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On the Representation Theory of Modular Hecke Algebras*

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The Cartan invariants and decomposition numbers of the endomorphism algebra of the permutation module on the cosets of a Sylow p-subgroup of a finite group of Lie type in characteristic p are computed. © 1992 Academic Press, Inc.

INTRODUCTION

Let G be a finite group with a split BN-pair of characteristic p. We shall be interested in (right) modules over the group algebra over a complete discrete valuation ring $\mathcal{O}$ of characteristic zero, its field of quotients K and its residual field k. We shall assume that k has characteristic p, and that K and k are splitting fields for all subgroups of G. For any $\mathcal{O}$-lattice$L$, we use the notations $L_K := L \otimes_{\mathcal{O}} K$ and $\bar{L} := L \otimes_{\mathcal{O}} k$. Let $Y = \text{ind}_{U,G}(\mathcal{O})$ be the permutation $\mathcal{O}G$-lattice on the right cosets of the Sylow p-subgroup U of G, and $E = \text{End}_{\mathcal{O}G}(Y)$. Then $Y_K = \text{ind}_{U,G}(K)$ and $\bar{Y} = \text{ind}_{U,G}(k)$. Moreover, E is an $\mathcal{O}$-form of $\text{End}_{KG}(Y_K)$ and $\bar{E} = \text{End}_{kG}(\bar{Y})$. In this paper we shall compute the Cartan invariants of $\bar{E}$, and the decomposition numbers of the triple $(E, E_K, \bar{E})$ in terms of the Weyl group associated to G. Some earlier work in this direction has been done by Norton [12], who computed the Cartan matrix and blocks of $\text{End}_{kG}(\text{ind}_{B,G}(k))$, which is an algebra direct summand of $\bar{E}$, and by Carter [4], who described the decomposition numbers in type A_l, again for the subalgebras obtained by replacing the subgroup U by its normaliser B. Recently, Cabanes [2] has determined the extension groups and blocks of E. He has also given a different description of the Cartan invariants. The approach in all of these papers is via the well-known description of the algebra by generators and relations [2, Sect. 2]. We shall instead exploit the fact that $\bar{E}$ is the endomorphism ring of a

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module about which much is known [5, 14, 17]. The operation of $\bar{E}$ on the left of $\bar{Y}$ defines an $(\bar{E}, kG)$-bimodule structure on $\bar{Y}$. It is well known [8, Prop. 6.3] that the restrictions of the functors $\mathrm{Hom}_{kG}(\bar{Y}, -)$ and $(-) \otimes_{\bar{E}} \bar{Y}$ are inverse k-equivalences between the full subcategory add $\bar{Y}$ of finitely generated right kG-modules with indecomposable direct summands isomorphic to those of $\bar{Y}$ and the category of finitely generated projective right $\bar{E}$-modules. (Corresponding equivalences also hold over $\mathcal{O}$ and K.) It is also known [17] that $\bar{E}$ is a Frobenius algebra. It follows that the Cartan invariants are nothing other than the intertwining numbers between the various indecomposable direct summands of $\bar{Y}$. Since these summands are trivial source modules, the calculation of their intertwining numbers is reduced to that of the inner products of the ordinary characters afforded by lifts to characteristic zero of these modules. We shall compute these using an induction formula, proved by N. Tinberg [17] for characters and extended to the Green ring in [14]. There, we computed these numbers for the submodule $\mathrm{ind}_{B,G}(k)$, and observed a duality phenomenon. This case had previously been calculated in [12]. In Section 4, we prove that at least for the summands of the permutation module on the cosets of the subgroup B, this duality is the restriction of the one studied by Curtis [6] and Alvis [1]. This follows from the formula we shall obtain for the decomposition numbers of the triple $(E, E_K, \bar{E})$. This formula is the same as one which appears in work of Stanley [15], and so the decomposition numbers have the combinatorial description given to this formula there. In particular, for type A_l, our calculation of the decomposition numbers explains why the same numbers appear in [15, 4].

1. Preliminaries

We recall some notation and basic facts (cf. [3]); thus we have subgroups B, N, H, U, with $U \in \mathrm{Syl}_p(G)$, $B = UH$, H being abelian of order prime to p, $H = B \cap N \trianglelefteq N$, and a Weyl group

$$W = N/H = \langle w_i \mid i \in R \rangle,$$

where R is the set of fundamental roots corresponding to B in the root system Φ on which W operates. For each subset S of Φ we denote by W_S the subgroup of W generated by the elements w_s, $s \in S$.

For each subset $J \subseteq R$, we have a standard parabolic subgroups N_J of N and $G_J = \langle B, N_J \rangle$ of G such that $G_\varnothing = B$ and $G_R = G$, and a parabolic subgroup $W_J = \langle w_i \mid i \in J \rangle$. It follows from the remarks after [3, p. 60, Corollary 2.6.2] that it is possible to pick coset representatives n_w, $w \in W$ so that each lies in the kernel of every character $\chi: G_J \to k^*$ of every standard parabolic subgroup to which it belongs. We shall fix such a choice throughout. Let w_R be the longest element of W and let U_w be the subgroup $U \cap U^{n_w R^{n_w}}$. If Ψ is a root system with base Δ and Weyl group W_1, we define for $J \subseteq L \subseteq \Delta$,

$$X(\Psi)_J = \{w \in W_1 \mid w^{-1}(J) > 0\},$$

$$Z(\Psi)_{L,J} = \{w \in W_1 \mid w^{-1}(J) > 0,\ w^{-1}(L \backslash J) < 0\}.$$

The set $X(\Psi)_J$ is a set of right coset representatives of W_{1J} in W_1 and the set $X(\Psi)_J \cap X(\Psi)_K^{-1}$ is a set of (W_{1J}, W_{1K}) double coset representatives in W_1 [3, Proposition 2.7.3]. The set $\{n_w \mid w \in X(\Phi)_J\}$ is a set of right coset representatives of G_J in G, and for J, $K \subseteq R$, the set

$$\{n_w \mid w \in X(\Phi)_J \cap X(\Phi)_K^{-1}\}$$

is a set of representatives for (G_J, G_K) double cosets in G and for (N_J, N_K) in N [3, Proposition 2.8.1]. We shall identify the group of K-characters and the group of k-characters of H via the natural map $\mathcal{O} \to k$, and denote the group by $\hat{H}$. No confusion will arise since we shall be interested in the action of W on these groups by $\lambda^w(h) = \lambda(n_w h n_w^{-1})$, where λ is a (K- or) k-character of H, and the map above is clearly a W-map. Let X_α be the root subgroup corresponding to the root α (see [3, p. 50]), and for $i \in R$, denote by H_i the subgroup $H \cap \langle X_i, X_{-i}\rangle$. For $\lambda \in \mathrm{Hom}(H, k^*)$, we define the set $M(\lambda) = \{i \in R \mid \lambda|_{H_i} = 1\}$.

Let W_λ be the stabilizer of λ and $W(\lambda)$ the (normal) subgroup of W_λ generated by its reflections. Then $W(\lambda)$ is the Weyl group of a root system $\Phi_\lambda \subseteq \Phi$ with base R_λ containing $M(\lambda)$ (but the inclusion $R_\lambda \subseteq R$ need not hold). Moreover, W_λ is the semidirect product of $W(\lambda)$ with the subgroup $C_\lambda := \{w \in W_\lambda \mid w(R_\lambda) = R_\lambda\}$. These facts are due to Kilmoyer [10] and Howlett–Kilmoyer [9] and can be found in [3, Proposition 10.6.3].

Curtis [5] has shown that the module $\bar{Y}$ has the following decomposition into pairwise nonisomorphic indecomposable direct summands:

$$\bar{Y} = \bigoplus_{(\chi, J)} \bar{Y}(\chi, J).$$

Here the indices (χ, J) run through all pairs consisting of an element χ of $\hat{H}$ and a subset J of $M(\chi)$. Let L_χ be a kB-module affording χ. Then [5], L_χ extends (uniquely) to $G_{M(\chi)}$, and we have the following formula in the Green ring of kG [14, Theorem 2]:

$$\bar{Y}(\chi, J) = \sum_{J \subseteq K \subseteq M(\chi)} (-1)^{|K \backslash J|} \operatorname{ind}_{G_K, G}(L_\chi). \qquad (*)$$

Finally, we shall use "(,)" for the usual inner product of (ordinary) characters, and if α is a character of a subgroup D of a group A, we shall write α_D^A for the induced character.

2. Computation of the Cartan Invariants

We denote the projective indecomposable $\bar{E}$-modules by

$$\bar{E}(\chi, J) := \mathrm{Hom}_{kG}(\bar{Y}, \bar{Y}(\chi, J)).$$

Thus, we have

$$\bar{E} = \bigoplus_{(\chi, J)} \bar{E}(\chi, J),$$

where the indices run over all admissible pairs as above.

The Cartan invariants are given by

$$C_{(\lambda, K),(\chi, J)} = \dim_k(\mathrm{Hom}_{\bar{E}}(\bar{E}(\chi, J), \bar{E}(\lambda, K))).$$

Suppose λ and χ are elements of $\hat{H}$ with $\chi = \lambda^w$ for some element $w \in W$. Then clearly $W(\chi) = W(\lambda)^w$ and so there is an element $w' \in W(\chi)$ such that $w'w(R_\lambda) = R_\chi$. For each subset $K \subseteq R_\lambda$ we let $K' := w'w(K)$. Then K' depends on the choice of w but different choices give sets which are conjugate by an element of C_λ. The aim of this section is to prove:

Theorem 1. (a) *If χ and λ are not W-conjugate then all of the $C_{(\lambda, K),(\chi, J)}$ are zero.*

(b) *If χ and λ are W-conjugate, then*

$$C_{(\lambda, K),(\chi, J)} = \sum_{c \in C_\chi} |Z(\Phi_\chi)_{M(\chi), J} \cap Z(\Phi_\chi)^{-1}_{M(\lambda)'^c, K'^c}|.$$

The following result is well known and stems from an idea of Steinberg [16, Lemma 2.2].

Lemma 2. *For $\lambda, \chi \in \hat{H}$, $K \subseteq M(\lambda)$, and $J \subseteq M(\chi)$, we have*

$$(\chi_{G_J}^G, \lambda_{G_K}^G) = (\chi_{N_J}^N, \lambda_{N_K}^N).$$

Proof. We may take the same set of double coset representatives on both sides (cf. Section 1) and then by the Mackey formula, both sides are equal to the number of these double coset representatives which map χ to λ.

LEMMA 3. *Suppose χ and λ are W-conjugate. Then for $J \subseteq M(\chi)$ and $K \subseteq M(\lambda)$ we have in the notation of Section* 1,

$$(\chi_{N_J}^N, \lambda_{N_K}^N) = \sum_{c \in C_\chi} |X(\Phi_\chi)_J \cap X(\Phi_\chi)_{K'^c}^{-1}|.$$

Proof. We have

$$(\chi_{N_J}^N, \lambda_{N_K}^N) = (\chi_{N_J}^N, \chi_{N_{K'}}^N).$$

By Clifford theory, the last inner product is equal to

$$(\chi_{N_J}^{N_\chi}, \chi_{N_{K'}}^{N_\chi}),$$

where N_χ is the stabiliser of χ in N, which is clearly the inverse image in N of W_χ. Now the result follows from the Mackey formula since the elements cx, as c and x run through C_χ and $X(\Phi_\chi)_J \cap X(\Phi_\chi)_{K'^c}^{-1}$ respectively, are a set of double coset representatives for $(N_J, N_{K'})$ in N_χ.

Proof of Theorem 1. By the remarks in the introduction, the Cartan invariant $C_{(\lambda, K),(\chi, J)}$ is equal to the inner product of the characters of $Y(\chi, J)_K$ and $Y(\lambda, K)_K$. If χ and λ are not W-conjugate, then it follows from the Mackey formula that $(\chi_B^G, \lambda_B^G) = 0$, proving (a). We suppose then that χ and λ are W-conjugate. By the formula (*), this is

$$\sum_{J \subseteq S \subseteq M(\chi)} \sum_{K \subseteq T \subseteq M(\lambda)} (-1)^{|S \setminus J|} (-1)^{|T \setminus K|} (\chi_{G_S}^G, \lambda_{G_T}^G).$$

If χ and λ are not W-conjugate, then it follows by Lemmas 2 and 3, this becomes

$$\sum_{c \in C_\chi} \sum_{J \subseteq S \subseteq M(\chi)} \sum_{K'^c \subseteq T \subseteq M(\chi)'^c} (-1)^{|S \setminus J|} (-1)^{|T \setminus K'^c|} |X(\Phi_\chi)_J \cap X(\Phi_\chi)_{K'^c}^{-1}|.$$

Next, we note that if $I \subseteq S$ and $Q \subseteq T$ then

$$\begin{array}{ccc} X(\Phi_\chi)_S \cap X(\Phi_\chi)_T^{-1} & \subseteq & X(\Phi_\chi)_I \cap X(\Phi_\chi)_T^{-1} \\ \cap\vert & & \cap\vert \\ X(\Phi_\chi)_S \cap X(\Phi_\chi)_Q^{-1} & \subseteq & X(\Phi_\chi)_I \cap X(\Phi_\chi)_Q^{-1}. \end{array}$$

Thus, the contribution to the sum from an element $w \in X(\Phi_\chi)_J \cap X(\Phi_\chi)_{K'^c}^{-1}$ is zero unless w lies in none of the sets $X(\Phi_\chi)_S \cap X(\Phi_\chi)_T^{-1}$ for $J \subsetneq S$ or $K'^c \subsetneq T$, in which case the contribution is 1. This proves Theorem 1.

COROLLARY 4. $\dim_k \bar{E}(\chi, J) = |W : W_{M(\chi)}| \, |Z(\Phi_{M(\chi)})_{M(\chi), J}|$.

Proof. We sum the Cartan invariants $C_{(\lambda,K),(\chi,J)}$ over all λ in the W-orbit of χ and all subsets $K\subseteq M(\lambda)$. For fixed λ and $c\in C_\chi$ we have

$$\sum_{K\subseteq M(\lambda)} |Z(\Phi_\chi)_{M(\chi),J}\cap Z(\Phi_\chi)^{-1}_{M(\lambda)'^c,K'^c}| = |Z(\Phi_\chi)_{M(\chi),J}|.$$

Thus,

$$\dim_k \bar{E}(\chi, J)=|W:W_\chi|\;|C_\chi|\;|Z(\Phi_\chi)_{M(\chi),J}|.$$

Now,

$$\begin{aligned}|Z(\Phi_\chi)_{M(\chi),J}| &= \sum_{J\subseteq S\subseteq M(\chi)} (-1)^{|S\setminus J|}\,|W(\chi):W_S| \\ &= |W(\chi):W_{M(\chi)}| \sum_{J\subseteq S\subseteq M(\chi)} (-1)^{|S\setminus J|}\,|W_{M(\chi)}:W_S| \\ &= |W(\chi):W_{M(\chi)}|\;|Z(\Phi_{M(\chi)})_{M(\chi),J}|.\end{aligned}$$

The result follows.

3. Duality

We may now generalise [14, Theorem 8].

THEOREM 5. *The mapping $Y(\chi, J)\to Y(\chi, M(\chi)\backslash J)$ defines an isometry of order two on the subgroup of the character ring of G spanned by the characters of $\{Y(\chi, J)_K \mid \chi\in\hat{H}, J\subseteq M(\chi)\}$.*

Proof. Let w_0 be the longest element of $W(\chi)$ (see [3]). Then for each $S\subseteq R_\chi$, we have $W_S^{w_0}=W_{-w_0(S)}$. Thus the sets $X(\Phi_\chi)_J\cap X(\Phi_\chi)^{-1}_{K'^c}$ and $X(\Phi_\chi)_J\cap X(\Phi_\chi)^{-1}_{-w_0(K'^c)}$ of double coset representatives have the same size. It follows that the sets $Z(\Phi_\chi)_{M(\chi),J}\cap Z(\Phi_\chi)^{-1}_{M(\lambda)'^c,K'^c}$ and $Z(\Phi_\chi)_{M(\chi),J}\cap Z(\Phi_\chi)^{-1}_{-w_0(M(\lambda)'^c),-w_0(K'^c)}$ also have the same size and now right multiplication by w_0 maps the latter set bijectively onto $Z(\Phi_\chi)_{M(\chi),M(\chi)\backslash J}\cap Z(\Phi_\chi)^{-1}_{M(\lambda)'^c,(M(\lambda)\backslash K)'^c}$, so the result follows from Theorem 1.

In Section 4, we shall give an alternative proof of Theorem 5 which will give the connection with the Alvis–Curtis map.

4. Decomposition Numbers

We have

$$Y=\bigoplus_{\chi\in\hat{H}} Y^\chi,$$

where Y^χ is the module induced from the one-dimensional representation of B affording χ. Let E^χ be the endomorphism ring of Y^χ. It has a natural basis $\{T_\omega \mid w \in W_\chi\}$ (see [3, 10.8]). As before, we set $Y^\chi_K := Y^\chi \otimes_{\mathcal{O}} K$ and $E^\chi_K := E^\chi \otimes_{\mathcal{O}} K$. We need to recall some facts about the generic algebra associated to E_χ (see [3]). It is an associative algebra $A(t_\alpha)$ over the ring $K[t_\alpha]$ of polynomials over K in $|R_\chi|$ variables, one for each $\alpha \in R_\chi$, with basis $\{a_w \mid w \in W_\chi\}$ and satisfying certain relations (see [3, p. 357]). A *specialisation* $A(l_\alpha)$ of $A(t_\alpha)$ is the algebra obtained from a homomorphism $K[t_\alpha] \to K$, $t_\alpha \mapsto l_\alpha$. It is known [3, 10.6 and Theorem 10.8.5] that if one chooses $p_\alpha = |X_\alpha|$ then the algebra $A(p_\alpha)$ is isomorphic to E^χ_K by an isomorphism mapping the image of a_w to T_w. Also, the specialisation $t_\alpha \mapsto 1$ gives an isomorphism $A(1) \cong KW_\chi$ in which the image of a_w is mapped to w (χ extends to W_χ by [9, Theorem 2.18]).

Let F be an algebraic closure of the field of quotients of $K[t_\alpha]$. By Tits' theorem on specialisations of $A(t_\alpha)$, the isomorphism classes of simple modules of each of the algebras $A(t_\alpha)_F := A(t_\alpha) \otimes_{K[t_\alpha]} F$, E^χ_K and KW_χ correspond bijectively via the above specialisations. They are therefore indexed by the irreducible K-characters $\mathrm{Irr}(W_\chi)$ of W_χ. For $\phi \in \mathrm{Irr}(W_\chi)$, denote by V_ϕ the corresponding E^χ_K-module and by D_ϕ the corresponding $A(t_\alpha)_F$-module. Thus, the image of the primitive idempotent of $A(t_\alpha)_F$ generating the right ideal isomorphic to D_ϕ maps under the above specialisations to those idempotents corresponding to the module V_ϕ and the character ϕ, respectively.

By the equivalances of categories considered in the introduction, the simple KG-modules which occur as constituents of Y^χ_K are in natural bijection with the simple E^χ_K-modules, so they are also indexed by $\mathrm{Irr}(W_\chi)$. We write M_ϕ for the simple KG-module $V_\phi \otimes_{E^\chi_K} Y^\chi_K$. Similarly, the simple E_K-modules are in bijection with the simple KG-modules which are constituents of Y_K. Since $Y^\chi_K \cong_{KG} Y^\lambda_K$ if and only if χ and λ are W-conjugate, we see that the simple E_K-modules are indexed by the set

$$\bigcup_\chi \mathrm{Irr}(W_\chi),$$

where χ is a set of representatives of W-orbits on $\hat{H}$. Moreover, if the simple module V corresponds to $\phi \in \mathrm{Irr}(W_\chi)$, then the restriction of V to the subalgebra E^χ is just the module V_ϕ of the preceeding paragraph, so we shall use the notation V_ϕ for both the E_K-module and its restriction, bearing in mind that the two E-modules V_ϕ, $\phi \in \mathrm{Irr}(W_\chi)$ and V_{ϕ^w}, $\phi^w \in \mathrm{Irr}(W^w_\chi)$ are the same.

We take an E-invariant $\mathcal{O}$-lattice $\tilde{V}_\phi$ in V_ϕ and consider the $\bar{E}$-module $\tilde{V}_\phi \otimes_{\mathcal{O}} k$. Let $L(\lambda, J)$ be the simple $\bar{E}$-module in the head of $\bar{E}(\lambda, J)$. Then the decomposition number $d_{\phi,(\lambda,J)}$ is the multiplicity of $L(\lambda, J)$ as a composition factor of $\tilde{V}_\phi \otimes_{\mathcal{O}} k$. By "Brauer Reciprocity" (see [13]), $d_{\phi,(\lambda,J)}$ is also the multiplicity of M_ϕ in $Y(\lambda, J)_K$. Since Y_K^χ and Y_K^λ have no common constituents if χ and λ are not W-conjugate, we see that $d_{\phi,(\lambda,J)}=0$ in this case. Thus, we only have to determine the multiplicities of the modules M_ϕ in the modules $Y(\chi, J)_K$ when ϕ is an irreducible character of W_χ. The following lemma is a slight extension of a well-known result [7, Proposition 8.4].

LEMMA 6. *Let $J \subseteq M(\chi)$ and $\phi \in \mathrm{Irr}(W_\chi)$. Then (identifying KG-modules with their characters) we have*

$$(\chi_{G_J}^G, V_\phi) = (1_{W_J}^{W_\chi}, \phi).$$

Proof. The module Y_K^χ which affords χ_B^G may be identified with the right ideal of KG generated by

$$e_B := |B|^{-1} \sum_{b \in B} \chi(b^{-1})b.$$

Under this identification we have $E_K^\chi = e_B KG e_B$, and a map $f \in E_K^\chi$ becomes identified with $f(e_B)$.

We show that the idempotent

$$e_{W_J} := |W_J|^{-1} \sum_{w \in W_J} w$$

of KW_χ which generates the right ideal of KW corresponding to the induced character $1_{W_J}^{W_\chi}$ and the idempotent

$$e_{G_J} := |G_J|^{-1} \sum_{g \in G_J} \chi(g^{-1})g$$

of E_K^χ corresponding to the induced character $\chi_{G_J}^G$ are both specialisations of the idempotent

$$e_J := \left(\sum_{w \in W_J} t_w\right)^{-1} \sum_{w \in W_J} a_w$$

of the generic algebra $A(t_\alpha)_{K(t_\alpha)}$ under the specialisations which give ϕ and V_ϕ, respectively (for $w \in W_{M(\chi)}$ the monomial t_w is defined to be $\prod_{\alpha_i} t_{\alpha_i}$ with the product taken over $\alpha_i \in M(\chi)$ in a reduced expression $w = w_{\alpha_1} \cdots w_{\alpha_m}$ (cf. [3, p. 361])). It follows from this that the multiplicities in the statement are both equal to the multiplicity of D_ϕ in the right ideal of $A(t_\alpha) \otimes_{K[t_\alpha]} F$ generated by e_J.

It is clear that e_J maps to e_{W_J}. In order to prove that e_J maps to e_{G_J}, we note first that when p_α is substituted for t_α in $\sum_{w\in W_J} t_w$ it becomes $|G_J : B|$. So we must show that

$$e_{G_J} = |G_J : B|^{-1} \sum_{w\in W_J} T_w.$$

When $w \in W_{M(\chi)}$, the basis element T_w of E_K^χ is the map sending e_B to $|U_w|\, e_B n_w$ (see [3, 10.8]), and using the fact that $\chi(n_w) = 1$ (see Section 1) this may be seen to be the map which sends e_B to

$$|B|^{-1} \sum_{g\in Bn_wB} \chi(g^{-1})\, g.$$

It then follows from the Bruhat decomposition $G_J = \bigcup_{w\in W_J} Bn_wB$, and the above identifications that e_{G_J} is the specialisation of e_J.

THEOREM 7. *For $\chi \in \hat{H}$, $J \subseteq M(\chi)$, and $\phi \in \mathrm{Irr}(W_\chi)$ we have*

$$d_{\phi,(\chi,J)} = \sum_{J\subseteq S\subseteq M(\chi)} (-1)^{|S\setminus J|} (1_{W_S}^{W_\chi}, \phi).$$

Proof. From the discussion at the beginning of this section, we have

$$d_{\phi,(\chi,J)} = (Y(\chi, J)_K, V_\phi).$$

The result now follows by applying the formula (*) and Lemma 6.

Next we point out the connection with Alvis–Curtis duality:

THEOREM 8. *Let ε be the sign character of W. Then*

$$d_{\varepsilon\phi,(\chi,J)} = d_{\phi,(\chi,M(\chi)\setminus J)}.$$

Proof. Write J^* for $M(\chi)\setminus J$. By Theorem 7, it suffices to prove

$$\sum_{J\subseteq S\subseteq M(\chi)} (-1)^{|S\setminus J|}\, \varepsilon_{W_S}^{W_\chi} = \sum_{J^*\subseteq T\subseteq M(\chi)} (-1)^{|T\setminus J^*|}\, 1_{W_T}^{W_\chi}.$$

Clearly, we need only prove the same equation with W_χ replaced by $W_{M(\chi)}$, which is a Coxeter group. We may therefore use Solomon's formula for ε [3, Prop. 6.2.1],

$$\varepsilon_{W_S} = \sum_{Q\subseteq S} (-1)^{|Q|}\, 1_{W_Q}^{W_S}.$$

The left hand side of the desired equation becomes

$$\sum_{J\subseteq S\subseteq M(\chi)} (-1)^{|S\setminus J|} \sum_{Q\subseteq S} (-1)^{|Q|}\, 1_{W_Q}^{W_{M(\chi)}}.$$

For each subset Q the term $1_{W_Q}^{W_{M(\chi)}}$ appears for each T containing $Q \cup J^*$ with a sign, so that the coefficient of $1_{W_Q}^{W_{M(\chi)}}$ is zero unless $Q \supseteq J^*$, in which case we must have $T = M(\chi)$. This shows that the desired equation holds.

Remarks. (1) Curtis [6, Theorem 1.7] has proved that under his duality map a constituent M_ϕ of Y_K^1 is mapped to $M_{\varepsilon\phi}$. Thus, the above theorem shows that for these characters our duality map coincides with his.

(2) Another way to prove that our map gives an isometry, without computing the Cartan invariants, would be to show that it is the composition of the following isometries. Let f be the isometry of Lemma 2 and m multiplication by ε (considered as a character of N). Then one can show using the argument in the proof of Theorem 8, that the map $Y(\chi, J) \mapsto Y(\chi, M(\chi \backslash J))$ is $f^{-1}mf$.

(3) In the case $\chi = 1$, write $d_{\phi,J}$ for the decomposition number $d_{\phi,(1,J)}$, $J \subseteq R$, $\phi \in \operatorname{Irr} W$. The formula for $d_{\phi,J}$ reduces to

$$\sum_{J \subseteq S \subseteq R} (-1)^{|S \backslash J|} (1_{W_S}^W, \phi).$$

This formula has been studied by Stanley [15], for Weyl groups of types A and B/C, in another connection. He gives the following interesting combinatorial interpretations: In type A, W is the symmetric group of degree $n = |R| + 1$ and Irr W is indexed by partitions of n. The number $d_{\phi,J}$ is then the number of standard Young tableaux whose underlying partition is the one corresponding to ϕ and whose "descent set" is J. In type B, Irr W is indexed by pairs (A, B) of partitions such that $|A| + |B| = n$, and $d_{\phi,J}$ is then the number of pairs of standard tableaux of shape (A, B) with "descent set" J. Carter [4], computed the decomposition numbers of E_K^1 for type A and obtained the same answer, so Theorem 7 explains why the answer was the same. Furthermore, the well-known equation $D^t D = C$ relating the decomposition and Cartan matrices can be described in terms of the Robinson–Schensted map in type A [4, p. 101], and a generalisation of this defined in [15, p. 145] for type B. A description of the multiplicities $(1_{W_J}^W, \phi)$ for type B also appears in [11]. Some of the decomposition numbers have been obtained independently by Khammash [18].

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