_{r+2}$. This implies that there are no nonzero maps from $X(S_T, C_I \otimes L_J)$ to (i) and that there are none to (ii) unless either $(T, I, J) = (\emptyset, \{r, r+2\}, \{r+1\})$ or $(T, I, J) = (\{r+2\}, \{r\}, \{r+1, r+2\})$. In both these cases it is easy to check that there are no maps from $X(S_T, C_I \otimes L_J)$ into the original module $L_{r+1} \otimes (C_{I\setminus\{r\}} \otimes L_J)$. In (iii), if $T = \emptyset$, there will be no maps from $X(S_T, C_I \otimes L_J)$ to (iii) unless $I = \{r\}$ and $J = \{r+1, r+2\}$, in which case there are no maps into the original module $L_{r+1} \otimes (C_{I\setminus\{r\}} \otimes L_J)$. Suppose finally that $T = \{r+2\}$. If $r+2 \in I$, we have

$$\begin{aligned}&\mathrm{Hom}_{FG}(S_{r+2}, L_{r+2} \otimes C_{I\setminus\{r\}} \otimes L_{J\setminus\{r+1\}})\\ &\qquad\qquad \cong \mathrm{Hom}_{FG}(S_{r+2} \otimes C_{r+2}, C_{I\setminus\{r,r+2\}} \otimes L_{J\cup\{r+2\}\setminus\{r+1\}})\end{aligned}$$

which will be zero unless $I = \{r, r+2\}$ and $J = \{r+1\}$, in which case there will be no maps from $X(S_{r+2}, C_{\{r,r+2\}} \otimes L_{\{r+1\}})$ into the original module $L_{r+1} \otimes (C_{r+2} \otimes L_{r+1})$. If $r+2 \in J$, we note that the only composition factors of $S_{r+2} \otimes L_{r+2}$ which have $L_{r+2}$ as a tensor factor are $L_{r+2}$ and $C_{r+3} \otimes L_{r+2}$. Therefore, there will be nonzero maps of $S_{r+2} \otimes L_{r+2}$ to $C_{I\setminus\{r\}} \otimes L_{J\setminus\{r+1\}}$ only if $(I, J) = (\{r\}, \{r+1, r+2\})$ or $(\{r, r+3\}, \{r+1, r+2\})$. In both cases one can check there are no maps from $X(S_T, C_I \otimes L_J)$ into the original module $L_{r+1} \otimes (C_{r+2} \otimes L_{r+1})$.

(6). This can be dealt with by calculations like those in (5).

This completes the proof of Case A.

Case **B.** $r \in J$. The proof proceeds by arguments and calculations, of which we omit the details, which are entirely analogous to those of Case A.

## §4. Completion of the proof

The results of the last section leave the groups $H^1(G, C_I \otimes L_J \otimes S_K)$, where $|I \cup J| \leq 2$ and $|K| \leq 2$, to be determined. This is the only place where we shall need $n > 6$, the sole reason being to apply Theorem 7.1 of [3], which gives sufficient conditions on $n$ and the weights $\nu$ and $\mu$ for the restriction map (1.1) to be an isomorphism. From this and the structure of the Weyl modules $V(\nu_1)$, $V(\nu_2)$, $V(\nu_1+\nu_2)$ and $V(2\nu_2)$ (found by elementary computations) we obtain

$$\begin{aligned}&\mathrm{Ext}^1_{FG}(F, L) = \mathrm{Ext}^1_{FG}(F, S) = \mathrm{Ext}^1_{FG}(S, C)\\ &= \mathrm{Ext}^1_{FG}(S, L) = \mathrm{Ext}^1_{FG}(C, L) = \mathrm{Ext}^1_{FG}(C_1, C) = 0,\\ &\mathrm{Ext}^1_{FG}(F, C) \cong F \cong \mathrm{Ext}^1_{FG}(C_1, L)\end{aligned} \tag{4.0.1}$$

as long as[2] $n > 6$. We now compute the remaining cohomology groups $\mathrm{Ext}^1_{FG}(S_K, C_I \otimes L_J)$ in the separate cases $|I \cup J| = 0,1$ and 2, back under the weaker assumption $n > 2$.

**4.1.** $I \cup J = \emptyset$. Since $G$ is simple, we have $\mathrm{Ext}^1_{FG}(F, F) = 0$, and $\mathrm{Ext}^1_{FG}(F, S_k)$ is in the list (4.0.1). The case $|K| = 2$ is covered by the following.

[2] This requirement is probably unnecessarily strong. For example we know from [5] that $H^1(G, C) \cong F$ for all $n$.

LEMMA 4.1. *$\mathrm{Ext}^1_{FG}(S_P, S_Q) = 0$ if $|P| \geq |Q| + 2$.*

PROOF: We may assume $P \supseteq Q$. We argue by backwards induction on $|P|$, starting at the injective module $S_N$. By means of Lemma 3.1, we shall show

$$d(S_P, S_Q) \leq d(S_{P\cup\{r\}}, S_{Q\cup\{r\}}),$$

for $r \in N \setminus P$, which will supply the inductive step. By Lemma 2.1, $\mathrm{mass}(S_r \otimes S_r) = 7 < 10$, so by Lemma 2.2, $S_P$ is not a composition factor of $S_r \otimes S_{Q\cup\{r\}}$. Thus, Lemma 3.1 applies.

**4.2.** $I \cup J = \{i\}$. The cases where $K = \emptyset$ and $K = \{i\}$ occur in (4.0.1).

If $|K| = 2$, then Lemma 3.1 can be used in conjunction with some character calculations to prove

$$d(S_K, C_i) \leq d(S_{K\cup\{i\}}, L_i) \quad \text{and} \quad d(S_K, L_i) \leq d(S_{K\cup\{i\}}, C_i).$$

The result then follows from Lemma 3.2.

Suppose then that $K = \{k\}$ for $k \neq i$. First, we shall show with the aid of Lemma 3.1 that

$$d(S_k, C_i) \leq d(S_{\{i,k\}}, L_i) \tag{4.2.1}$$

and

$$d(S_k, L_i) \leq d(S_{\{i,k\}}, C_i). \tag{4.2.2}$$

By Lemma 2.5(a), $L_i$ is a quotient of $S_i \otimes C_i$, so (4.2.1) follows from the fact that $S_k$ is not a composition factor of $S_i \otimes L_i$. Similarly, $C_i$ is a quotient of $S_i \otimes L_i$, and $S_k$ is not a composition factor of $S_i \otimes C_i$, proving (4.2.2). The following lemma shows that the right hand members of (4.2.1) and (4.2.2) are zero, which finishes (4.2).

LEMMA 4.2. *Let $i$ and $k$ be distinct elements of $R \subseteq N$. Then*

$$\mathrm{Ext}^1_{FG}(S_R, S_{R\setminus\{i,k\}} \otimes C_i) = 0 = \mathrm{Ext}^1_{FG}(S_R, S_{R\setminus\{i,k\}} \otimes L_i).$$

PROOF: The lemma will be proved by downward induction once we show that for $t \in N \setminus R$ we have

$$d(S_R, S_{R\setminus\{i,k\}} \otimes C_i) \leq d(S_{R\cup\{t\}}, S_{(R\cup\{t\})\setminus\{i,k\}} \otimes C_i) \tag{4.2.3}$$

and

$$d(S_R, S_{R\setminus\{i,k\}} \otimes L_i) \leq d(S_{R\cup\{t\}}, S_{(R\cup\{t\})\setminus\{i,k\}} \otimes L_i). \tag{4.2.4}$$

As usual, we wish to invoke Lemma 3.1. For (4.2.3) we need check only that $\sigma_R$ is not a constituent of $\sigma_t^2\sigma_{R\setminus\{i,k\}}\chi_i$ and for (4.2.4) that $\sigma_R$ is not a constituent of $\sigma_t^2\sigma_{R\setminus\{i,k\}}\lambda_i$. Both of these are easily verified using Lemma 2.2.

**4.3.** $I \cup J = \{i,j\}$. Let us consider first the case $|K| = 2$. Let $K = \{k,l\}$. Then $S_K$ is not a composition factor of $S_i \otimes M$ for $M \in \{C_j \otimes L_i, C_{\{i,j\}}, L_{\{i,j\}}\}$, by Lemma 2.3. Therefore by Lemmas 3.1 and 3.2 we have $d(S_K, M) \leq d(S_{K\cup[i]}, M') = 0$, where $M' \in \{C_{\{i,j\}}, L_i \otimes C_j, C_i \otimes L_j\}$ is a simple quotient of $S_i \otimes M$. Suppose next that $K$ contains a single element, say $k$. Arguments analogous to those of (4.2) enable one to show

(4.3.1) $d(S_k, C_{\{i,j\}}) \leq d(S_{\{i,k\}}, C_j \otimes L_i) \leq d(S_{\{i,j,k\}}, L_{\{i,j\}})$,
(4.3.2) $d(S_k, C_i \otimes L_j) \leq d(S_{\{j,k\}}, C_{\{i,j\}}) \leq d(S_{\{i,j,k\}}, L_i \otimes C_j)$ and
(4.3.3) $d(S_k, L_{\{i,j\}}) \leq d(S_{\{j,k\}}, C_j \otimes L_i) \leq d(S_{\{i,j,k\}}, C_{\{i,j\}})$.

The right hand terms are all zero by Lemma 3.2.

Suppose now $K = \emptyset$. If $|i-j| > 2$, our standard arguments yield

$$d(F, C_{\{i,j\}}) \leq d(S_i, L_i \otimes C_j) \leq d(S_{\{i,j\}}, L_{\{i,j\}}) \tag{4.3.4}$$

and for $i,j \in R \subseteq N$ and $t \in N \setminus R$,

$$d(S_R, S_{R\setminus\{i,j\}}) \leq d(S_{R\cup\{t\}}, S_{(R\cup\{t\})\setminus\{i,j\}} \otimes L_{\{i,j\}}). \tag{4.3.5}$$

These imply $d(F, C_{\{i,j\}}) = 0$. With appropriate modifications, one obtains chains of inequalities to prove $d(F, L_{\{i,j\}}) = 0$.

If $|i-j| = 1$, then by Galois conjugation, we may assume $\{i,j\} = \{0,1\}$. The groups $\mathrm{Ext}^1_{FG}(C_1, C)$ and $\mathrm{Ext}^1_{FG}(C_1, L)$ appeared in (4.0.1). The standard argument shows

$$\begin{aligned} d(F, C\otimes L_1) &\leq d(S, L_{\{0,1\}}) \leq d(S_{\{0,1\}}, C_1 \otimes L) \\ &\leq d(S_{\{0,1,2\}}, C_1 \otimes K \otimes S_2) \leq \cdots \leq d(S_N, C_1 \otimes L \otimes S_{N\setminus\{0,1\}}) = 0. \end{aligned}$$

In order to prove $\mathrm{Ext}^1_{FG}(L_1, L) = 0$, we wish to show

$$\begin{aligned} d(L_1, L) &\leq d(S, C\otimes L_1) \leq d(S_{\{0,1\}}, C_{\{0,1\}}) \\ &\leq d(S_{\{0,1,2\}}, C_{\{0,1\}} \otimes S_2) \leq \cdots \leq d(S_N, C_{\{0,1\}} \otimes S_{N\setminus\{0,1\}}) = 0. \end{aligned} \tag{4.3.6}$$

The only difficulty arises in proving the first inequality in (4.3.6), so we will discuss this further. Since $d(S, C\otimes L_1) = d(S\otimes L_1, C)$, we can try to apply Lemma 3.1 by showing

$$\mathrm{Hom}_{FG}(X(L_1, L), S\otimes C) \cong \mathrm{Hom}_{FG}(X(L_1, L)\otimes S, C) = 0.$$

By Lemma 2.5(c), it suffices to show $\mathrm{Hom}_{FG}(X(L_1, L), (L\otimes C)\otimes C) = 0$. Consideration of the composition factors of $C\otimes C$ (and the composition factors of their tensor products by $L$) further reduces the problem to proving

$$\mathrm{Hom}_{FG}(X(L_1, L), L\otimes L) = 0. \tag{4.3.7}$$

Now $L = V(\nu_1)$ and it is well known (see for example [4] II.4.19) that $V(\nu_1)\otimes V(\nu_1)$ has a submodule isomorphic to $V(2\nu_1)$. The latter module has a simple head isomorphic to $L_1$ and by elementary computations one sees that it also has $C_1$ as a composition factor. But $\mathrm{Hom}_{FG}(C_1, L\otimes L) = 0$ which implies that $V(2\nu_1)$ has socle length at least 3. Therefore, the unique composition factor of $L\otimes L$ which is isomorphic to $L_1$ is not a composition factor of $\mathrm{soc}^2(L\otimes L)$ and so (4.3.7) is finally proved. Note that we used the injectivity of the map (1.1) for this argument, which requires only $n > 2$ for the modules involved.

Finally we must turn to the case $|i-j| = 2$. Here, we may assume $\{i,j\} = \{0,2\}$. We may also assume $n > 3$ or else we would be back in the case $|i-j| = 1$. The vanishing of $\mathrm{Ext}^1_{FG}(L_2, C)$, $\mathrm{Ext}^1_{FG}(C_2, L)$ and $\mathrm{Ext}^1_{FG}(L_2, L)$ is not difficult to prove by finding chains of inequalities similar to (4.3.4) and (4.3.5). This requires nothing but the usual character calculations, though the assumption $n > 3$ is used in this step. For example, a suitable chain for $\mathrm{Ext}^1_{FG}(L_2, C)$ would be

$$\begin{aligned} d(L_2, C) &\leq d(S\otimes L_2, L) = d(S, L_{\{0,2\}}) \leq d(S_{\{0,2\}}, C_2\otimes L) \\ &\leq d(S_{\{0,1,2\}}, C_2 \otimes L \otimes S_1) \leq \cdots \leq d(S_N, C_2\otimes L\otimes S_{N\setminus\{0,2\}}) = 0. \end{aligned}$$

The only group left to be computed is $H^1(G, C_{\{0,2\}}) \cong \mathrm{Ext}^1_{FG}(C_2, C)$. We shall show $d(C_2, C) \leq d(S, C_2\otimes L)$. It will then be routine to check

$$\begin{aligned} d(S, C_2\otimes L) &\leq d(S_{\{0,1\}}, C_2 \otimes L_\otimes S_1) \\ &\leq d(S_{\{0,1,2\}}, L_{\{0,2\}} \otimes S_1) \leq \cdots \leq d(S_N, L_{\{0,2\}}\otimes S_{N\setminus\{0,2\}}) = 0. \end{aligned}$$

We know by Lemma 2.5(c) that

$$\dim_F \mathrm{Hom}_{FG}(X(C_2, C) \otimes S, L) \leq \dim_F \mathrm{Hom}_{FG}(X(C_2, C), (C \otimes L) \otimes L).$$

Now of all the composition factors of $L \otimes L$ only $C_{\{0,1\}}$ has the property that its tensor product with $C$ has $C_2$ as a compostion factor. But it is clear from Lemma 2.6 that $\mathrm{Hom}_{FG}(X(C_2, C), C \otimes C_{\{0,1\}}) = 0$, hence also $\mathrm{Hom}_{FG}(X(C_2, C) \otimes S, L) = 0$ and the desired inequality follows from Lemma 3.1. This completes the proof for (4.3) and also the proof of the main theorem.

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