NEW BOUNDS FOR PARTIAL SPREADS OF $H(2d-1, q^2)$ AND PARTIAL OVOIDS OF THE REE-TITS OCTAGON

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ABSTRACT. Our first result is that the size of a partial spread of $H(3, q^2)$ is at most $\left(\frac{2p^3+p}{3}\right)^t + 1$, where $q = p^t$, $p$ is a prime. For fixed $p$ this bound is in $o(q^3)$, which is asymptotically better than the previous best known bound of $(q^3 + q + 2)/2$. We give similar bounds for partial spreads of $H(2d-1, q^2)$, $d$ even. Our second result is that the size of a partial ovoid of the Ree-Tits octagon $O(2^t)$ is at most $26^t + 1$. This bound, in particular, implies that the Ree-Tits octagon $O(2^t)$ does not have an ovoid.

1. Introduction

Determining clique numbers of graphs is a traditional topic of combinatorics. Partial spreads of polar spaces are cliques of the oppositeness graph defined on the generators of the polar space. While bounds for partial spreads and partial ovoids are a long studied topic since Thas popularized the problem in [10], many questions remain open. The previous best known bound of polar spaces are cliques of the oppositeness graph defined on the generators of the polar space. While bounds for partial spreads and partial ovoids are a long studied topic since Thas conjectured that the Ree-Tits octagon $O(2^t)$ does not have an ovoid.

The only known thick finite generalized octagons are the Ree-Tits octagons $O(2^t)$, $t$ odd, and their duals, which were defined in [11]. For $O(2)$, it was already shown [5] that a partial ovoid has at most 27 points, while for the $t \geq 3$ cases even the existence of ovoids, which are partial ovoids of size $64^t + 1$, was an open question. Coolsaet and Van Maldeghem [5, p. 108] conjectured that the Ree-Tits octagon $O(2^t)$ does not have an ovoid. We prove this conjecture by giving the upper bound $26^t + 1$ on the size of a partial ovoid of $O(2^t)$, $t$ odd. This nonexistence result further implies that the Ree-Tits octagon does not admit a line-domestic collineation [8, p. 582].

A Hermitian polar space $H(2d-1, q^2)$ is the incidence geometry arising from a non-degenerate Hermitian form $f$ of $\mathbb{F}_{q^2}^{2d}$. Here the flats of $H(2d-1, q^2)$ consist of all nonzero totally isotropic subspaces of $\mathbb{F}_{q^2}^{2d}$ with respect to the form $f$; incidence is the inclusion relation of the flats of $H(2d-1, q^2)$. The maximal totally isotropic subspaces of $\mathbb{F}_{q^2}^{2d}$ with respect to the form $f$ are called generators of $H(2d-1, q^2)$. A partial spread of $H(2d-1, q^2)$ is a set of pairwise disjoint generators of $H(2d-1, q^2)$. A simple double counting argument shows that a partial spread of $H(2d-1, q^2)$ has size at most $q^{2d-1} + 1$.

Our first result is the following.

Theorem 1.1. Let $q = p^t$ with $p$ prime and $t \geq 1$. Let $Y$ be a partial spread of $H(2d-1, q^2)$, where $d$ is even.

(a) If $d = 2$, then $|Y| \leq \left(\frac{2p^3+p}{3}\right)^t + 1$.
(b) If $d = 2$ and $p = 3$, then $|Y| \leq 19^t$.

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(c) If \( d > 2 \), then \(|Y| \leq \left( p^{2d-2} - p^{2d-2-1} \right)^t + 1 \).

For fixed \( p \) these bounds are, unlike all other known bounds, asymptotically better than \( o(q^{2d-1}) \). For \( d = 2 \) the previous best known bound is \( (q^3 + q + 2)/2 \) \([6]\). An easy calculation shows that this old bound is better than the bound in part (a) of Theorem 1.1 if \( p = 2 \) and \( t = 2 \) or if \( t = 1 \). But for fixed \( p \) (and let \( q = p^t \)), the bound in part (a) of Theorem 1.1 is in \( o(q^3) \), which is asymptotically better than the bound of \( (q^3 + q + 2)/2 \).

A generalized \( n \)-gon of order \((s, r)\) is a triple \( \Gamma = (P, L, I) \), where elements of \( P \) are called points, elements of \( L \) are called lines, and \( I \subseteq P \times L \) is an incidence relation between the points and lines, which satisfies the following axioms \([12]\):

(a) Each line is incident with \( s + 1 \) points.
(b) Each point is incident with \( r + 1 \) lines.
(c) The incidence graph has diameter \( n \) and girth 2\( n \).

Here the incidence graph is the bipartite graph with \( P \cup L \) as vertices, \( p \in P \) and \( \ell \in L \) are adjacent if \((p, \ell) \in I\). A partial ovoid of a generalized \( n \)-gon \( \Gamma \) is a set of points pairwise at distance \( n \) in the incidence graph. An easy counting argument shows that the size of a partial ovoid of a generalized octagon of order \((s, r)\) is at most \( (st)^2 + 1 \). A partial ovoid of a generalized octagon of order \((s, r)\) is called an ovoid if it has the maximum possible size \((st)^2 + 1 \). The Ree-Tits octagon \( O(2^t) \) is a generalized octagon of order \((2^t, 4^t)\), so the size of a partial ovoid is at most \( 64^t + 1 \).

Our second result is the following.

**Theorem 1.2.** The size of a partial ovoid of the Ree-Tits octagon \( O(2^t) \), \( t \) odd, is at most \( 26^t + 1 \).

This extends a result by Coolsaet and Van Maldeghem \([5]\), who established an upper bound of 27 for \( O(2) \). Theorem 1.2 implies that the Ree-Tits octagon \( O(2^t) \) does not have an ovoid. A computer search showed that for \( O(2) \) the largest partial ovoid has size 24, so Theorem 1.2 is not tight.

## 2. Two Association Schemes

We need some basic properties of an association scheme defined on the generators of \( H(2d - 1, q^2) \). A complete introduction to association schemes can be found in \([3, \text{Ch. 2}]\).

**Definition 2.1.** Let \( X \) be a finite set of size \( n \). An association scheme with \( d + 1 \) classes is a pair \((X, \mathcal{R})\), where \( \mathcal{R} = \{R_0, \ldots, R_d\} \) is a set of symmetric binary relations on \( X \) with the following properties:

(a) \( \mathcal{R} \) is a partition of \( X \times X \).
(b) \( R_0 \) is the identity relation.
(c) There are numbers \( p_{ij}^k \) such that for \( x, y \in X \) with \( xR_ky \) there are exactly \( p_{ij}^k \) elements \( z \) with \( xR_iz \) and \( zR_jy \).

The relations \( R_i \) are described by their adjacency matrices \( A_i \in \mathbb{C}^{n \times n} \) defined by

\[
(A_i)_{xy} = \begin{cases} 
1, & \text{if } xR_iy, \\
0, & \text{otherwise}.
\end{cases}
\]
In this paper the matrix \( A_d \) is referred to as the \textit{oppositeness matrix}. Denote the all-ones matrix by \( J \). There exist (see [3, p. 45]) idempotent Hermitian matrices \( E_j \in \mathbb{C}^{n,n} \) with the properties

\[
\sum_{j=0}^{d} E_j = I, \quad E_0 = n^{-1}J,
\]

\[
A_j = \sum_{i=0}^{d} P_{ij} E_i, \quad E_j = \frac{1}{n} \sum_{i=0}^{d} Q_{ij} A_i,
\]

where \( P = (P_{ij}) \in \mathbb{C}^{d+1,d+1} \) and \( Q = (Q_{ij}) \in \mathbb{C}^{d+1,d+1} \) are the so-called eigenmatrices of the association scheme. The \( P_{ij} \) are the eigenvalues of \( A_j \). The multiplicity \( f_i \) of \( P_{ij} \) satisfies \( f_i = \text{rank}(E_i) = \text{tr}(E_i) = Q_{0i} \).

In this paper we consider the association schemes corresponding to the dual polar graph of \( H(2d - 1, q^2) \) and the Ree-Tits octagon. For the dual polar graph of \( H(2d - 1, q^2) \) we have the following situation. Here \( X \) is the set of generators of \( H(2d - 1, q^2) \) and two generators \( a, b \) are in relation \( R_i \), where \( 0 \leq i \leq d \), if and only if \( a \) and \( b \) intersect in codimension \( i \). The dual polar graph of \( H(2d - 1, q^2) \) has diameter \( d \) and is distance regular.

As in [7], for the dual polar graph of \( H(2d - 1, q^2) \) we obtain

\[
f_d = q^{2d}q^{1-2d} + 1 \quad q + 1 = q^{2d-1} - q^{2d-2} - 1 \quad q + 1,
\]

and

\[
Q_{id} = \frac{P_{di}}{P_{0i}} Q_{0d} = f_d(-q)^{-i}.
\]

For the Ree-Tits octagon \( O(2^t) \) we consider the following association scheme: the set \( X \) is the set of points of the octagon, and two points \( a \) and \( b \) are in relation \( R_i \), where \( 0 \leq i \leq 4 \), if their distance in the incidence graph of \( O(2^t) \) is \( 2i \). Notice that \( A_4 \) is the oppositeness matrix in this case.

3. \textit{p-Ranks of the Oppositeness Matrices}

We will use the following observation.

\textbf{Lemma 3.1.} Let \( A \) and \( B \) be \( n \times n \) integer matrices and \( p \) be a prime. If \( A \equiv B \mod p \), then \( \text{rank}_p(A) \leq \text{rank}(B) \).

\textit{Proof.} We have \( \text{rank}_p(A) = \text{rank}_p(B) \leq \text{rank}(B) \).

\textbf{Lemma 3.2.} (a) The \( p \)-rank of the oppositeness matrix of lines of \( H(3, p^2) \) is \( \frac{2p^3+p}{3} \).

(b) The \( p \)-rank of the oppositeness matrix of generators of \( H(2d - 1, p^2) \), \( d \) even, is at most \( p^{2d-1} - p^{2d-2} - 1 \).

\textit{Proof.} Notice that \( H(3, p^2) \) is dual to \( Q^-(5, p) \). By [9, Example 6.2], the \( p \)-rank of the oppositeness matrix \( A_2 \) is equal to the dimension of \( L((p - 1)\omega_1) \). This value was calculated in [2, Theorem 1.2]. Applying Theorem 1.2 of [2] with \( t = 3 \) and \( r = p - 1 \), for the oppositeness matrix \( A_2 \) we obtain

\[
\text{rank}_p(A_2) = \frac{2p^3 + p}{3}.
\]

Part (a) of the lemma follows.
For (b) notice that the matrix $E_d$ of the dual polar graph of $H(2d - 1, p^2)$ has rank

$$f_d = p^{2d-1} - p^{2d-2} - 1.$$  

The matrix $np^{2d-3}E_d$ has only integer entries and we have $A_d \equiv np^{d-1}E_d \mod p$ for $d$ even. By Lemma 3.1, $\text{rank}_p(A_d) \leq \text{rank}(E_d) = p^{2d-1} - p^{2d-2} - 1$. $\square$

**Proposition 3.3** (Sin [9, Prop. 5.2]). Let $A_R(q)$ denote the oppositeness matrix for objects of one fixed type in a building with root system $R$ over $\mathbb{F}_q$, where $q = p^t$, $p$ is a prime. Then

$$\text{rank}_p(A_R(q)) = \text{rank}_p(A_R(p))^t.$$  

According to [9, p. 282, para. 3], the 2-rank of the oppositeness matrix of $O(2)$ corresponds to the dimension of a simple module of the group $F_4$ ($\omega_4$ in the notation of [9]). Veldkamp calculated all simple modules of the group $F_4$ in characteristic 2 [15, Table 2]. As for this particular module it is the modulo 2 reduction of the corresponding Weyl module in characteristic zero, whose dimension was known before. It is also easy to compute the oppositeness matrix of $O(2)$ and its 2-rank by computer. Hence, we obtain the following result.

**Lemma 3.4.** The 2-rank of the oppositeness matrix of $O(2)$ is 26.

### 4. Proofs of the Results

Our main result will follow from the following lemma.

**Lemma 4.1.** Let $(X, \sim)$ be a graph. Let $A$ be the adjacency matrix of $X$. Let $Y$ be a clique of $X$. Then

$$|Y| \leq \begin{cases} \text{rank}_p(A) + 1, & \text{if } p \text{ divides } |Y| - 1, \\ \text{rank}_p(A), & \text{otherwise}. \end{cases}$$  

**Proof.** Let $I$ be the all-ones matrix of size $|Y| \times |Y|$. Let $I$ be the identity matrix of size $|Y| \times |Y|$. As $Y$ is a clique, the submatrix $A'$ of $A$ indexed by $Y$ is $J - I$. Hence, the submatrix has $p$-rank $|Y| - 1$ if $p$ divides $|Y| - 1$, and it has $p$-rank $|Y|$ if $p$ does not divide $|Y| - 1$. As $\text{rank}_p(A') \leq \text{rank}_p(A)$, the assertion follows. $\square$

**Proof of Theorem 1.1 and Theorem 1.2.** Combining Lemma 3.2, Proposition 3.3 and Lemma 4.1 shows Theorem 1.1. Notice here that $2p^3 + p$ is divisible by $p$ unless $p = 3$. Combining Proposition 3.3, Lemma 3.4 and Lemma 4.1 shows Theorem 1.2. $\square$

### 5. Final Remarks

The used technique also works for partial ovoids of the twisted triality hexagon of order $(q, q^3)$, where we obtain the $p$-rank $((4p^3 + p)/5)^t$, but there a better bound of order $q^3 + 1$ is known [14, Theorem 6.4.19]. For all other known generalized polygons $p$-ranks do not improve any known bounds for partial ovoids.

The bounds on partial spreads $H(2d - 1, q^2)$, $d > 2$ even, in Theorem 1.1 can be improved by calculating exact $p$-ranks. We expect these bounds to be much better than the given ones as for $H(5, q^2)$ the corresponding $p$-rank is $((11p^3 + 5p^3 + 4p)/20)^t$, while an argument as in Lemma 3.2 (b) only yields $(p^3 - p^t + p^3 - p^2 + p)^t$ as an upper bound. Unfortunately, calculating the exact $p$-rank seems to be too complicated. Notice that this is a finite problem for fixed $p$ and fixed $d$ as one only has to calculate the $p$-rank for $t = 1$ by Proposition 3.3.
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REFERENCES


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