THE MODULO 2 STRUCTURE OF RANK 3 PERMUTATION MODULES FOR ODD CHARACTERISTIC SYMPLECTIC GROUPS

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Abstract. This paper studies the permutation representation of the symplectic group $\mathrm{Sp}(2\mathrm{m}, \mathbb{F}_q)$, where q is odd, on the 1-spaces of its natural module. The complete submodule lattice for the modulo ℓ reduction of this permutation module is known for all odd primes ℓ not dividing q. In this paper we determine the complete submodule lattice for the mod 2 reduction. Similar results are then obtained for the orthogonal group $\mathrm{O}(5, \mathbb{F}_q)$.

1. Introduction

Let p be an odd prime and let $q = p^f$. Throughout the following V will be a $2m$-dimensional $\mathbb{F}_q$-vector space equipped with a non-singular alternating bilinear form $(\ , \ )$. We shall assume $m \geqslant 2$ to avoid trivial exceptions. For $1 \leqslant r \leqslant m$, let $\mathcal{L}_r$ denote the set of r-dimensional isotropic subspaces of V. Then $\mathcal{L}_1$ is the set of all 1-dimensional subspaces of V, and $\mathcal{L}_m$ is the set of all maximal isotropic subspaces of V. The group $G := \mathrm{Sp}(2\mathrm{m}, \mathbb{F}_q)$ acts transitively with rank 3 on $\mathcal{L}_1$.

Let $\mathbb{F}$ be any field of characteristic coprime to p, and let $\mathbb{F}^{\mathcal{L}_r}$ denote the $\mathbb{F}G$-permutation module on $\mathcal{L}_r$. We ask for its submodule lattice in the case where $r = 1$. This has been determined in [9] by Liebeck in all cases except when char $\mathbb{F} = 2$. In [2] Bagchi et al have conjectured the submodule structure of $\mathbb{F}_2^{\mathcal{L}_1}$ for the special case where $m = 2$.

In this paper we determine the complete $\mathbb{F}G$-submodule lattice of $\mathbb{F}^{\mathcal{L}_1}$, where $\mathbb{F}$ is a field of characteristic 2. Our approach is to first restrict the action of G to that of a maximal parabolic subgroup. The composition factors of this restricted action are determined and using a recent result [5], we are then able to determine the composition factors for the action of the full group. This puts us in a position to obtain the submodule lattice (see Theorem 2.13). Taking $m = 2$ in our work, we see that the conjecture in [2] is correct only if $q \equiv \pm 3 \bmod 8$.

In fact, for the case $m = 2$, Bagchi et al also conjectured in [2] the submodule structure of $\mathbb{F}_2^{\mathcal{L}_2}$. We are able to use our results along with results

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[10] of White and [9] of Liebeck to show that this conjecture is true only if $q \equiv \pm 3 \mod 8$ (see Theorem 3.1 and its corollary).

For simplicity, we will always work over an algebraically closed field of characteristic 2, which we denote by k.

We mention here that other rank 3 permutation modules for finite classical groups will be treated in a forthcoming paper.

2. THE SUBMODULE STRUCTURE OF $k^{\mathcal{L}_1}$

2.1. **Restriction to a maximal parabolic subgroup.** Fix a symplectic basis $e_1, \ldots, e_m, f_1, \ldots, f_m$ for V over $\mathbb{F}_q$, so that

$$(e_i, e_j) = (f_i, f_j) = 0 \text{ and } (e_i, f_j) = -(f_j, e_i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Let $M := \langle e_1, \ldots, e_m \rangle$ and $P := \langle f_1, \ldots, f_m \rangle$ be maximal isotropic subspaces of V.

Let G_M denote the set-wise stabilizer of M in G. Then

$$G_M = S \rtimes L \tag{1}$$

where

$$S = \left\{ \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \mid A = A^t, A \in \mathrm{Hom}(P, M) \right\} \tag{2}$$

and

$$L = \left\{ \begin{pmatrix} g & 0 \\ 0 & g^{-t} \end{pmatrix} \mid g \in \mathrm{GL}(M) \right\}. \tag{3}$$

Here I is the $m \times m$ identity matrix and 0 is the $m \times m$ zero matrix.

To determine the kG-composition factors of $k^{\mathcal{L}_1}$ we will first need to determine the composition factors of $\mathrm{Res}^G_{G_M}\, k^{\mathcal{L}_1}$. We start by noting that G_M has two orbits on $\mathcal{L}_1$:

$$\mathcal{O}_1 := \left\{ \omega \in \mathcal{L}_1 \,\middle|\, \omega \subset M \right\}$$

and

$$\mathcal{O}_2 := \left\{ \omega \in \mathcal{L}_1 \,\middle|\, \omega \nsubseteq M \right\}.$$

Now for any subset $X \subseteq \mathcal{L}_1$, we will let k^X denote the k-span of the elements of X. Then we have the following decomposition of $\mathrm{Res}^G_{G_M}\, k^{\mathcal{L}_1}$ as a direct sum of kG_M-submodules:

$$\mathrm{Res}^G_{G_M}\, k^{\mathcal{L}_1} = k^{\mathcal{O}_1} \oplus k^{\mathcal{O}_2}. \tag{4}$$

Thus, to determine the composition factors of $\mathrm{Res}^G_{G_M}\, k^{\mathcal{L}_1}$ we may separately study the summands in (4).

The first summand is easily handled:

For $v \in V$, write $v = \begin{pmatrix} x \\ y \end{pmatrix}$, where $x \in M$ and $y \in P$. Then

$$\langle v \rangle \in \mathcal{O}_2 \quad \text{if and only if} \quad y \neq 0. \tag{5}$$

With this notation, the computation

$$\begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + Ay \\ y \end{pmatrix} \tag{6}$$

shows that S acts trivially on $\mathcal{O}_1$, i.e. S acts trivially on $k^{\mathcal{O}_1}$. ¿From (1), (2), and (3) we see that the induced action of $G_M/S \simeq \mathrm{GL}(M)$ on $k^{\mathcal{O}_1}$ is the usual action of $\mathrm{GL}(M)$ on the 1-spaces of M. Thus, the kG_M-submodule lattice of $k^{\mathcal{O}_1}$ is known from [8]. Explicitly, if we put

$$\mathbf{1}_{\mathcal{O}_1} := \sum_{\omega \in \mathcal{O}_1} \omega \in k^{\mathcal{O}_1}$$

and

$$\mathcal{K} := \langle \omega - \alpha \ \mid \ \omega, \alpha \in \mathcal{O}_1 \rangle_k \,,$$

where $\langle\ \rangle_k$ denotes k-span, then we have

Lemma 2.1. *(a) If m is odd, then $\mathcal{K}$ is simple and*

$$k^{\mathcal{O}_1} = \langle \mathbf{1}_{\mathcal{O}_1} \rangle_k \ \oplus \ \mathcal{K}$$

(b) If m is even, then $k^{\mathcal{O}_1}$ is uniserial with composition series

$$\begin{array}{c} k^{\mathcal{O}_1} \\ | \\ \mathcal{K} \\ | \\ \langle \mathbf{1}_{\mathcal{O}_1} \rangle_k \\ | \\ \{0\} \end{array}$$

□

In the situation of Lemma 2.1.b, put $\mathcal{K}' := \mathcal{K} / \langle \mathbf{1}_{\mathcal{O}_1} \rangle_k$.

We will indicate the composition factors of $k^{\mathcal{O}_1}$ informally by writing

$$k^{\mathcal{O}_1} = \begin{cases} k + \mathcal{K} & \text{if } m \text{ is odd} \\ (2)k + \mathcal{K}' & \text{if } m \text{ is even} \end{cases} \tag{7}$$

Of course, here k denotes the simple trivial module.

To determine the kG_M-composition factors of the second summand in (4), we will once again begin by restricting the action of G_M to that of its normal subgroup S, i.e. first we will determine the composition factors of $\mathrm{Res}_S^{G_M}\ k^{\mathcal{O}_2}$. We will then use Clifford's theorem along with a result of Higman's (see [6]) to recover the G_M-composition factors.

2.2. **The composition factors of** $\mathrm{Res}_S^{G_M}\ k^{\mathcal{O}_2}$**.** Using elementary linear algebra we see that given any non-zero $y \in P$ and any $z \in M$ we can always find a symmetric transformation $A \in \mathrm{Hom}(P, M)$ which sends y to z. Therefore, it follows from (6) that the S-orbits on $\mathcal{O}_2$ are indexed by the 1-spaces in P. Explicitly, let $\langle y_1 \rangle, \ldots, \left\langle y_{\frac{q^m-1}{q-1}} \right\rangle$ be a list of the 1-spaces in P. Then the S-orbits on $\mathcal{O}_2$ are the sets

$$\mathcal{O}_{\langle y_i \rangle} := \left\{ \left\langle \begin{pmatrix} x \\ y_i \end{pmatrix} \right\rangle \mid x \in M, \right\}.$$

Thus, we have the following decomposition of $\mathrm{Res}_S^{G_M}\ k^{\mathcal{O}_2}$ as a direct sum of kS-submodules:

$$\mathrm{Res}_S^{G_M}\ k^{\mathcal{O}_2} = \bigoplus_{i=1}^{\frac{q^m-1}{q-1}} k^{\mathcal{O}_{\langle y_i \rangle}} \tag{8}$$

Let $S_{y_i} \leqslant S$ be the stabilizer of $\left\langle \begin{pmatrix} 0 \\ y_i \end{pmatrix} \right\rangle \in O_{\langle y_i \rangle}$. So

$$S_{y_i} = \left\{ \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \in S \ \middle|\ Ay_i = 0 \right\}.$$

Then

$$k^{\mathcal{O}_{\langle y_i \rangle}} = \mathrm{Ind}_{S_{y_i}}^S k,$$

so that by (8) we may write

$$\mathrm{Res}_S^{G_M}\ k^{\mathcal{O}_2} = \bigoplus_{i=1}^{\frac{q^m-1}{q-1}} \mathrm{Ind}_{S_{y_i}}^S k. \tag{9}$$

We now pause to establish a correspondence between the irreducible kS-characters and the symmetric bilinear forms on M. This correspondence will be the key to determining the composition factors of $\mathrm{Res}_S^{G_M}\ k^{\mathcal{O}_2}$.

We start by noting that

$$P \simeq V/M = V/M^{\perp} \simeq M^*,$$

(where M^* denotes the dual space of M) so we may identify P with M^*. If we also identify M with $(M^*)^*$, then we can identify $\mathrm{Hom}(P, M)$ with $\mathrm{Hom}(M^*, (M^*)^*)$, i.e. we may regard $\mathrm{Hom}(P, M)$ as the set of all bilinear forms on M^*.

The correspondence

$$\begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \mapsto A \tag{10}$$

then identifies S with the set of symmetric bilinear forms on M^*. Under this identification S_{y_i} corresponds to the set of all symmetric bilinear forms on M^* which have y_i in their radical, i.e. S_{y_i} corresponds to the symmetric bilinear forms on $(Ker\ y_i)^*$.

Now let ζ be a primitive p-th root of unity in algebraically closed k. The corresondence

$$f(\cdot) \mapsto \zeta^{Trace_{\mathbb{F}_q/\mathbb{F}_p}(f(\cdot))} \tag{11}$$

allows us to identify the linear functionals on the $\mathbb{F}_q$-vector space S with the irreducible k-characters of the elementary abelian p-group S. Since S is the set of symmetric bilinear forms on M^*, we see that S^* is the set of symmetric bilinear forms on M. Thus, we may identify the irreducible characters of S with the symmetric bilinear forms on M.

Remark 2.2. Let N be an irreducible submodule of $\mathrm{Ind}_{S_{y_i}}^S\ k$ and let $f \in S^*$ be the linear functional which corresponds under (11) to the character of N. By Frobenius reciprocity, we know that S_{y_i} acts trivially on N. This means that $Trace_{\mathbb{F}_q/\mathbb{F}_p}(f(A)) = 0$ for every $A \in S_{y_i}$, from which it follows that $f(A) = 0$ for all $A \in S_{y_i}$. But as S_{y_i} is the set of symmetric bilinear forms on $(Ker\ y_i)^*$, this means that the symmetric bilinear form on M which corresponds to f must be isotropic on the hyperplane $Ker\ y_i \subset M$.

Thus, the irreducible characters in $\mathrm{Ind}_{S_{y_i}}^S k$ are the symmetric bilinear forms on M which are isotropic on $Ker\ y_i$. Again using Frobenius reciprocity, we see that each such form occurs with multiplicity one. In particular, the zero form (which corresponds to the trivial character) occurs exactly once in each $\mathrm{Ind}_{S_{y_i}}^S k$. In fact, it is easily seen that the unique trivial submodule of $\mathrm{Ind}_{S_{y_i}}^S k$ is

$$T_i := \left\langle \mathbf{1}_{\mathcal{O}_{\langle y_i \rangle}} \right\rangle_k \tag{12}$$

where

$$\mathbf{1}_{\mathcal{O}_{\langle y_i \rangle}} := \sum_{\omega \in \mathcal{O}_{\langle y_i \rangle}} \omega \in k^{\mathcal{O}_{\langle y_i \rangle}}. \tag{13}$$

Now let B be a non-zero symmetric bilinear form on M which has an isotropic hyperplane. Then B has either rank 1 or 2. If B has rank 1, then the radical of B, denoted by Rad B, is the unique isotropic hyperplane for B. If B has rank 2 then $M/\mathrm{Rad}\ B$ is hyperbolic and therefore has precisely two isotropic lines for the form induced from B, i.e. M has precisely two isotropic hyperplanes for B.

In light of (9), the above then gives us all of the composition factors of $\mathrm{Res}_S^{G_M}\ k^{\mathcal{O}_2}$. We record this information as

Lemma 2.3. *Under the identification in (11),* $\mathrm{Res}_S^{G_M}\ k^{\mathcal{O}_2}$ *has the following composition factors:*

(a) The zero form, i.e. the trivial character, which occurs with multiplicity $\frac{q^m-1}{q-1}$.
(b) The rank 1 symmetric bilinear forms, where each occurs with multiplicity 1.
(c) The rank 2 symmetric bilinear forms having isotropic hyperplane, where each occurs with multiplicty 2.

2.3. **The kG_M-composition factors of $k^{\mathcal{O}_2}$.** We start by examining the S-fixed points of $k^{\mathcal{O}_2}$. Define

$$T := \bigoplus_{i=1}^{\frac{q^m-1}{q-1}} T_i \tag{14}$$

where the T_i are as in (12). Now it is easily seen from (1) that $G_M/S \simeq \mathrm{GL}(M)$ permutes the vectors in (13) in the usual way that $\mathrm{GL}(M)$ acts on the 1-spaces of M^*, i.e. in the usual way that $\mathrm{GL}(M)$ acts on the hyperplanes of M. Thus, if we let $\mathcal{L}_{m-1}$ denote the set of hyperplanes in M, then the kG_M-module T can be naturally identified with the $k\mathrm{GL}(M)$-module $k^{\mathcal{L}_{m-1}}$.

It is well-known that the permutation modules on the 1-spaces and the hyperplanes, respectively, of M are isomorphic over a field of characteristic zero. Therefore, from a general principle of modular representation theory (see [4], Theorem 17.7) we know that $k^{\mathcal{L}_{m-1}}$ and $k^{\mathcal{O}_1}$ have the same composition factors. Therefore, it follows from (7) that

$$T = \begin{cases} k + \mathcal{K} & \text{if } m \text{ is odd} \\ (2)k + \mathcal{K}' & \text{if } m \text{ is even} \end{cases} \tag{15}$$

We remark here that it can actually be shown that $k^{\mathcal{L}_{m-1}}$ and $k^{\mathcal{O}_1}$ are isomorphic for G.

To find the remaining composition factors, we now consider the action of G_M on the irreducible characters of S. We start by observing that as S acts trivially on its characters, we need only consider the induced action of $G_M/S \simeq \mathrm{GL}(M)$. Now $\mathrm{GL}(M)$ acts by congruence transformations on S. Therefore, if we view the elements of S^* as symmetric matrices, then the action of $\mathrm{GL}(M)$ is again by congruence transformations. We then see that under correspondence (11), $\mathrm{GL}(M)$ acts by congruence transformations on the characters of S.

There are two $\mathrm{GL}(M)$ congruence classes of rank 1 symmetric bilinear forms, represented by

$$diag(1, 0, \ldots, 0) \tag{16}$$

and

$$diag(\alpha, 0, \ldots, 0) \tag{17}$$

where α is a non-square in $\mathbb{F}_q^\times$ (see [1]). The stabilizer of both classes is

$$\left\{ \begin{pmatrix} \pm 1 & 0 \\ * & * \end{pmatrix} \right\}$$

which has index $\frac{q^m-1}{2}$ in $\mathrm{GL}(M)$.

Let B_1 denote the congruence class of (16) and B_α denote the congruence class of (17). Let W_1 denote the external direct sum of the S-characters which correspond to the forms in B_1, and let W_2 denote the external direct sum of the S-characters which correspond to the forms in B_α. Then it follows from Lemma 2.3.b and Clifford's Theorem [4] that $k^{\mathcal{O}_2}$ has composition factors, call them $\mathcal{W}_1$ and $\mathcal{W}_2$, which when restricted to S are isomorphic to W_1 and W_2, respectively. Note that

$$dim_k\ \mathcal{W}_1 = dim_k\ \mathcal{W}_2 = \frac{q^m-1}{2} \tag{18}$$

but $\mathcal{W}_1 \ncong \mathcal{W}_2$.

Now, there is one congruence class of rank 2 symmetric bilinear forms having isotropic hyperplane, represented by

$$\begin{pmatrix} \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) & 0 \\ 0 & 0 \end{pmatrix}. \tag{19}$$

The stabilizer in $\mathrm{GL}(M)$ of this class is

$$\left\{ \begin{pmatrix} C & 0 \\ * & * \end{pmatrix} \right\}$$

where C is a 2×2 monomial matrix. This subgroup has index $\frac{q(q^m-1)(q^{m-1}-1)}{2(q-1)}$ in $\mathrm{GL}(M)$.

Let D denote the external direct sum of the S-characters which correspond to these forms. Note that $dim_k\ \mathsf{D} = \frac{q(q^m-1)(q^{m-1}-1)}{2(q-1)}$. It then follows from Lemma 2.3.c and Clifford's Theorem that exactly one of the following cases holds for $k^{\mathcal{O}_2}$:

Case A: $k^{\mathcal{O}_2}$ has precisely two composition factors, call them $\mathcal{D}_1$ and $\mathcal{D}_{-1}$, which when restricted to S are isomorphic to D.

Case B: $k^{\mathcal{O}_2}$ has a single composition factor, call it $\mathcal{D}_0$, which when restricted to S is isomorphic to $\mathsf{D} \oplus \mathsf{D}$.

We now show that the former is true. We start by establishing some notation which we will use throughout the remainder of the paper:

For any field $\mathbb{F}$, we let $\mathbb{F}^{\mathcal{L}_r}$ denote, as usual, the $\mathbb{F}G$-permutation module on $\mathcal{L}_r$. Let

$$\eta_{r,s} : \mathbb{F}^{\mathcal{L}_r} \longrightarrow \mathbb{F}^{\mathcal{L}_s}$$

be the $\mathbb{F}G$-module homomorphism which sends each isotropic r-space to the (formal) sum of the isotropic s-spaces which are incident with it.

Define

$$\mathbf{1} := \sum_{\omega \in \mathcal{L}_1} \omega \in \mathbb{F}^{\mathcal{L}_1}.$$

For $\omega \in \mathcal{L}_1$ put

$$\Delta(\omega) := \{\alpha \in \mathcal{L}_1 \mid \alpha \nsubseteq \omega^\perp\},$$

and define an element $s_{\Delta(\omega)} \in \mathbb{F}^{\mathcal{L}_1}$ as follows:

$$s_{\Delta(\omega)} := \sum_{\alpha \in \Delta(\omega)} \alpha. \tag{20}$$

Define a non-singular symmetric bilinear form $[-,-]_{\mathbb{F}}$ by demanding that the elements of $\mathcal{L}_1$ form an orthonormal basis. For any subset $S \subseteq \mathbb{F}^{\mathcal{L}_1}$ put

$$S^\perp := \left\{ v \in k^{\mathcal{L}_1} \;\; | \;\; [v,s]_{\mathbb{F}} = 0, \text{ for all } s \in S \right\}$$

Note that we have used the same notation for orthogonal complements in V, but no confusion should arise. Note also that if S is a $\mathbb{F}G$-submodule of $\mathbb{F}^{\mathcal{L}_1}$, then so is $S^\perp$.

Now let $\mathbb{Q}_2$ denote the field of 2-adic numbers and let $\overline{\mathbb{Q}_2}$ be its algebraic closure. Then $\mathcal{F}$ will be the maximal unramified extension of $\mathbb{Q}_2$ in $\overline{\mathbb{Q}_2}$ (see [7], pg.37), and $\mathcal{R}$ will be the valuation ring of $\mathcal{F}$. Note that $\mathcal{F}$ has residue field k. ¿From [6] we have that

$$\mathcal{F}^{\mathcal{L}_1} = \langle \mathbf{1} \rangle_{\mathcal{F}} \;\oplus\; M_{-1} \;\oplus\; M_1, \tag{21}$$

where $M_{\pm 1}$ are irreducible $\mathcal{F}G$-submodules with

$$dim_{\mathcal{F}}\; M_{-1} = \frac{q(q^m-1)(q^{m-1}+1)}{2(q-1)} \tag{22}$$

and

$$dim_{\mathcal{F}}\; M_1 = \frac{q(q^m+1)(q^{m-1}-1)}{2(q-1)}. \tag{23}$$

Let $\overline{M_{\pm 1}}$ be the reductions modulo 2 of $M_{\pm 1}$. Their restrictions to G_M must be collections of the composition factors described above. By (22) and (23) the dimensions of the composition factors of $\overline{M_{\pm 1}}$ add up to $\frac{q(q^m \pm 1)(q^{m-1} \mp 1)}{2(q-1)}$. Assume now that $(m,q) \neq (2,3)$. Then

$$\frac{q(q^m \pm 1)(q^{m-1} \mp 1)}{2(q-1)} < 2(dim_k\; \mathsf{D}). \tag{24}$$

So it cannot be that either $\mathrm{Res}^G_{G_M}\; \overline{M_{\pm 1}}$ contains a composition factor which when restricted to S is isomorphic to $\mathsf{D} \oplus \mathsf{D}$. Thus, we deduce that Case A holds. If $(m,q) = (2,3)$, then $dim_{\mathcal{F}}\; M_{-1} = 2(dim_k\; \mathsf{D})$. However, it is easy to see (e.g. by considering degrees) that M_{-1} is the unique non-trivial composition factor which is common to both $\mathcal{F}^{\mathcal{L}_1}$ and $\mathcal{F}^{\mathcal{L}_2}$. Since $\mathcal{F}^{\mathcal{L}_2} = \mathrm{Ind}^G_{G_M}\; \mathcal{F}$, it then follows from Frobenius reciprocity that G_M (and hence

S) has a non-zero fixed point on M_{-1}. This then implies that $\mathrm{Res}^G_{G_M}\ \overline{M_{-1}}$ contains a trivial composition factor. Since S has no fixed points on $\mathsf{D} \oplus \mathsf{D}$, we deduce that Case A holds in this case as well. We mention here that it will be shown in §2.4 (see (32)) that $\mathcal{D}_1 \simeq \mathcal{D}_{-1}$.

Hence, we have found all the kG_M-composition factors of $k^{\mathcal{O}_2}$. Combining this information with Lemma 2.1 and using the informal notation of (7) and (15), we may now state

Lemma 2.4. *(a) If m is odd, then*

$$\mathrm{Res}^G_{G_M}\ k^{\mathcal{L}_1} = (2)k + (2)\mathcal{K} + \mathcal{W}_1 + \mathcal{W}_2 + \mathcal{D}_1 + \mathcal{D}_{-1}.$$

(b) If m is even, then

$$\mathrm{Res}^G_{G_M}\ k^{\mathcal{L}_1} = (4)k + (2)\mathcal{K}' + \mathcal{W}_1 + \mathcal{W}_2 + \mathcal{D}_1 + \mathcal{D}_{-1}.$$

2.4. **The kG-composition factors of $k^{\mathcal{L}_1}$.** Let $\mathcal{F}, \mathcal{R}, M_{\pm 1}$, and $\overline{M_{\pm 1}}$ be as in §2.3. It follows from (24) and the remarks following it that each of $\mathcal{D}_{\pm 1}$ occurs in exactly one of $\mathrm{Res}^G_{G_M}\ \overline{M_{\pm 1}}$, and that the $\mathcal{D}_{\pm 1}$ do not occur together. Thus, we may assume that our notation is chosen so that $\mathcal{D}_{\pm 1}$ is a composition factor of $\mathrm{Res}^G_{G_M}\ \overline{M_{\pm 1}}$. Also, since

$$\begin{aligned} dim_k\ \overline{M_1} - dim_k\ \mathcal{D}_1 &= \frac{q(q^m+1)(q^{m-1}-1)}{2(q-1)} - \frac{q(q^m-1)(q^{m-1}-1)}{2(q-1)} \\ &= \frac{q^m-1}{q-1} - 1 \\ &< \frac{q^m-1}{2} \\ &= dim_k\ \mathcal{W}_1 = dim_k\ \mathcal{W}_2, \end{aligned}$$

we deduce upon inspection of Lemma 2.4 that

$$\mathrm{Res}^G_{G_M}\ \overline{M_1} = \begin{cases} \mathcal{K} + \mathcal{D}_1 & (m \text{ odd}) \\ k + \mathcal{K}' + \mathcal{D}_1 & (m \text{ even}) \end{cases} \tag{25}$$

Suppose for the sake of contradiction that $\overline{M_1}$ has a kG-composition factor, call it K, which when restricted to G_M is isomorphic to $\mathcal{K}$. Since S acts trivially on $\mathcal{K}$, it is contained in the kernel, call it J, of the representation of G on K. Since S is not contained in the center of G, and since the center of G is the only non-trivial normal subgroup of G, we deduce that J must be all of G. But G_M acts non-trivially on $\mathcal{K}$, a contradiction. It follows that $\overline{M_1}$ has no such composition factor for G, and therefore $\overline{M_1}$ *is irreducible if m is odd.* If m is even, then similar reasoning allows us to conclude that either $\overline{M_1}$ is irreducible or else $\overline{M_1} = k + X$, with X irreducible. We now show that the latter is true.

Using the notation in (20), we define a kG-module homomorphism

$$\varphi : k^{\mathcal{L}_1} \to k^{\mathcal{L}_1} \tag{26}$$

by

$$\omega \mapsto \omega + s_{\Delta(\omega)},$$

where $\omega \in \mathcal{L}_1$. Put

$$U := Im\ \varphi. \tag{27}$$

Now define

$$U' := \langle u_1 + u_2 \mid u_1, u_2 \in U \rangle_k . \tag{28}$$

¿From [9] we have the following non-trivial fact:

Lemma 2.5. *(Theorem 1.1 of* [9]*) Every kG-submodule of $k^{\mathcal{L}_1}$ not contained in $\langle \mathbf{1} \rangle_k$ must contain U'.*

□

It is easily seen that

$$M_1 \cap \mathcal{R}^{\mathcal{L}_1}$$

is an $\mathcal{R}$-form of M_1 and a pure $\mathcal{R}G$-submodule of $\mathcal{R}^{\mathcal{L}_1}$. Therefore,

$$\overline{M_1 \cap \mathcal{R}^{\mathcal{L}_1}}$$

is a mod 2 reduction of M_1 as well as a kG-submodule of $k^{\mathcal{L}_1}$. Since $\overline{M_1 \cap \mathcal{R}^{\mathcal{L}_1}}$ is certainly not contained in $\langle \mathbf{1} \rangle_k$, we see from Lemma 2.5 that

$$U' \subseteq \overline{M_1 \cap \mathcal{R}^{\mathcal{L}_1}},$$

and therefore $\overline{M_1}$ contains the composition factors of U'.

We require the following result:

Lemma 2.6. *If m is even, then $\langle \mathbf{1} \rangle_k \subset U'$.*

Proof. Let $M \in L_m$. Then an easy computation shows that

$$\sum_{\substack{\omega \subseteq M \\ \omega \in \mathcal{L}_1}} \omega + s_{\Delta(\omega)} = \mathbf{1}. \tag{29}$$

Now the number of 1-spaces in M is $\frac{q^m - 1}{q-1}$, which is an even number since m is even. Thus we may group the summands in the left-hand side of (29) into pairs. The result then follows from the definition (28) of U'.

□

Since $\overline{M_1 \cap \mathcal{R}^{\mathcal{L}_1}}$ has at most 2 composition factors, it follows from Lemma 2.6 that

$$\overline{M_1 \cap \mathcal{R}^{\mathcal{L}_1}} = U'.$$

We may summarize the above as

Lemma 2.7. $\overline{M_1}$ *and* U' *have the same composition factors.*
(a) If m *is odd, then* U' *is simple.*
(b) If m *is even, then* U' *is uniserial with composition series*

$$\begin{array}{c} U' \\ | \\ \langle \mathbf{1} \rangle_k \\ | \\ \{0\} \end{array}$$

□

In the situation of Lemma 2.7.b, we put $U'/\langle \mathbf{1} \rangle_k := X$. In the situation of Lemma 2.7.a, we will use X' to denote the composition factor isomorphic to U'.

In view of Lemma 2.7, we have only to determine the composition factors of $\overline{M_{-1}}$. We do this now:

A simple matrix computation shows that

$$\varphi^2 = 0. \tag{30}$$

Thus,

$$U \subseteq Ker\ \varphi, \tag{31}$$

where U is as in (27).

Since φ is symmetric, we see that $Ker\ \varphi = U^{\perp}$. It then follows that

$$U \simeq k^{\mathcal{L}_1}/Ker\ \varphi = k^{\mathcal{L}_1}/U^{\perp} \simeq U^*,$$

i.e. U is self-dual. So from the structure of U' given in Lemma 2.7 we deduce

Lemma 2.8. *(a) If* m *is odd, then*

$$U = \langle \mathbf{1} \rangle_k \oplus U'.$$

(b) If m *is even, then* U *is uniserial with composition series*

$$\begin{array}{c} U \\ | \\ U' \\ | \\ \langle \mathbf{1} \rangle_k \\ | \\ \{0\} \end{array}$$

□

We next observe that $\overline{M_{-1}}$ has the same composition factors as $k^{\mathcal{L}_1}/U$. But since $U \subset U^{\perp}$, and since $k^{\mathcal{L}_1}/U^{\perp} \simeq U$, we see that $k^{\mathcal{L}_1}/U$ contains the composition factors of U. We pause now to note that this implies that $\mathcal{D}_1$

is a composition factor of $\mathrm{Res}^G_{G_M}\ \overline{M_{-1}}$. Since $\mathcal{D}_{-1}$ is the only composition factor of $\mathrm{Res}^G_{G_M}\ \overline{M_{-1}}$ of the same dimension as $\mathcal{D}_1$, we deduce that

$$\mathcal{D}_1 \simeq \mathcal{D}_{-1}, \tag{32}$$

as was promised in §2.3.

It remains to determine the kG-composition factors of $U^{\perp}/U$. By inspecting Lemma 2.4, we see that

$$\mathrm{Res}^G_{G_M}\ U^{\perp}/U = \mathcal{W}_1 + \mathcal{W}_2.$$

We now show that $\overline{M_{-1}}$ has kG-composition factors, call them W_1 and W_2, which when restricted to G_M are isomorphic to $\mathcal{W}_1$ and $\mathcal{W}_2$. We will need to consider the conformal symplectic group

$$\mathrm{CSp}(2\mathrm{m}, q) := \left\{ T \in GL(V)\ \mid \exists\ \alpha \in \mathbb{F}_q^{\times} \text{ so that } (Tv, Tw) = \alpha(v,w),\ \forall v, w \in V \right\}.$$

For brevity, we put $\Gamma := \mathrm{CSp}(2\mathrm{m}, q)$. Then $\Gamma \simeq G \rtimes \mathbb{F}_q^{\times}$ and it is easy to see that U is a module for Γ. Therefore, $U^{\perp}/U$ is also a $k\Gamma$-module.

We claim that $U^{\perp}/U$ is simple for Γ. Suppose not. Then it follows that $U^{\perp}/U$ has $k\Gamma$-composition factors, call them $\widehat{\mathcal{W}_1}$ and $\widehat{\mathcal{W}_2}$, which when restricted to G_M are isomorphic to $\mathcal{W}_1$ and $\mathcal{W}_2$, respectively, and (hence) when restricted to S are isomorphic to W_1 and W_2, respectively. Since W_1 and W_2 are not isomorphic as kS-modules, we see that the following result then leads to a contradiction:

Lemma 2.9. *The kS-modules W_1 and W_2 are conjugate for Γ.*

Proof. Let $\beta \in \mathbb{F}_q$ be a non-square and consider the element $\tilde{g} \in \Gamma$ whose matrix representation with respect to the basis in §2.1 is

$$\tilde{g} := \begin{pmatrix} \beta I & 0 \\ 0 & I \end{pmatrix} \in N_{\Gamma}(S),$$

where $N_{\Gamma}(S)$ denotes the normalizer of S in Γ. If $h := \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \in S$, then an easy computation shows that

$$\tilde{g} h \tilde{g}^{-1} = \begin{pmatrix} I & \beta A \\ 0 & I \end{pmatrix},$$

i.e. $\tilde{g}$ acts as multiplication by β on S. It then follows that $\tilde{g}$ acts (on the left) on S^* as multiplication by β^{-1}. Under the identification in (11), this means that $\tilde{g}$ acts as multiplication by β^{-1} on the characters of S. Taking $\beta = \alpha$, where α is as in (17), it is now easy to see that the conjugate by $\tilde{g}$ of the form in (17) is the form in (16). The result now follows from the construction of the $W_i's$ in §2.3. □

Thus, $U^{\perp}/U$ is simple for Γ, and it follows from Clifford's theorem that $U^{\perp}/U$ is semi-simple for G. Now, either $U^{\perp}/U$ is a simple kG-module, or else $U^{\perp}/U \simeq W_1 \oplus W_2$, where W_1 and W_2 are simple kG-modules which when restricted to G_M are isomorphic to $\mathcal{W}_1$ and $\mathcal{W}_2$, respectively.

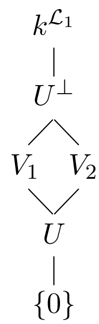
Consider the following result from [5]:

Lemma 2.10. *(Theorem 2.1,* [5]*)*
Any irreducible kG-module of dimension less than $\frac{(q^m-1)(q^m-q)}{2(q+1)}$ is either the trivial module, or a Weil module of dimension $\frac{(q^m\pm1)}{2}$.

This result shows that we must have $U^\perp/U \simeq W_1 \oplus W_2$, where W_1 and W_2 are irreducible Weil modules of dimension $\frac{q^m-1}{2}$. Hence, we now have all of the kG-composition factors of $k^{\mathcal{L}_1}$.

If we let V_1 and V_2 be submodules such that $U \subset V_1, V_2 \subset U^\perp$ and $V_1/U \simeq W_1$ and $V_2/U \simeq W_2$, then the above arguments yield the following filtration of $k^{\mathcal{L}_1}$:

(33)



2.5. **The kG-submodule lattice of $k^{\mathcal{L}_1}$.** By the minimality of U' (see Lemma 2.5) it suffices to determine the submodule structure of $(U')^\perp/U'$. We start by defining submodules C and C^+ as follows:

$$C := Im\ \eta_{m,1}$$

and

$$C^+ := \langle x + y \ \mid\ x, y \in C\rangle_k \,.$$

We will need the following results:

Lemma 2.11. *(a)* $C^+ \subsetneq C$
(b) $\mathrm{Hom}_{kG}(k^{\mathcal{L}_m}, W_i) = \{0\}$, *for* $i = 1, 2$.
(c) C has no quotient isomorphic to W_i, for $i = 1, 2$.
(d) $\mathrm{Hom}_{kG}(k^{\mathcal{L}_m}, k) \simeq k$
(e) C^+ is the unique maximal submodule of C.

Proof. An easy computation shows that

$$[\omega + s_{\Delta(\omega)},\ \eta_{m,1}(M)]_k = 1,$$

for all $\omega \in \mathcal{L}_1$ and all $M \in L_m$. From the definition of C^+, we then deduce that $C^\perp \subsetneq (C^+)^\perp$. Now (a) follows by taking orthogonal complements.

We have $\mathrm{Res}^G_S\ W_i \simeq \mathsf{W}_i$, for $i = 1, 2$. But from our construction of the W_i in §2.3, we know that they are fixed point free for S. Therefore, G_M has no fixed points on the $\mathrm{Res}^G_{G_M}\ W_i \simeq \mathcal{W}_i$, i.e.

$$\mathrm{Hom}_{kG_M}(k, \mathrm{Res}^G_{G_M}\ W_i) = \{0\},$$

for $i = 1, 2$. Since

$$k^{\mathcal{L}_m} = \mathrm{Ind}^G_{G_M}\ k,$$

the assertions in (b) follow from Frobenius reciprocity. Since C is a homomorphic image of $k^{\mathcal{L}_m}$, we see that (c) is an immediate consequence of (b).

Again by Frobenius reciprocity, we have

$$\mathrm{Hom}_{kG}(k^{\mathcal{L}_m}, k) \simeq \mathrm{Hom}_{kG_M}(k, k) \simeq k.$$

This proves part (d).

It follows from (a) and (d) that C^+ is the unique maximal submodule of C with trivial quotient. ¿From Lemma 2.5, we have $U' \subseteq C$. Using the inner product computation at the start of the proof, we have $C \subseteq (U')^\perp$. Thus,

$$U' \subseteq C \subseteq (U')^\perp.$$

Since

$$(U')^\perp/(U') = k + k + W_1 + W_2,$$

we know that any maximal submodule of C with non-trivial quotient must have quotient W_1 or W_2, which is impossible by (c). Then (e) follows.

□

Since C^+ is not orthogonal to C, we get

$$C \cap C^\perp \subsetneq C,$$

and hence

$$C \cap C^\perp \subsetneq C^+.$$

by Lemma 2.11.e. Thus, the quotient $C/(C \cap C^\perp)$ has at least 2 composition factors. Furthermore, $C/(C \cap C^\perp)$ has a unique maximal submodule, namely $C^+/(C \cap C^\perp)$.

Lemma 2.12. *(a) $C/(C \cap C^\perp)$ is self-dual.*
(b) $C/(C \cap C^\perp)$ has a unique maximal submodule and a unique simple submodule. Both the head and socle of $C/(C \cap C^\perp)$ are trivial.

Proof. The form induced by $[-,-]_k$ on the quotient $C/(C \cap C^\perp)$ is nonsingular and therefore induces an isomorphism between $C/(C \cap C^\perp)$ and its dual. Since the form is G-invariant, this is actually a kG-isomorphism, and (a) follows. Part (b) then follows immediately from the remarks following Lemma 2.11.

□

In light of Lemma 2.9, it follows from Clifford's theorem that any $k\Gamma$-module having at least one of the W_i as a composition factor for G must have the other as well. Since C and $C^\perp$ are modules for Γ, we deduce from Lemma 2.12 that either

$$C/(C \cap C^{\perp}) = k + k \tag{34}$$

or

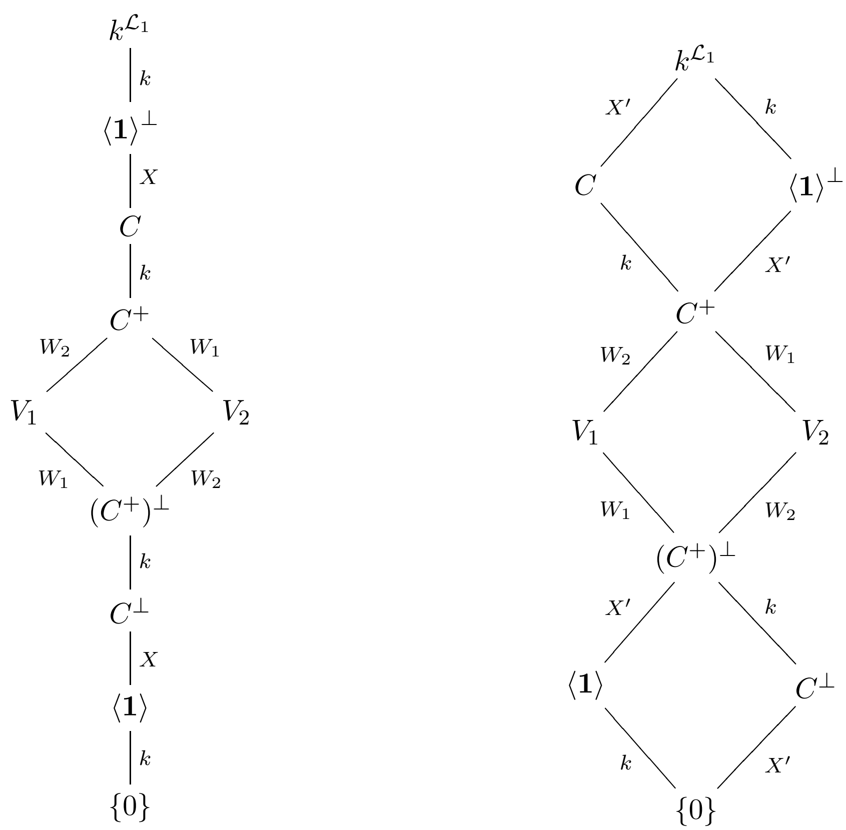
$$C/(C \cap C^{\perp}) = k + k + W_1 + W_2. \tag{35}$$

Suppose by way of contradiction that (34) holds. By Lemma 2.12.b it must then be the case that $C/(C \cap C^{\perp})$ is uniserial. But as G is perfect, it has no module which is a non-split extension of the simple trivial module by itself. So (35) holds and it follows that $C = (U')^{\perp}$ and $C^{+} = U^{\perp}$.

We may now state our main result:

Theorem 2.13. *Using the above notation, $k^{\mathcal{L}_1}$ has the following submodule lattice:*

<u>*m even*</u> <u>*m odd*</u>



Proof. Let N be a kG-submodule of $k^{\mathcal{L}_1}$. Assume $N \neq \{0\}$ or $\langle \mathbf{1} \rangle$. Then we know from Lemma 2.5 that $U' \subseteq N$. If we assume that $N \neq k^{\mathcal{L}_1}$ or $\langle \mathbf{1} \rangle^{\perp}$, then we have that $N^{\perp} \neq \{0\}$ or $\langle \mathbf{1} \rangle$. But then from Lemma 2.5 we have

$U' \subseteq N^{\perp}$, i.e. $N \subseteq (U')^{\perp}$. Thus,

$$U' \subseteq N \subseteq (U')^{\perp}.$$

¿From the remarks immediately following Lemma 2.12, we know that $U' = C^{\perp}$ and $U^{\perp} = C^{+}$. Thus, if $N \neq U'$ or $(U')^{\perp}$, it follows from Lemma 2.12.b that

$$U \subseteq N \subseteq U^{\perp}.$$

But as $U^{\perp}/U \simeq W_1 \oplus W_2$, and since $W_1 \not\cong W_2$, we see that V_1 and V_2 are the only kG-submodules between U and $U^{\perp}$, i.e. $N = V_1$ or V_2. □

Although the dimensions of the submodules pictured above have been given earlier, for convenience we recall here that

$$dim_k\ C = 1 + \frac{q(q^m - 1)(q^{m-1} + 1)}{2(q - 1)}$$

and

$$dim_k\ V_1 = dim_k\ V_2 = \frac{q^{2m} - 1}{2(q - 1)}.$$

It has already been noted (see the comments immediately following Lemma 2.9) that $C^{+}/(C^{+})^{\perp}$ is a simple $k\Gamma$-module. In the sequel, we shall denote this simple quotient by W. Since all of the kG-submodules of $k^{\mathcal{L}_1}$ except for the V_i are also $k\Gamma$-submodules, we then have

Corollary 2.14. The pictures in Theorem 2.13 are the Hasse diagrams for $\Gamma = \mathrm{CSp}(2\mathrm{m}, q)$, except that the quotient $W = C^{+}/(C^{+})^{\perp}$ is irreducible. □

By abuse of notation, we shall also denote by X and X' the Γ-composition factors which when restricted to G are isomorphic to the composition factors X and X', respectively, which are mentioned above. However, we caution the reader that these G-modules need not have unique extensions to Γ-modules.

Remark 2.15. ¿From [5] we know that the Weil modules can be realized over $\mathbb{F}_2$ if and only if $q \equiv \pm 1 \bmod 8$. If $q \equiv \pm 3 \bmod 8$, then the smallest field of definition for the Weil modules is $\mathbb{F}_4$. With this insight, we may deduce from Theorem 2.13 the complete kG-submodule lattice of $\mathbb{F}^{\mathcal{L}_1}$ for any field $\mathbb{F}$ of characteristic 2. Explicitly, if $q \equiv \pm 1 \bmod 8$ and $\mathbb{F}$ is arbitrary, or if $q \equiv \pm 3 \bmod 8$ and $\mathbb{F}_4 \subseteq \mathbb{F}$, then the submodule lattice of $\mathbb{F}^{\mathcal{L}_1}$ is as pictured in Theorem 2.13. However, if $q \equiv \pm 3 \bmod 8$ and $\mathbb{F}_4 \not\subseteq \mathbb{F}$, then the submodule lattice is as pictured in Theorem 2.13 except that the quotient $C^{+}/(C^{+})^{\perp}$ is irreducible.

Remark 2.16. In [2] (see pg. 353), Bagchi et al conjectured the submodule lattice of $\mathbb{F}_2^{\mathcal{L}_1}$ for G. We now see from Theorem 2.13 and the preceding remark that their conjectured structure is correct in all cases except when $q \equiv \pm 1 \bmod 8$, in which case the Weil modules have been neglected. However, their structure is correct for the conformal group Γ.

3. THE $\mathrm{Sp}(4,q)$-SUBMODULE STRUCTURE OF $k^{\mathcal{L}_2}$

Throughout this section we take $m = 2$, i.e. V is a 4-dimensional non-singular symplectic $\mathbb{F}_q$-vector space and $G = \mathrm{Sp}(4,q)$. So $\mathcal{L}_1$ is the set of 1-spaces in V, and $\mathcal{L}_2$ is the set of maximal isotropic subspaces in V. As usual, k is an algebraically closed field of characteristic 2.

Let $M \in \mathcal{L}_2$ and define

$$\Phi(M) := \left\{ N \in \mathcal{L}_2 \;\mid\; dim_{\mathbb{F}_q}\, M \cap N = 1 \right\}.$$

Now let

$$s_{\Phi(M)} := \sum_{N \in \Phi(M)} N \in k^{\mathcal{L}_2}.$$

Define submodules $\mathcal{C}$ and $\mathcal{P}$ as follows:

$$\mathcal{C} := \langle M \;\mid\; M \in \mathcal{L}_2 \rangle_k$$

and

$$\mathcal{P} := \left\langle s_{\Phi(M)} \;\mid\; M \in \mathcal{L}_2 \right\rangle_k.$$

Now put

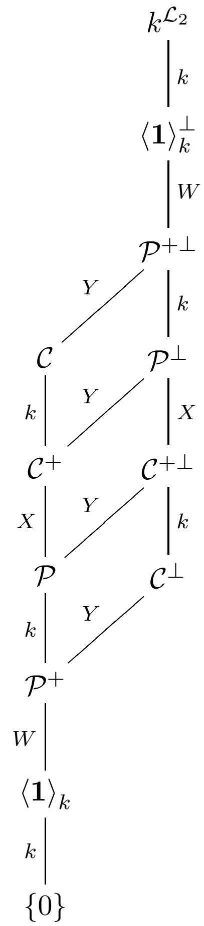
$$\mathcal{C}^+ := \langle M + N \;\mid\; M, N \in \mathcal{L}_2 \rangle_k$$

and

$$\mathcal{P}^+ := \langle P_1 + P_1 \;\mid\; P_1, P_2 \in \mathcal{P} \rangle_k$$

Finally, denote $Y := \mathcal{C}^{\perp}/\mathcal{P}^+$. We now prove

Theorem 3.1. *Using the above notation, $k^{\mathcal{L}_2}$ has the following submodule lattice for $\Gamma = \mathrm{CSp}(4,q)$:*



Proof. That all the containments pictured above actually hold has been proven in [2] by Bagchi et al. Furthermore, in the same paper the authors have determined the dimensions of all the submodules pictured above, and we will freely use that information here.

The incidence map $\eta_{1,2} : k^{\mathcal{L}_1} \to k^{\mathcal{L}_2}$ induces an isomorphism $k^{\mathcal{L}_1}/C^{\perp} \simeq \mathcal{C}$. By Corollary 2.14, we then have that $\mathcal{C}$ is uniserial with composition factors as indicated in the picture above. The incidence map $\eta_{2,1} : k^{\mathcal{L}_2} \to k^{\mathcal{L}_1}$ induces an isomorphism $k^{\mathcal{L}_2}/\mathcal{C}^{\perp} \simeq C$. By Corollary 2.14, we then have that $k^{\mathcal{L}_2}/\mathcal{C}^{\perp}$ is uniserial with composition factors as indicated in the picture above. We now see that $\mathrm{Res}_G^{\Gamma}\, k^{\mathcal{L}_2}$ has the Weil modules, W_1 and W_2, as composition factors, and that each occurs with multiplicity at least 2. Now from [10] we know that $k^{\mathcal{L}_2}$ has composition length 10 for G. It then follows that $Y(= \mathcal{C}^{\perp}/\mathcal{P}^{+})$ is simple for G, and hence simple for Γ.

Now from Theorem 2.2 of [9], we know that every submodule of $k^{\mathcal{L}_2}$ which is not contained in $\langle \mathbf{1} \rangle_k$ must contain $\mathcal{P}^{+}$. Thus, to verify the conjectured structure, it suffices to prove that $\mathcal{P}^{+\perp}/\mathcal{P}^{+}$ is as pictured.

Since $\mathcal{P} \cap \mathcal{C}^{\perp} = \mathcal{P}^{+}$ we see that

$$\mathcal{P}^{+\perp}/\mathcal{P}^{+} = (\mathcal{C}/\mathcal{P}^{+}) \oplus (\mathcal{C}^{\perp}/\mathcal{P}^{+}), \tag{36}$$

i.e. the quotient $\mathcal{P}^{+\perp}/\mathcal{P}^{+}$ is the direct sum of a uniserial module and a simple module.

It is determined in [2] that $dim_k\ (\mathcal{C}^{\perp}/\mathcal{P}^{+}) = \frac{q(q-1)^2}{2}$, so that $\mathcal{C}^{\perp}/\mathcal{P}^{+}$ is non-trivial. Since $\mathcal{C}/\mathcal{P}^{+}$ has trivial socle and trivial head, we see that we have have found all the submodules of $\mathcal{P}^{+\perp}/\mathcal{P}^{+}$. The result follows.

□

In the isomorphism $k^{\mathcal{L}_1}/C^{\perp} \simeq \mathcal{C}$, let $\mathcal{V}_1$ and $\mathcal{V}_2$ denote the images in $\mathcal{C}$ of $V_1/C^{\perp}$ and $V_2/C^{\perp}$, respectively. Thus,

$$\langle \mathbf{1} \rangle_k \subset \mathcal{V}_1, \mathcal{V}_2 \subset \mathcal{P}^{+}$$

and

$$\mathcal{P}^{+}/\langle \mathbf{1} \rangle_k \simeq W_1 \oplus W_2.$$

Let $\mathcal{V}_3 := \mathcal{V}_1^{\perp}$ and $\mathcal{V}_4 := \mathcal{V}_2^{\perp}$. Thus,

$$\langle \mathbf{1} \rangle_k^{\perp} \supset \mathcal{V}_3, \mathcal{V}_4 \supset \mathcal{P}^{+\perp}$$

and

$$\langle \mathbf{1} \rangle_k^{\perp}/\mathcal{P}^{+\perp} \simeq W_1 \oplus W_2.$$

Corollary 3.2. The picture in Theorem 3.1 shows every kG-submodule of $k^{\mathcal{L}_2}$, except for the modules $\mathcal{V}_i$, for $i = 1$ to 4.

Proof. By Theorem 2.2 of [9], we know that every submodule of $k^{\mathcal{L}_2}$ which is not contained in $\langle \mathbf{1} \rangle_k$ must contain one of $\mathcal{V}_1$ or $\mathcal{V}_2$. Therefore, in light of Theorem 3.1, it suffices to show that any submodule which properly contains either $\mathcal{V}_1$ or $\mathcal{V}_2$ must contain the other one as well, i.e. must contain $\mathcal{P}^{+}$.

Let N be a kG-submodule of $k^{\mathcal{L}_2}$ which properly contains $\mathcal{V}_1$ but does not contain $\mathcal{V}_2$. Assume that N is chosen minimal with respect to this property, i.e. assume no submodule of N has this property also. Denote by $\tilde{N}$ the Γ-module generated by N.

Assume first that $N \subseteq \mathcal{P}^{+\perp}$. Since N properly contains $\mathcal{V}_1$, we have $\mathcal{P}^{+} \subsetneq N + \mathcal{P}^{+}$. Thus,

$$\mathcal{P}^{+} \subsetneq N + \mathcal{P}^{+} \subseteq P^{+\perp}.$$

But we know that, in general, any kG-submodule A such that $\mathcal{P}^{+} \subseteq A \subseteq P^{+\perp}$ is actually a $k\Gamma$-submodule. Hence, $N + \mathcal{P}^{+}$ is a $k\Gamma$-submodule. Since $N + \mathcal{P}^{+} \subseteq \tilde{N}$, we deduce

$$N + \mathcal{P}^{+} = \tilde{N}.$$

We have

$$\begin{aligned}(N + \mathcal{P}^{+})/N &\simeq \mathcal{P}^{+}/(\mathcal{P}^{+} \cap N)\\ &= \mathcal{P}^{+}/\mathcal{V}_1\\ &\simeq W_2.\end{aligned}$$

Therefore,

$$\mathrm{Hom}_{kG}(\mathrm{Res}^{\Gamma}_{G}\ N + \mathcal{P}^{+},\ \mathrm{Res}^{\Gamma}_{G}\ W) \neq \{0\}.$$

Let H be the subgroup (of index 2) of Γ which is generated by G along with all of the scalar matrices. We then have

$$\mathrm{Hom}_{kH}(\mathrm{Res}^{\Gamma}_{H}\ N + \mathcal{P}^{+},\ \mathrm{Res}^{\Gamma}_{H}\ W) \neq \{0\}.$$

Now H acts trivially on $\mathrm{Hom}_{kH}(\mathrm{Res}^{\Gamma}_{H}\ N+\mathcal{P}^{+},\ \mathrm{Res}^{\Gamma}_{H}\ W)$, so we may consider the induced action of the 2-group Γ/H on $\mathrm{Hom}_{kH}(\mathrm{Res}^{\Gamma}_{H}\ N+\mathcal{P}^{+}, \mathrm{Res}^{\Gamma}_{H}\ W)$. But the action of a 2-group on a vector space over a field of characteristic 2 always has a fixed point. Thus,

$$\mathrm{Hom}_{k\Gamma}(N + \mathcal{P}^{+},\ W) \neq \{0\},$$

i.e. $N+\mathcal{P}^{+}$ has a quotient isomorphic to W. But from Theorem 3.1 we know that no submodule of $\mathcal{P}^{+\perp}$ which contains $\mathcal{P}^{+}$ has such a quotient. Thus, we have obtained a contradiction.

Now assume that $N \nsubseteq \mathcal{P}^{+\perp}$. Then $(N + \mathcal{P}^{+}) \nsubseteq \mathcal{P}^{+\perp}$ either. But

$$(N + \mathcal{P}^{+})^{\perp} = N^{\perp} \cap \mathcal{P}^{+\perp}$$

is, of course, a kG-submodule of $\mathcal{P}^{+\perp}$. From the first part, we know all such submodules. Obviously, we cannot have $\mathcal{P}^{+} \subseteq (N + \mathcal{P}^{+})^{\perp}$, so it must be the case that $(N + \mathcal{P}^{+})^{\perp} = \{0\}, \langle \mathbf{1}\rangle_k, \mathcal{V}_1,$ or $\mathcal{V}_2$.

Consider the following composition series for $N + \mathcal{P}^{+}$:

$$\{0\} \subset \langle \mathbf{1}\rangle_k \subset \mathcal{V}_1 \subset N \subset N + \mathcal{P}^{+}.$$

Note that the simplicity of $Q := N/\mathcal{V}_1$ is a consequence of the minimality of N. We know all of the kG-composition factors of $k^{\mathcal{L}_2}$, but we do not know the isomorphism class of Q. However, by considering the various possibilities for Q, we may see that regardless of its isomorphism class, we always have that the dimension of $(N + \mathcal{P}^{+})^{\perp}$ is strictly greater than the dimensions of $\{0\}, \langle \mathbf{1}\rangle_k, \mathcal{V}_1,$ and $\mathcal{V}_2$. Thus, we have reached a contradiction and the assertion has been established.

□

Remark 3.3. In view of Remark 2.15, we see that we may deduce from Theorem 3.1 and its corollary the submodule structure of $\mathbb{F}^{\mathcal{L}_2}$ where $\mathbb{F}$ is *any* field of characteristic 2.

Remark 3.4. In [2] (see pg. 352), Bagchi et al conjectured the submodule lattice of $\mathbb{F}_2^{L_2}$ for G. We now see from Corollary 3.2 and the preceding remark that their conjectured structure is correct in all cases except when $q \equiv \pm 1 \bmod 8$, in which case the Weil modules have been neglected. However, their structure is correct for the conformal group Γ.

Remark 3.5. Let E be a 5-dimensional vector space over $\mathbb{F}_q$ which is endowed with a non-singular orthogonal geometry. Denote by $\mathrm{O}(5, q)$ the corresponding full orthogonal group and by $\Omega(5, q)$ its derived subgroup. We will write $\mathcal{L}_1(E)$ for the set of isotropic 1-spaces in E. There is a natural identification (see [3]) of the elements of $\mathcal{L}_2$ with the elements of $\mathcal{L}_1(E)$. This identification carries Γ onto $\mathrm{O}(5,q)$ and G onto $\Omega(5,q)$. Thus, we see that the above results give us the submodule structure of $\mathbb{F}^{\mathcal{L}_1(E)}$, where $\mathbb{F}$ is any field of characteristic 2.

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