Classical Modules, Simple Modules and Incidence Matrices

Peter Sin

University of Florida

Finite Groups, Representations, and Related Topics
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Outline

Classical Weyl modules

Classical polar spaces over finite fields

Weyl modules and Jantzen Sum Formula

Oppositeness in buildings
\[ G = \text{SL}(n, k), \ k \text{ algebraically closed.} \]

\[ V \text{ standard module, } P = \text{max parabolic subgroup stabilizing 1-dimensional subspace.} \]

Then \[ G/P \cong \mathbf{P}(V), \] homogeneous coordinate ring is
\[ S(V^*) = \bigoplus_{d \geq 0} S^d(V^*) \]

The modules \[ S^d(V^*) \] are simple if \( k \) has characteristic 0.

In characteristic \( p > 0 \) \( S^d(V^*) \) are simple for \( d < p \).

The \( G \)-submodule lattice of every \( S^d(V^*) \) was described by Doty, Krop, (1980).

\[ G = \text{Sp}(n, k), \ S^d(V^*) \text{ are simple for } d < p. \]

Submodule lattice of \( S^d(V^*) \) for certain \( d \geq p \) by Lahtonen, (1990).
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Simple algebraic groups

- $G$ simple algebraic group over $k$, $\lambda$ dominant weight
- $V(\lambda)$, $P =$ max parabolic stabilizing highest weight vector.
- $G/P$ embeds into $\mathbf{P}(V(\lambda))$, homogeneous coordinate ring is $S = \bigoplus_{d \geq 0} H^0(d(-w_0\lambda)) = \bigoplus_{d \geq 0} V(d\lambda)^*$ (Ramanan-Ramanathan, 1985).
- When $\lambda = \omega_1$, the $V(r\omega_1)$ will be called classical Weyl modules.
- $G = \text{Spin}_n(k, f)$, $V(\omega_1)$ is the orthogonal module, $G/P$ embedded as a quadric in $\mathbf{P}(V(\omega_1))$, $S = S(V(\omega_1)^*)/(f)$, $V(r\omega_1)^* \cong S^r(V(\omega_1)^*)/(S^{r-2}(V(\omega_1)^*) \cdot f)$
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This talk is about results describing the $G$-submodule lattice of $V(r\omega_1)$, for $r \leq p - 1$, when $G$ is a classical group or of type $E_6$.

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Oppositeness in buildings
Finite Polar Spaces

- $V$ a vector space over $\mathbb{F}_q$, $q = p^t$ with nonsingular form $b(-, -)$.
- $b$ may be alternating or symmetric or hermitian.
- $P = \{\text{singular 1-dimensional subspaces}\}$, “points”
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An ovoid is a set of points of such that each max. tot. singular subspace contains exactly one point of the set. Any ovoid must have size

\[ N := \frac{\#\{\text{max. tot. sing. subspaces}\}}{\#\{\text{max. tot. sing. subspaces containing a given point}\}} \]

An general problem is to determine which polar spaces have ovoids.

Let \( A \) be the 0-1 incidence matrix with rows indexed by \( P \) and columns indexed by \( P^* \) and entry 1 iff the point lies on the hyperplane.

(Moorhouse) An ovoid determines an identity \( N \times N \) submatrix of \( A \).

\( A \) may be considered over any field. If for some prime \( \ell \) we have \( \text{rank}_\ell A < N \), then ovoids do not exist.

The smallest rank occurs when \( \ell = p \).

(Moorhouse, 2006) What is \( \text{rank}_p A \)?
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Permutation modules

- $G(q)$ = group of linear transformations preserving $b(-, -)$.
- $k[P], kG(q)$-permutation module on $P$.
- $k[P] \cong k.1 \oplus Y_P$,
- head($Y_P$) $\cong$ soc($Y_P$), a simple module $L$. 
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The incidence matrix $A$ corresponds to a $kG(q)$-module homomorphism

$$
\phi \in \text{End}_{kG(q)}(k[P]), \quad \phi(p) = \sum_{p' \in p^\perp} p'.
$$

$$
\text{Im} \phi = k.1 \oplus L.
$$

Outcome: $\text{rank}_p A = 1 + \text{dim } L.$
Identifying the simple module $L$

$L \cong L((q - 1)\omega)$,

where $\omega = \omega_1$ in the orthogonal and symplectic cases, and $\omega_1 + \omega_\ell$ in the unitary case.

By Steinberg’s Tensor Product Theorem,

$L((q - 1)\omega) = L((p - 1)\omega) \otimes L((p - 1)\omega)^{(p)} \cdots \otimes L((p - 1)\omega)^{(p^{t-1})}$

Conclusion: $\text{rank}_p A = 1 + (\dim L((p - 1)\omega))^t$.

Note that for even-dimensional orthogonal groups, the $p$-rank is independent of Witt index.

Ovoids $\leadsto$ $p$-ranks $\leadsto$ simple modules $L((p - 1)\omega)$

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Oppositeness in buildings
O. Arslan, P.S., (2011) treat the following groups and highest weights.

(B) $G$ of type $B_\ell$, ($\ell \geq 2$) $\lambda = r(\omega_1)$, $0 \leq r \leq p - 1$;

(D) $G$ of type $D_\ell$, ($\ell \geq 3$) $\lambda = r(\omega_1)$, $0 \leq r \leq p - 1$;

(A) $G$ of type $A_\ell$, ($\ell \geq 3$) $\lambda = r(\omega_1 + \omega_\ell)$, $0 \leq r \leq p - 1$;

For type $A$ and type $C$, the Weyl modules $V(r\omega_1)$ are simple for $0 \leq r \leq p - 1$. 
Theorem B

Let $G$ be of type $B_\ell$, $\ell \geq 2$. Let $\omega_1$ be the highest weight of the standard orthogonal module of dimension $2\ell + 1$. Assume $0 \leq r \leq p - 1$. Then the following hold.

(a) $H^0(r\omega_1)$ is simple unless (i) $p = 2$ and $r = 1$ or (ii) $p > 2$ and there exists a positive odd integer $m$ such that

$$r + 2\ell - 1 \leq mp \leq 2r + 2\ell - 2.$$ 

(b) If (i) holds then the quotient $H^0(\omega_1)/L(\omega_1)$ is the one-dimensional trivial module.

(c) If (ii) holds then $m$ is unique and

$$H^0(r\omega_1)/L(r\omega_1) \cong H^0(r_1\omega_1),$$

where $r_1 = mp - 2\ell + 1 - r$. Furthermore the module $H^0(r_1\omega_1)$ is simple.
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Theorem D
Let $G$ be of type $D_{\ell}$, $\ell \geq 3$. Let $\omega_1$ be the highest weight of the standard orthogonal module of dimension $2\ell$. Assume $0 \leq r \leq p - 1$. Then the following hold.

(a) Suppose that there exists a positive even integer $m$ such that

$$r + 2\ell - 2 \leq mp \leq 2r + 2\ell - 3.$$ 

Then $m$ is unique and

$$H^0(r\omega_1)/L(r\omega_1) \cong H^0(r_1\omega_1),$$

where $r_1 = mp - 2\ell + 2 - r$. Furthermore the module $H^0(r_1\omega_1)$ is simple.

(b) Otherwise, $H^0(r\omega_1)$ is simple.
Theorem A
Let \( G \) be of type \( A_\ell, \ell \geq 3 \). Assume \( 0 \leq r \leq p - 1 \). Then the following hold.

(a) Suppose that here exists a positive integer \( m \) such that

\[
 r + \ell \leq mp \leq 2r + \ell - 1.
\]

Then \( m \) is unique and

\[
 H^0(r(\omega_1 + \omega_\ell))/L(r(\omega_1 + \omega_\ell)) \cong H^0(r_1(\omega_1 + \omega_\ell)),
\]

where \( r_1 = mp - \ell - r \). Furthermore the module \( H^0(r_1(\omega_1 + \omega_\ell)) \) is simple.

(b) Otherwise, \( H^0(r(\omega_1 + \omega_\ell)) \) is simple.
The Jantzen filtration \( V(\lambda)^i, i > 0 \), of \( V(\lambda) \) satisfies

\[
V(\lambda)^1 = \text{rad} \, V(\lambda), \quad \text{so} \quad V(\lambda)/V(\lambda)^1 \cong L(\lambda).
\]

and

\[
\sum_{i>0} \text{Ch} \, (V(\lambda)^i) = - \sum_{\alpha>0} \sum_{\{m:0<mp<\langle \lambda+\rho,\alpha \vee \rangle \}} v_p(mp) \chi(\lambda - mp\alpha)
\]

\( \chi(\mu) \) are either 0 or \( \pm \) the character of a Weyl module of “lower” weight than \( \lambda \).

Iterate the process on these Weyl module terms.
Jantzen Sum Formula

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Jantzen Sum Formula

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and

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Remarks on the proofs of Theorems B, D and A

\[ \sum_{i>0} \text{Ch}(V(r\omega)^i) = - \sum_{\alpha>0} \sum_{\{m: 0 < mp < \langle r\omega + \rho, \alpha \rangle\}} v_p(mp) \chi(r\omega - mp\alpha) \]

- Need to handle the weight combinatorics uniformly with respect to parameters \( r, p, \ell \).
- The sum formula overestimates composition multiplicities.
- Nearly all root multiples contribute nothing, i.e. \( \chi(\lambda - mp\alpha) = 0 \).
- The Weyl characters which do occur in the Sum formula are of the form \( \text{Ch} V(r\omega) \).
- Theory of good filtrations (Donkin, Wang, Mathieu) reduces certain computations to the complex case, and then we can apply formulae from classical invariant theory.
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Outline

Classical Weyl modules

Classical polar spaces over finite fields

Weyl modules and Jantzen Sum Formula

Oppositeness in buildings
Oppositeness

Let \((\Delta(q), S)\) be the spherical Tits building of a finite group of Lie type.

Two types \(I, J \subseteq S\) are opposite if \(I^{w_0} = J\).

If \(w_0 = -1\) then every type is its own opposite.

Assume \(I\) and \(J\) are opposite types. We say the cosets \(gP_I\) and \(hP_J\) of the parabolic subgroups are opposite iff \(P_ig^{-1}hP_J = P_iw_0P_J\).

Oppositeness map:

\[
\eta : \text{ind}_{P_I}^{G(q)}(k) \rightarrow \text{ind}_{P_J}^{G(q)}(k), \quad gP_I \mapsto \sum_{hP_J \subseteq gP_Iw_0P_J} hP_J
\]

By Carter and Lusztig (1976) \(\text{Im} \eta\) is a simple module of the form

\[
L((q-1)\omega_I) = L((p-1)\omega_I) \otimes L((p-1)\omega_I)^{(p)} \otimes \cdots \otimes L((p-1)\omega_I)^{(p^{t-1})}
\]

The complements of the points vs. polar hyperplanes relations for classical groups are oppositeness relations, with \(\omega_I = \omega\).
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The complements of the points vs. polar hyperplanes relations for classical groups are oppositeness relations, with $\omega_I = \omega$. 
Classical Weyl modules for $E_6$

- $G = E_6(q)$, group of linear automorphisms preserving a certain cubic form on a 27-dimensional vector space $V(\omega_1)$.
- The geometry of this space has been studied in great detail. (Dickson, Aschbacher, Buekenhout, Cohen, Cooperstein, Pasini.)
- Objects of type 1 are singular points of $V(\omega_1)$ and objects of the opposite type 6 are distinguished hyperplanes. A singular point $\langle \nu \rangle$ is opposite a distinguished hyperplane $H$ if and only if $\nu \notin H$. 

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\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_5 \quad \alpha_6
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\alpha_4
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[Diagram showing the root system of $E_6$]
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Objects of type 1 are singular points of \( V(\omega_1) \) and objects of the opposite type 6 are distinguished hyperplanes. A singular point \( \langle \nu \rangle \) is opposite a distinguished hyperplane \( H \) if and only if \( \nu \notin H \).
Structure of $V(r\omega_1)$, $0 \leq r \leq p - 1$

- Oppositeness module $L((q - 1)\omega_1)$ leads to Weyl modules $V(r\omega_1)$, and some others.
- If $0 \leq r \leq p - 4$ the Weyl module $V(r\omega_1)$ is simple.
- Results for $r = p - 3$, $r = p - 2$, $r = p - 1$ are similar in form, so just look at $r = p - 1$. 
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- Oppositeness module $L((q - 1)\omega_1)$ leads to Weyl modules $V(r\omega_1)$, and some others.
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- Results for $r = p - 3$, $r = p - 2$, $r = p - 1$ are similar in form, so just look at $r = p - 1$. 
(i) If $p \leq 5$ the the Weyl module $V((p - 1)\omega_1)$ is simple.

(ii) If $p = 7$, there is an exact sequence

$$0 \to V(3\omega_6) \to V(6\omega_1) \to L(6\omega_1) \to 0.$$

(iii) For $p \geq 11$ there is an exact sequence

$$0 \to V((p - 11)\omega_1 + 2\omega_2) \to V((p - 10)\omega_1 + \omega_2 + \omega_5) \to V((p - 9)\omega_1 + \omega_3 + \omega_6) \to V((p - 8)\omega_1 + \omega_4 + 2\omega_6) \to V((p - 7)\omega_1 + 3\omega_6) \to V((p - 1)\omega_1) \to L((p - 1)\omega_1) \to 0.$$

(First and last nonzero terms simple, other terms have two composition factors.)
Let $J(\lambda)$ denote the Jantzen sum

$$
\sum_{i>0} \chi(V(\lambda)^i) = - \sum_{\alpha>0} \sum_{\{m:0<mp<\langle\lambda+\rho,\alpha^\vee\rangle\}} v_p(mp) \chi(\lambda - mp\alpha)
$$

$$
J((p-1)\omega_1) = \chi((p-7)\omega_1+3\omega_6) - \chi((p-8)\omega_1+\omega_4+2\omega_6) + \chi((p-9)\omega_1+\omega_3+\omega_6) - \chi((p-10)\omega_1+\omega_2+\omega_5) + \chi((p-11)\omega_1+2\omega_2)
$$

$$
J((p-11)\omega_1+2\omega_2) = 0.
$$

$$
J((p-10)\omega_1+\omega_2+\omega_5) = \text{Ch} L((p-11)\omega_1+2\omega_2).
$$

$$
J((p-9)\omega_1+\omega_3+\omega_6) = \chi((p-10)\omega_1+\omega_2+\omega_5) - \chi((p-11)\omega_1+2\omega_2)
$$

$$
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J((p-11)\omega_1 + 2\omega_2) = 0.
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\]

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Let $J(\lambda)$ denote the Jantzen sum

$$\sum_{i>0} \text{Ch}(V(\lambda)^i) = - \sum_{\alpha>0} \sum_{\{m: 0<m \rho<\langle \lambda+\rho,\alpha \rangle\}} v_p(mp) \chi(\lambda - mp\alpha)$$

$$J((p-1)\omega_1) = \chi((p-7)\omega_1 + 3\omega_6) - \chi((p-8)\omega_1 + \omega_4 + 2\omega_6) + \chi((p-9)\omega_1 + \omega_3 + \omega_6) - \chi((p-10)\omega_1 + \omega_2 + \omega_5) + \chi((p-11)\omega_1 + 2\omega_2) \quad (1)$$

$$J((p-11)\omega_1 + 2\omega_2) = 0.$$  

$$J((p-10)\omega_1 + \omega_2 + \omega_5) = \text{Ch} L((p-11)\omega_1 + 2\omega_2).$$

$$J((p-9)\omega_1 + \omega_3 + \omega_6) = \chi((p-10)\omega_1 + \omega_2 + \omega_5) - \chi((p-11)\omega_1 + 2\omega_2) = \text{Ch} L((p-10)\omega_1 + \omega_2 + \omega_5). \quad (2)$$
Let $J(\lambda)$ denote the Jantzen sum
\[
\sum_{i>0} \text{Ch}(V(\lambda)^i) = - \sum_{\alpha>0} \sum_{\{m:0<mp<\langle\lambda+\rho,\alpha^\vee\rangle\}} v_p(mp) \chi(\lambda - mp\alpha)
\]

\[
J((p-1)\omega_1) = \chi((p-7)\omega_1 + 3\omega_6) - \chi((p-8)\omega_1 + \omega_4 + 2\omega_6) + \chi((p-9)\omega_1 + \omega_3 + \omega_6) - \chi((p-10)\omega_1 + \omega_2 + \omega_5) + \chi((p-11)\omega_1 + 2\omega_2) \quad (1)
\]

\[
J((p-11)\omega_1 + 2\omega_2) = 0.
\]

\[
J((p-10)\omega_1 + \omega_2 + \omega_5) = \text{Ch} \ L((p-11)\omega_1 + 2\omega_2).
\]

\[
J((p-9)\omega_1 + \omega_3 + \omega_6) = \chi((p-10)\omega_1 + \omega_2 + \omega_5) - \chi((p-11)\omega_1 + 2\omega_2) = \text{Ch} \ L((p-10)\omega_1 + \omega_2 + \omega_5). \quad (2)
\]
Sum formula calculation

Let $J(\lambda)$ denote the Jantzen sum

$$
\sum_{i > 0} \text{Ch}(V(\lambda)^i) = - \sum_{\alpha > 0} \sum_{\{m:0 < mp < (\lambda + \rho, \alpha^\vee)\}} v_p(mp) \chi(\lambda - mp\alpha)
$$

$$
J((p-1)\omega_1) = \chi((p-7)\omega_1 + 3\omega_6) - \chi((p-8)\omega_1 + \omega_4 + 2\omega_6) + \chi((p-9)\omega_1 + \omega_3 + \omega_6) - \chi((p-10)\omega_1 + \omega_2 + \omega_5) + \chi((p-11)\omega_1 + 2\omega_2) \quad (1)
$$

$$
J((p-11)\omega_1 + 2\omega_2) = 0.
$$

$$
J((p-10)\omega_1 + \omega_2 + \omega_5) = \text{Ch} L((p-11)\omega_1 + 2\omega_2).
$$

$$
J((p-9)\omega_1 + \omega_3 + \omega_6) = \chi((p-10)\omega_1 + \omega_2 + \omega_5) - \chi((p-11)\omega_1 + 2\omega_2) = \text{Ch} L((p-10)\omega_1 + \omega_2 + \omega_5). \quad (2)
$$
\[ J((p - 8)\omega_1 + \omega_4 + 2\omega_6) = \chi((p - 9)\omega_1 + \omega_3 + \omega_6) \\
- \chi((p - 10)\omega_1 + \omega_2 + \omega_5) + \chi((p - 11)\omega_1 + 2\omega_2) \\
= \text{Ch} \, L((p - 9)\omega_1 + \omega_3 + \omega_6), \quad (3) \]

\[ J((p - 7)\omega_1 + 3\omega_6) = \chi((p - 8)\omega_1 + \omega_4 + 2\omega_6) \\
- \chi((p - 9)\omega_1 + \omega_3 + \omega_6) + \chi((p - 10)\omega_1 + \omega_2 + \omega_5) \\
- \chi((p - 11)\omega_1 + 2\omega_2) \\
= \text{Ch} \, L((p - 8)\omega_1 + \omega_4 + 2\omega_6). \quad (4) \]

\[ J((p - 1)\omega_1) = \text{Ch} \, L((p - 7)\omega_1 + 3\omega_6). \]
\[ J((p - 8)\omega_1 + \omega_4 + 2\omega_6) = \chi((p - 9)\omega_1 + \omega_3 + \omega_6) - \chi((p - 10)\omega_1 + \omega_2 + \omega_5) + \chi((p - 11)\omega_1 + 2\omega_2) = \text{Ch} L((p - 9)\omega_1 + \omega_3 + \omega_6), \quad (3) \]

\[ J((p - 7)\omega_1 + 3\omega_6) = \chi((p - 8)\omega_1 + \omega_4 + 2\omega_6) - \chi((p - 9)\omega_1 + \omega_3 + \omega_6) + \chi((p - 10)\omega_1 + \omega_2 + \omega_5) - \chi((p - 11)\omega_1 + 2\omega_2) = \text{Ch} L((p - 8)\omega_1 + \omega_4 + 2\omega_6). \quad (4) \]

\[ J((p - 1)\omega_1) = \text{Ch} L((p - 7)\omega_1 + 3\omega_6). \]
\[ J((p - 8)\omega_1 + \omega_4 + 2\omega_6) = \chi((p - 9)\omega_1 + \omega_3 + \omega_6) \]

\[ - \chi((p - 10)\omega_1 + \omega_2 + \omega_5) + \chi((p - 11)\omega_1 + 2\omega_2) \]

\[ = Ch L((p - 9)\omega_1 + \omega_3 + \omega_6), \quad (3) \]

\[ J((p - 7)\omega_1 + 3\omega_6) = \chi((p - 8)\omega_1 + \omega_4 + 2\omega_6) \]

\[ - \chi((p - 9)\omega_1 + \omega_3 + \omega_6) + \chi((p - 10)\omega_1 + \omega_2 + \omega_5) \]

\[ - \chi((p - 11)\omega_1 + 2\omega_2) \]

\[ = Ch L((p - 8)\omega_1 + \omega_4 + 2\omega_6). \quad (4) \]

\[ J((p - 1)\omega_1) = Ch L((p - 7)\omega_1 + 3\omega_6). \]
Simple modules for $E_6$

\[0 \to V((p-11)\omega_1 + 2\omega_2) \to V((p-10)\omega_1 + \omega_2 + \omega_5) \to V((p-9)\omega_1 + \omega_3 + \omega_6) \to V((p-8)\omega_1 + \omega_4 + 2\omega_6) \to V((p-7)\omega_1 + 3\omega_6) \to V((p-1)\omega_1) \to L((p-1)\omega_1) \to 0\]

The characters of the $L(\mu)$ can then be found using Weyl’s Character Formula. For example, the $p$-rank of the point-hyperplane oppositeness matrix is

\[
\dim L((p-1)\omega_1) = \frac{1}{2^7 \cdot 3 \cdot 5 \cdot 11} p(p+1)(p+3) \\
\times \left(3p^8 - 12p^7 + 39p^6 + 320p^5 - 550p^4 + 1240p^3 + 2080p^2 - 1920p + 1440\right)
\]
Thank you for your attention!