Classical Modules, Simple Modules and Incidence Matrices

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Finite Groups, Representations,
and Related Topics
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Outline

$\mathrm{SL}(V)$

- $G = \mathrm{SL}(n, k)$, k algebraically closed.
- V standard module, P=max parabolic subgroup stabilizing 1-dimensional subspace.
- Then $G/P \cong \mathbf{P}(V)$, homogeneous coordinate ring is $S(V^*) = \bigoplus_{d \geq 0} S^d(V^*)$
- The modules $S^d(V^*)$ are simple if k has characteristic 0.
- In characteristic $p > 0$ $S^d(V^*)$ are simple for $d < p$.
- The G-submodule lattice of every $S^d(V^*)$ was described by Doty, Krop, (1980).
- $G = \mathrm{Sp}(n, k)$, $S^d(V^*)$ are simple for $d < p$.
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- $V(\lambda)$, P= max parabolic stabilizing highest weight vector.
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- When $\lambda = \omega_1$, the $V(r\omega_1)$ will be called *classical Weyl modules*.
- $G = Spin_n(k, f)$, $V(\omega_1)$ is the orthogonal module, G/P embedded as a quadric in $\mathbf{P}(V(\omega_1))$, $S = S(V(\omega_1)^*)/(f)$, $V(r\omega_1)^* \cong S^r(V(\omega_1)^*)/(S^{r-2}(V(\omega_1)^*) \cdot f)$
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- This talk is about results describing the G-submodule lattice of $V(r\omega_1)$, for $r \leq p-1$, when G is a classical group or of type E_6.
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Outline

Classical Weyl modules

Classical polar spaces over finite fields

Weyl modules and Jantzen Sum Formula

Oppositeness in buildings

Finite Polar Spaces

- V a vector space over $\mathbb{F}_q$, $q = p^t$ with nonsingular form $b(-,-)$.
- b may be alternating or symmetric or hermitian.
- $P = \{\text{singular 1-dimensional subspaces}\}$, “points”
- $P^* = \{p^\perp \mid p \in P\}$, “polar hyperplanes”.

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Ovoids

- An *ovoid* is a set of points of such that each max. tot. singular subspace contains exactly one point of the set. Any ovoid must have size

$$N := \frac{\#\{\text{max. tot. sing. subspaces}\}}{\#\{\text{max. tot. sing. subspaces containing a given point}\}}$$

- An general problem is to determine which polar spaces have ovoids.
- Let A be the 0-1 incidence matrix with rows indexed by P and columns indexed by P^* and entry 1 iff the point lies on the hyperplane.
- (Moorhouse) An ovoid determines an identity $N \times N$ submatrix of A.
- A may be considered over any field. If for some prime ℓ we have $\mathrm{rank}_\ell A < N$, then ovoids do not exist.
- The smallest rank occurs when $\ell = p$.
- (Moorhouse, 2006) What is $\mathrm{rank}_p A$?

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Permutation modules

- $G(q) =$ group of linear transformations preserving $b(-,-)$.
- $k[P]$, $kG(q)$-permutation module on P.
- $k[P] \cong k.\mathbf{1} \oplus Y_P$,
- $\mathrm{head}(Y_P) \cong \mathrm{soc}(Y_P)$, a simple module L.

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Incidence map

- The incidence matrix A corresponds to a $kG(q)$-module homomorphism

$$\phi \in \mathrm{End}_{kG(q)}(k[P]), \quad \phi(p) = \sum_{p' \in p^{\perp}} p'.$$

- $$\operatorname{Im} \phi = k.\mathbf{1} \oplus L.$$

- Outcome: $\mathrm{rank}_p A = 1 + \dim L$.

Identifying the simple module L

- $$L \cong L((q-1)\omega),$$

  where $\omega = \omega_1$ in the orthogonal and symplectic cases, and $\omega_1 + \omega_\ell$ in the unitary case.
- By Steinberg's Tensor Product Theorem,

  $$L((q-1)\omega) = L((p-1)\omega) \otimes L((p-1)\omega)^{(p)} \cdots \otimes L((p-1)\omega)^{(p^{t-1})}$$
- Conclusion: $\mathrm{rank}_p A = 1 + (\dim L((p-1)\omega))^t$.
- Note that for even-dimensional orthogonal groups, the p-rank is independent of Witt index.
- Ovoids $\rightsquigarrow$ p-ranks $\rightsquigarrow$ simple modules $L((p-1)\omega)$ $\rightsquigarrow$ Weyl modules $V((p-1)\omega)$, $\omega = \omega_1$ or $\omega_1 + \omega_\ell$.

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Submodule structure of classical Weyl modules

O. Arslan, P.S., (2011) treat the following groups and highest weights.

(B) G of type B_ℓ, ($\ell \geq 2$) $\lambda = r(\omega_1)$, $0 \leq r \leq p-1$;

(D) G of type D_ℓ, ($\ell \geq 3$) $\lambda = r(\omega_1)$, $0 \leq r \leq p-1$;

(A) G of type A_ℓ, ($\ell \geq 3$) $\lambda = r(\omega_1 + \omega_\ell)$, $0 \leq r \leq p-1$;

For type A and type C, the Weyl modules $V(r\omega_1)$ are simple for $0 \leq r \leq p-1$.

Theorem B

Let G be of type B_ℓ, $\ell \geq 2$. Let ω_1 be the highest weight of the standard orthogonal module of dimension $2\ell + 1$. Assume $0 \leq r \leq p - 1$. Then the following hold.

(a) $H^0(r\omega_1)$ is simple unless (i) $p = 2$ and $r = 1$ or (ii) $p > 2$ and there exists a positive *odd* integer m such that

$$r + 2\ell - 1 \leq mp \leq 2r + 2\ell - 2.$$

(b) If (i) holds then the quotient $H^0(\omega_1)/L(\omega_1)$ is the one-dimensional trivial module.

(c) If (ii) holds then m is unique and

$$H^0(r\omega_1)/L(r\omega_1) \cong H^0(r_1\omega_1),$$

where $r_1 = mp - 2\ell + 1 - r$. Furthermore the module $H^0(r_1\omega_1)$ is simple.

Theorem B

Let G be of type B_ℓ, $\ell \geq 2$. Let ω_1 be the highest weight of the standard orthogonal module of dimension $2\ell + 1$. Assume $0 \leq r \leq p - 1$. Then the following hold.

(a) $H^0(r\omega_1)$ is simple unless (i) $p = 2$ and $r = 1$ or (ii) $p > 2$ and there exists a positive *odd* integer m such that

$$r + 2\ell - 1 \leq mp \leq 2r + 2\ell - 2.$$

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Theorem D

Let G be of type D_ℓ, $\ell \geq 3$. Let ω_1 be the highest weight of the standard orthogonal module of dimension 2ℓ. Assume $0 \leq r \leq p-1$. Then the following hold.

(a) *Suppose that there exists a positive even integer m such that*

$$r + 2\ell - 2 \leq mp \leq 2r + 2\ell - 3.$$

Then m is unique and

$$H^0(r\omega_1)/L(r\omega_1) \cong H^0(r_1\omega_1),$$

where $r_1 = mp - 2\ell + 2 - r$. Furthermore the module $H^0(r_1\omega_1)$ is simple.

(b) *Otherwise, $H^0(r\omega_1)$ is simple.*

Theorem A

Let G be of type A_ℓ, $\ell \geq 3$. Assume $0 \leq r \leq p-1$. Then the following hold.

(a) *Suppose that here exists a positive integer m such that*

$$r + \ell \leq mp \leq 2r + \ell - 1.$$

Then m is unique and

$$H^0(r(\omega_1 + \omega_\ell))/L(r(\omega_1 + \omega_\ell)) \cong H^0(r_1(\omega_1 + \omega_\ell)),$$

where $r_1 = mp - \ell - r$. Furthermore the module $H^0(r_1(\omega_1 + \omega_\ell))$ is simple.

(b) *Otherwise, $H^0(r(\omega_1 + \omega_\ell))$ is simple.*

Jantzen Sum Formula

- Jantzen (1977) , $p > h$, Andersen (1983), all p.
- The *Jantzen filtration* $V(\lambda)^i$, $i > 0$, of $V(\lambda)$ satisfies

$$V(\lambda)^1 = \mathrm{rad}\, V(\lambda), \quad \text{so} \quad V(\lambda)/V(\lambda)^1 \cong L(\lambda).$$

  and

$$\sum_{i>0} \mathrm{Ch}(V(\lambda)^i) = -\sum_{\alpha>0} \sum_{\{m:0<mp<\langle\lambda+\rho,\alpha^\vee\rangle\}} v_p(mp)\chi(\lambda - mp\alpha)$$

- $\chi(\mu)$ are either 0 or $\pm$ the character of a Weyl module of "lower" weight than λ.
- Iterate the process on these Weyl module terms.

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Remarks on the proofs of Theorems B, D and A

- $$\sum_{i>0} \mathrm{Ch}(V(r\omega)^i) = -\sum_{\alpha>0} \sum_{\{m:0<mp<\langle r\omega+\rho,\alpha^\vee\rangle\}} v_p(mp)\chi(r\omega - mp\alpha)$$
- Need to handle the weight combinatorics uniformly with respect to parameters r, p, ℓ.
- The sum formula *overestimates* composition multiplicities.
- Nearly all root multiples contribute nothing, i.e $\chi(\lambda - mp\alpha) = 0$.
- The Weyl characters which do occur in the Sum formula are of the form $\mathrm{Ch}\, V(r\omega)$.
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Outline

Classical Weyl modules

Classical polar spaces over finite fields

Weyl modules and Jantzen Sum Formula

Oppositeness in buildings

Oppositeness

- Let $(\Delta(q), S)$ be the spherical Tits building of a finite group of Lie type.
- Two types $I, J \subseteq S$ are *opposite* if $I^{w_0} = J$.
- If $w_0 = -1$ then every type is its own opposite.
- Assume I and J are opposite types. We say the cosets gP_I and hP_J of the parabolic subgroups are opposite iff $P_I g^{-1} h P_J = P_I w_0 P_J$.
- Oppositeness map:

$$\eta : \mathrm{ind}_{P_I}^{G(q)}(k) \to \mathrm{ind}_{P_J}^{G(q)}(k), \quad gP_I \mapsto \sum_{hP_J \subseteq gP_I w_0 P_J} hP_J$$

- By Carter and Lusztig (1976) $\mathrm{Im}\,\eta$ is a simple module of the form

$$L((q-1)\omega_I) = L((p-1)\omega_I) \otimes L((p-1)\omega_I)^{(p)} \cdots \otimes L((p-1)\omega_I)^{(p^{t-1})}$$

- The complements of the points vs. polar hyperplanes relations for classical groups are oppositeness relations, with $\omega_I = \omega$.

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Classical Weyl modules for E_6

- $G = E_6(q)$, group of linear automorphisms preserving a certain cubic form on a 27-dimensional vector space $V(\omega_1)$.
- The geometry of this space has been studied in great detail. (Dickson, Aschbacher, Buekenhout, Cohen, Cooperstein, Pasini.)
- Objects of type 1 are singular points of $V(\omega_1)$ and objects of the opposite type 6 are distinguished hyperplanes. A singular point $\langle v \rangle$ is opposite a distinguished hyperplane H if and only $v \notin H$.

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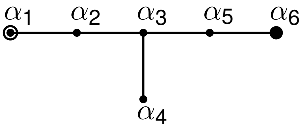

Structure of $V(r\omega_1)$, $0 \leq r \leq p-1$

- Oppositeness module $L((q-1)\omega_1)$ leads to Weyl modules $V(r\omega_1)$, and some others.
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$r = p - 1$

(i) If $p \leq 5$ the the Weyl module $V((p-1)\omega_1)$ is simple.

(ii) If $p = 7$, there is an exact sequence

$$0 \to V(3\omega_6) \to V(6\omega_1) \to L(6\omega_1) \to 0.$$

(iii) For $p \geq 11$ there is an exact sequence

$$0 \to V((p-11)\omega_1 + 2\omega_2) \to V((p-10)\omega_1 + \omega_2 + \omega_5)$$
$$\to V((p-9)\omega_1 + \omega_3 + \omega_6) \to V((p-8)\omega_1 + \omega_4 + 2\omega_6)$$
$$\to V((p-7)\omega_1 + 3\omega_6) \to V((p-1)\omega_1) \to L((p-1)\omega_1) \to 0$$

(First and last nonzero terms simple, other terms have two composition factors.)

Sum formula calculation

- Let $J(\lambda)$ denote the Jantzen sum

$$\sum_{i>0} \mathrm{Ch}(V(\lambda)^i) = -\sum_{\alpha>0} \sum_{\{m:0<mp<\langle\lambda+\rho,\alpha^\vee\rangle\}} v_p(mp)\chi(\lambda - mp\alpha)$$

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$$\begin{aligned} J((p-1)\omega_1) = \chi((p-7)\omega_1+3\omega_6)-\chi((p-8)\omega_1+\omega_4+2\omega_6) \\ + \chi((p-9)\omega_1+\omega_3+\omega_6) - \chi((p-10)\omega_1+\omega_2+\omega_5) \\ + \chi((p-11)\omega_1+2\omega_2) \quad (1) \end{aligned}$$

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- $J((p-10)\omega_1 + \omega_2 + \omega_5) = \mathrm{Ch}\, L((p-11)\omega_1 + 2\omega_2).$
-

$$J((p-9)\omega_1+\omega_3+\omega_6) = \chi((p-10)\omega_1+\omega_2+\omega_5) - \chi((p-11)\omega_1+2\omega_2) = \mathrm{Ch}\, L((p-10)\omega_1+\omega_2+\omega_5). \quad (2)$$

Sum formula calculation

- Let $J(\lambda)$ denote the Jantzen sum
$$\sum_{i>0} \mathrm{Ch}(V(\lambda)^i) = -\sum_{\alpha>0} \sum_{\{m:0<mp<\langle\lambda+\rho,\alpha^\vee\rangle\}} v_p(mp)\chi(\lambda - mp\alpha)$$
- $$\begin{aligned} J((p-1)\omega_1) = \chi((p-7)\omega_1+3\omega_6)-\chi((p-8)\omega_1+\omega_4+2\omega_6) \\ + \chi((p-9)\omega_1+\omega_3+\omega_6) - \chi((p-10)\omega_1+\omega_2+\omega_5) \\ + \chi((p-11)\omega_1+2\omega_2) \quad (1) \end{aligned}$$
- $J((p-11)\omega_1 + 2\omega_2) = 0.$
- $J((p-10)\omega_1 + \omega_2 + \omega_5) = \mathrm{Ch}\, L((p-11)\omega_1 + 2\omega_2).$
- $$\begin{aligned} J((p-9)\omega_1+\omega_3+\omega_6) &= \chi((p-10)\omega_1+\omega_2+\omega_5) \\ &\quad - \chi((p-11)\omega_1+2\omega_2) \\ &= \mathrm{Ch}\, L((p-10)\omega_1+\omega_2+\omega_5). \quad (2) \end{aligned}$$

- $$J((p-8)\omega_1+\omega_4+2\omega_6)=\chi((p-9)\omega_1+\omega_3+\omega_6)$$
  $$-\chi((p-10)\omega_1+\omega_2+\omega_5)+\chi((p-11)\omega_1+2\omega_2)$$
  $$=\operatorname{Ch}L((p-9)\omega_1+\omega_3+\omega_6), \quad (3)$$

- $$J((p-7)\omega_1+3\omega_6)=\chi((p-8)\omega_1+\omega_4+2\omega_6)$$
  $$-\chi((p-9)\omega_1+\omega_3+\omega_6)+\chi((p-10)\omega_1+\omega_2+\omega_5)$$
  $$-\chi((p-11)\omega_1+2\omega_2)$$
  $$=\operatorname{Ch}L((p-8)\omega_1+\omega_4+2\omega_6). \quad (4)$$

- $J((p-1)\omega_1)=\operatorname{Ch}L((p-7)\omega_1+3\omega_6)$.

- $$J((p-8)\omega_1+\omega_4+2\omega_6)=\chi((p-9)\omega_1+\omega_3+\omega_6) - \chi((p-10)\omega_1+\omega_2+\omega_5)+\chi((p-11)\omega_1+2\omega_2) = \mathrm{Ch}\, L((p-9)\omega_1+\omega_3+\omega_6), \quad (3)$$

- $$J((p-7)\omega_1+3\omega_6)=\chi((p-8)\omega_1+\omega_4+2\omega_6) - \chi((p-9)\omega_1+\omega_3+\omega_6)+\chi((p-10)\omega_1+\omega_2+\omega_5) - \chi((p-11)\omega_1+2\omega_2) = \mathrm{Ch}\, L((p-8)\omega_1+\omega_4+2\omega_6). \quad (4)$$

- $J((p-1)\omega_1)=\mathrm{Ch}\, L((p-7)\omega_1+3\omega_6)$.

- $$J((p-8)\omega_1+\omega_4+2\omega_6)=\chi((p-9)\omega_1+\omega_3+\omega_6) \\ -\chi((p-10)\omega_1+\omega_2+\omega_5)+\chi((p-11)\omega_1+2\omega_2) \\ =\operatorname{Ch} L((p-9)\omega_1+\omega_3+\omega_6), \quad (3)$$

- $$J((p-7)\omega_1+3\omega_6)=\chi((p-8)\omega_1+\omega_4+2\omega_6) \\ -\chi((p-9)\omega_1+\omega_3+\omega_6)+\chi((p-10)\omega_1+\omega_2+\omega_5) \\ -\chi((p-11)\omega_1+2\omega_2) \\ =\operatorname{Ch} L((p-8)\omega_1+\omega_4+2\omega_6). \quad (4)$$

- $J((p-1)\omega_1)=\operatorname{Ch} L((p-7)\omega_1+3\omega_6)$.

Simple modules for E_6

- $$0 \to V((p-11)\omega_1 + 2\omega_2) \to V((p-10)\omega_1 + \omega_2 + \omega_5)$$
$$\to V((p-9)\omega_1 + \omega_3 + \omega_6) \to V((p-8)\omega_1 + \omega_4 + 2\omega_6)$$
$$\to V((p-7)\omega_1 + 3\omega_6) \to V((p-1)\omega_1) \to L((p-1)\omega_1) \to 0$$

- The characters of the $L(\mu)$ can then be found using Weyl's Character Formula. For example, the p-rank of the point-hyperplane oppositeness matrix is

$$\dim L((p-1)\omega_1) = \frac{1}{2^7.3.5.11} p(p+1)(p+3)$$
$$\times (3p^8 - 12p^7 + 39p^6 + 320p^5$$
$$- 550p^4 + 1240p^3 + 2080p^2 - 1920p + 1440)$$

Thank you for your attention!