OPPOSITENESS IN BUILDINGS AND SIMPLE MODULES FOR FINITE GROUPS OF LIE TYPE

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Abstract. In the building of a finite group of Lie type we consider the incidence relations defined by oppositeness of flags. Such a relation gives rise to a homomorphism of permutation modules (in the defining characteristic) whose image is a simple module for the group. The p-rank of the incidence relation is then the dimension of this simple module. We give some general reductions towards the determination of the character of the simple module. Its highest weight is identified and the problem is reduced to the case of a prime field. The reduced problem can be approached through the representation theory of algebraic groups and the methods are illustrated for some examples.

1. Introduction

Let G be a finite group with a split BN-pair of characteristic p and rank ℓ, and let $I = \{1, \dots, \ell\}$. The Weyl group W is a euclidean reflection group in a real vector space V, containing a root system R and if $S = \{\alpha_i \mid i \in I\}$ is a set of simple roots then W is a Coxeter group with generators the fundamental reflections w_i, $i \in I$, where w_i is the reflection in the hyperplane perpendicular to the simple root α_i.

For $J \subset I$, let $W_J := \langle w_i \mid i \in J\rangle$ be the associated standard parabolic subgroup of W and $P_J = BW_JB$ the standard parabolic subgroup of G. By a *type* we simply mean a nonempty subset of I. An *object of type* $I \setminus J$, or *of cotype* J is, by definition, a right coset of P_J in G.

The associated building is the simplicial complex in which the simplices are the cosets of the standard parabolic subgroups of G, and the face relation is the reverse of inclusion. An object of type $I \setminus J$ is a $(|I \setminus J| - 1)$-simplex.

An example which has an elementary description is the building for $G = \mathrm{SL}(V)$ of type A_ℓ, where V is an $(\ell+1)$-dimensional vector space. A k-simplex is a flag of $k + 1$ nontrivial, proper subspaces of V, and its type is the set of dimensions of the subspaces in the flag.

Given two fixed types there are various incidence relations between objects of the respective types which are preserved by the action of G. In the above example, if we consider in V the complete flags (objects of type I) and the

i-dimensional subspaces (objects of type $\{i\}$), then an incidence relation can be specified by prescribing a sequence of ℓ dimensions for the intersections of an i-dimensional subspace with the subspaces in a complete flag.

An incidence relation $\mathcal{R}$ between the sets of objects of cotypes J and K can be encoded by an incidence matrix or, more conveniently for our purposes, by an *incidence map of permutation modules*. Let k be a commutative ring with 1. Let $\mathcal{F}_J$ denote the space of functions from the set $P_J\backslash G$ of objects of cotype J to k. Then $\mathcal{F}_J$ is a left kG-module by the rule

$$(xf)(P_Jg) = f(P_Jgx), \quad f \in \mathcal{F}_J, \quad g, x \in G. \tag{1}$$

Let δ_{P_Jg} denote the characteristic function of the object $P_Jg \in P_J\backslash G$. These characteristic functions are permuted transitively by G and form a basis for $\mathcal{F}_J$.

A G-invariant relation $\mathcal{R}$ defines a kG-homomorphism $\eta : \mathcal{F}_J \to \mathcal{F}_K$ given by

$$\eta(f)(P_Kh) = \sum_{P_Jg\mathcal{R}P_Kh} f(P_Jg). \tag{2}$$

The characteristic function of an object of cotype J is sent to the sum of the characteristic functions of all objects of cotype K which are incident with it.

Naturally, we would like to compute invariants of η. We can ask for its Smith Normal Form when $k = \mathbb{Z}$, its rank when k is a field, or its eigenvalues if $J = K$. For greater detail we can also consider the kG-module structure of the image of η. We have described the problem in great generality and, as might be expected, the nature and difficulty of a specific instance will depend very much on G, $\mathcal{R}$ and k. For example, if k is a field then whether or not its characteristic is the same as that of G is crucial, because the representation theory of G in its defining characteristic is closely related to the representation theory of reductive algebraic groups, while the cross-characteristic representation theory of G has a closer connection to its complex character theory and to Brauer's theory of blocks. This dichotomy can be observed in the case of generalized quadrangles by comparing the results in [3], [14] and [9]. In the first two papers k has characteristic 2, while G has odd characteristic in the first and even characteristic in the second. In the third, the group G and k have the same odd characteristic.

In this paper we restrict ourselves to the case where k is a field of the same characteristic as G and turn next to the choice of $\mathcal{R}$. We know from examples (e.g. [4]) that the permutation kG-module $\mathcal{F}_J$ can have many composition factors, growing with respect to the Lie rank of G, and also growing for fixed rank and fixed characteristic as the size of the field increases. The same may hold for the image of η. Even for rank 2, examples (e.g. [14]) demonstrate that the number of composition factors of $\operatorname{Im}\eta$ can grow without bound. In many cases, one knows neither the number of composition factors nor their dimensions, and in these cases we have no formula for $\operatorname{rank}\eta$. There are also examples of two problems which seem very similar on the surface, but one turns out to be much harder than the other. For example, the problem of computing the rank of the inclusion relation of 2-dimensional subspaces in $(n-2)$-dimensional subspaces in a vector space of dimension $n \geq 5$ is unsolved. In contrast, a simple formula is known ([15]) for the rank of the relation of nonzero intersection. It is therefore desirable, for a given $\mathcal{R}$, to have some idea *a priori* of the complexity of the kG-submodule lattice of $\operatorname{Im}\eta$. From this point of view, the oppositeness relations (defined below), which will be shown to give simple modules, can be considered to form the starting point of the theory.

2. Oppositeness in buildings

Let R^+ denote the set of positive roots. The length of an element $w \in W$, denoted $\ell(w)$, is the length of the shortest expression for w as a word in the generators w_i. This is also equal to the number of positive roots which w transforms to negative roots. There is a unique element of maximal length, denoted w_0, which sends all positive roots to negative roots.

Notions of oppositeness exist at the level of types and at the level of objects. Two types J and K are *opposite* if $\{-w_0(\alpha_i) \mid i \in J\} = \{\alpha_j \mid j \in K\}$, or, equivalently, if $\{w_0 w_i w_0 \mid i \in J\} = \{w_i \mid i \in K\}$. In other words, two types are opposite when they are mapped to each other by the symmetry of the Dynkin diagram induced by w_0. If $w_0 = -1$, then every type is opposite itself; this holds for all connected root systems except for those of type A_ℓ, D_ℓ (ℓ odd) and E_6. Let J and K be fixed opposite cotypes. An object $P_J g$ of cotype J and an object $P_K h$ of cotype K are *opposite* each other if $P_K h g^{-1} P_J = P_K w_0 P_J$ ($\iff P_K h \subseteq P_K w_0 P_J g \iff P_J g \subseteq P_J w_0 P_K h$).

The definition of oppositeness is a precise way to formulate the intuitive idea of two objects being in "general position". From now on we shall let J and K be fixed opposite cotypes.

Example 2.1. Consider a universal Chevalley group G type A_ℓ and $J = I \backslash \{i\}$. Then $G \cong \mathrm{SL}(V)$ for an $(\ell+1)$-dimensional vector space V. The objects of type $\{i\}$ can be identified with i-dimensional subspaces of V and objects of the opposite type $\{\ell+1-i\}$ can be identified with $\ell+1-i$-dimensional subspaces. A subspace of type $\{i\}$ is opposite one of type $\{\ell+1-i\}$ if their intersection is the zero subspace. A familar special case is when $\ell = 3$ and $i = \ell+1-i = 2$. If we think projectively, the objects are lines in space and the oppositeness relation is skewness.

More generally, an object of cotype $J = \{j_1, \dots, j_m\}$ is a flag

$$V_{j_1} \subset V_{j_2} \subset \cdots \subset V_{j_m}$$

of subspaces of V with $\dim V_{i_j} = i_j$. If $V'_{k_1} \supset V'_{k_2} \supset \cdots \supset V'_{k_m}$ is an object of the opposite type, then the two flags are opposite iff $V_{i_j} \cap V'_{k_j} = \{0\}$, for $j = 1$, $\dots$, m.

Example 2.2. Let G be of type B_ℓ or C_ℓ with $\ell \geq 2$ or D_ℓ with $\ell \geq 3$ and let $J = I \setminus \{1\}$. Then J is opposite to itself. In the B_ℓ case, objects of cotype J can be identified with singular points (one-dimensional subspaces) with respect to a nondegenerate quadratic form in a finite vector space of dimension $2\ell + 1$. Two singular points are opposite if and only if they are not orthogonal. Similarly, objects of cotype J can be viewed as singular points of a 2ℓ-dimensional vector space with respect to a symplectic symplectic form for type C_ℓ or a quadratic form for type D_ℓ, with oppositeness interpreted as non-orthogonality. Two points are opposite if and only if they do not lie on a singular line. Thus, the concept of oppositeness generalizes the concept of collinearity of singular points in these classical geometries.

Example 2.3. Consider a universal Chevalley group G of type $E_6(q)$. This group has a concrete description as the group of linear transformations which preserve a certain cubic form on a 27-dimensional vector space V over $\mathbf{F}_q$. The geometry of this space has been studied in great detail. (See [1], [10], [11].) Consider the objects of type 1 and the opposite type 6. (See Figure 1.) We can view these, respectively, as the singular points and singular (in a dual sense) hyperplanes of V. A singular point $\langle v \rangle$ is opposite a singular hyperplane H if and only $v \notin H$.

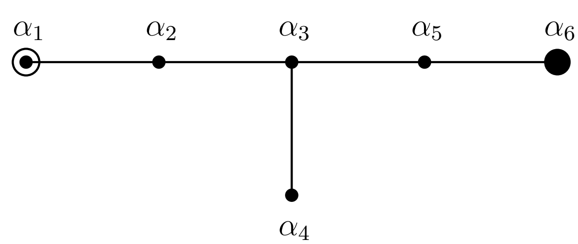

FIGURE 1

For further examples of oppositeness, we refer the reader to [5].

Let $A = A_{J,K}$ be the incidence matrix of the oppositeness relation, with rows indexed by objects of cotype J (in some order) and columns indexed by objects of cotype K. Suppose G is defined over $\mathbf{F}_q$, where q is a power of p.

Brouwer [5] has shown that each eigenvalue of AA' is a power of q, where A' is the transpose of A.

In the following sections, we show that the p-rank of the incidence matrix A is the dimension of an irreducible p-modular representation of G. This fact is derived as a corollary of a general theorem of Carter and Lusztig [8]. Then we describe the simple module in terms of its highest weight and show, using Steinberg's Tensor Product Theorem that, given a root system and choice of opposite types, the p-ranks can be computed for all q once they are known in the case $q = p$. We then discuss methods for computing the character of the simple module in some examples.

3. Some lemmas on double cosets

Let V_J be the subspace of V spanned by $S_J = \{\alpha_i \mid i \in J\}$. Then $R_J = R \cap V_J$ is a root system in V_J with simple system S_J and Weyl group W_J. For $w \in W_J$, its length as an element of W_J is the same as for W. Let w_J be the longest element in W_J.

The following is immediate from the definition of opposite types.

Lemma 3.1. *If $w_0 = w^* w_J = w_K w'$ then $w^* = w'$.*

□

For $w \in W$, we recall that $U_w^- = U \cap w^{-1} w_0 U w_0 w$. For $w \neq 1$ this group is a nontrivial p-group. We also have that given a choice of preimage $n_w \in N$ of w, each element of BwB can be written as a unique product $g = b n_w u$, with $b \in B$ and $u \in U_w^-$. Thus, $|U_w^-|$ equals the number of cosets Bg' in BwB, which is equal to $q^{\ell(w)}$ in the untwisted case. In the case of twisted groups, the equation $|U_w^-| = q^{\ell(w)}$ is also valid if q is taken to be the *level* [12, Definition 2.1.9] of the Frobenius endomorphism defining G (which is a prime power or, for Suzuki and Ree groups, the square root of an odd power of 2 or 3) and $\ell(w)$ means the length in the Weyl group of the untwisted root system. We refer to [12, Theorems 2.3.8 and 2.4.1] and [7]. We shall use only the obvious fact that U_w^- is a nontrivial p-group when $w \neq 1$.)

Lemma 3.2. *(*[5, Proposition 3.1]*) We have $P_K w_0 P_J = P_K w^* P_J = P_K w^* B$. The number of cosets $P_K g$ in $P_K w_0 P_J$ is $q^{\ell(w^*)}$.*

□

Lemma 3.3. *(cf.* [5, Corollary 3.2]*) Let $w \in W_J$. Then $B(w^* w)B \subseteq P_K w_0 P_J$. Let $x \in G$. For a given coset $P_K h \subseteq P_K w_0 P_J x$, the number of cosets $Bg \subseteq Bw^* wBx$ such that $Bg \subseteq P_K h$ is $q^{\ell(w)}$.*

Proof. The first assertion is true because $P_K w^* w P_J = P_K w^* P_J = P_K w_0 P_J$. The rest follows by counting. The number of cosets $P_K h \subseteq P_K w_0 P_J g$ is $q^{\ell(w^*)}$ and the number of cosets $Bg \subseteq Bw^* wBx$ is $q^{\ell(w^* w)} = q^{\ell(w^*)+\ell(w)}$. □

4. Permutation modules on flags and their oppositeness homomorphisms

Let k be an algebraically closed field of characteristic p. Recall from the Introduction that $\mathcal{F}_J$ denotes the space of functions from the set $P_J\backslash G$ of objects of cotype J to k, with left kG-module structure given by (1). Also, δ_{P_Jg} denotes the characteristic function of the object $P_Jg \in P_J\backslash G$ and these functions form a permutation basis of $\mathcal{F}_J$. (The module $\mathcal{F}_J$ is isomorphic to the kG-permutation module on the *l*eft cosets of P_J in G by the map sending δ_{P_Jg} to $g^{-1}P_J$.) The relation of oppositeness defines the kG-homomorphism $\eta : \mathcal{F}_J \to \mathcal{F}_K$ given by

$$\eta(f)(P_Kh) = \sum_{P_Jg\subseteq P_Jw_0P_Kh} f(P_Jg) \tag{3}$$

and we have

$$\eta(\delta_{P_Jg}) = \sum_{P_Kh\subseteq P_Kw_0PJg} \delta_{P_Kh}.$$

The following result is essentially a corollary of more general results in [8].

Theorem 4.1. *The image of η is a simple module, uniquely characterized by the property that its one-dimensional U-invariant subspace has full stablizer equal to P_J, which acts trivially on it.*

The next subsections describe some results in [8] and explain how Theorem 4.1 follows from them.

Fundamental endomorphisms of $\mathcal{F}_\emptyset$. Let $\mathcal{F} = \mathcal{F}_\emptyset$. For each $w \in W$ we define $T_w \in \mathrm{End}_k(\mathcal{F})$ by the formula

$$T_w(f)(Bg) = \sum_{Bg'\subseteq Bw^{-1}Bg} f(Bg').$$

Then

$$T_w \in \mathrm{End}_{kG}(\mathcal{F}), \quad \text{for all } w \in W.$$

One can show (see [8]) that

$$T_{ww'} = T_wT_{w'} \quad \text{if } \ell(ww') = \ell(w) + \ell(w').$$

Let $w \in W$ have reduced expression

$$w_{j_n} \cdots w_{j_1}.$$

We consider the partial products w_{j_1}, $w_{j_2}w_{j_1}$, $\dots w_{j_n} \cdots w_{j_1}$. The positive roots sent to negative roots by each partial product are also sent to negative roots by each of its successors and each partial product sends exactly one more positive root to a negative root than its predecessors. Explicitly, the new positive root sent by $w_{j_i} \cdots w_{j_1}$ to a negative root is $w_{j_1} \cdots w_{j_{i-1}}(r_{j_i})$. With this in mind, we can define the endomorphsim $\Theta^J_{w_0}$ of [8, p.363] for any J subset of I.

For any reduced expression

$$w_0 = w_{j_k} \cdots w_{j_1}$$

define

$$\Theta_{j_i} = \begin{cases} T_{w_{j_i}} & \text{if } w_{j_1} \cdots w_{j_{i-1}}(r_{j_i}) \notin V_J \\ I + T_{w_{j_i}} & \text{if } w_{j_1} \cdots w_{j_{i-1}}(r_{j_i}) \in V_J \end{cases} \tag{4}$$

and set

$$\Theta^J_{w_0} = \Theta_{j_k}\Theta_{j_{k-1}} \cdots \Theta_{j_1}.$$

The definition depends on the choice of reduced expression but it is shown in [8] that different expressions give the same endomorphism up to a nonzero scalar multiple.

We can now state the result from which Theorem 4.1 can be deduced. We recall that in every simple kG-module the subspace fixed by the subgroup U is one-dimensional. Therefore, the full stabilizer in G of this subspace must contain $N_G(U) = B$, so must be equal to some standard parabolic subgroup P_Q, $Q \subseteq I$. Thus this line is a one-dimensional kP_Q-module.

Theorem 4.2. *(*[8, Theorem 7.1]*) The image $\Theta^J_{w_0}(\mathcal{F})$ is a simple kG-module. The full stablizer of the one-dimensional subspace of U-fixed points in this module is P_J and the action of P_J on this one-dimensional subspace is trivial.*

(Our space $\mathcal{F}$ is the space denoted $\mathcal{F}_\chi$ in [8] when χ is the trivial character.)

Proof of Theorem 4.1. Let

$$w_J = w_{i_m} \cdots w_{i_2} w_{i_1} \tag{5}$$

be a reduced expression for w_J. The above expression can be extended to a reduced expresion

$$w_0 = w_{i_k} \cdots w_{i_{m+1}} w_{i_m} \cdots w_{i_1} \tag{6}$$

of w_0. Thus $m = |R^+_J|$ and $k = |R^+|$.

Then

$$w^* = w_{i_k} \cdots w_{i_{m+1}}. \tag{7}$$

is a reduced expression for w^*. We choose the special expression (6) for w_0 to define $\Theta^J_{w_0}$. Since w_J sends all positive roots in V_J to negative roots and w_0 sends all negative roots to positive roots, it is clear that for the first m partial products the new positive root sent to a negative root belongs to V_J, and that the new positive roots for the remaining partial products are the elements of $R^+ \setminus R_J^+$, so do not belong to V_J. Thus we have

$$\Theta_{w_0}^J = T_{w^*}(1 + T_{i_m}) \cdots (1 + T_{i_1}), \tag{8}$$

Since $\ell(w^* w) = \ell(w^*) + \ell(w)$ for all $w \in W_J$, we see that $\Theta_{w_0}^J$ is a sum of endomorphisms of the form $T_{w^* w}$, for certain elements $w \in W_J$, with exactly one term of this sum equal to T_{w^*}.

Let $\pi_J : \mathcal{F} \to \mathcal{F}_J$ be defined by

$$(\pi_J(f))(P_J g) = \sum_{Bh \subseteq P_J g} f(Bh)$$

and π_K defined similarly. It is easily checked that π_J and π_K are kG-module homomorphisms and they are surjective since $\pi_J(\delta_B) = \delta_{P_J}$.

We can now complete the proof of Theorem 4.1. The main step is to compare $\eta\pi_J$ with $\pi_K T_{w^* w}$ for $w \in W_J$. For $f \in \mathcal{F}$, we compute

$$\begin{aligned} [\eta(\pi_J(f))]\,(P_K g) &= \sum_{P_J h \subseteq P_J w^{*-1} P_K g} \; \sum_{Bx \subseteq P_J h} f(Bx) \\ &= \sum_{Bx \subseteq P_J w^{*-1} P_K g} f(Bx). \end{aligned} \tag{9}$$

and

$$\begin{aligned} [\pi_K(T_{w^* w}(f))]\,(P_K h) &= \sum_{Bg \subseteq P_K h} (T_{w^* w} f)(Bg) \\ &= \sum_{Bg \subseteq P_K h} \; \sum_{Bx \subseteq B(w^* w)^{-1} Bg} f(Bx) \\ &= \sum_{Bg \subseteq P_K h} \; \sum_{Bg \subseteq B(w^* w) Bx} f(Bx) \\ &= q^{\ell(w)} \sum_{Bx \subseteq P_J w^{*-1} P_K g} f(Bx). \end{aligned} \tag{10}$$

where the last equality follows from Lemma 3.3, since $Bx \subseteq B(w^* w)^{-1} Bg$ if and only if $Bg \subseteq Bw^* wBx$. Thus, we have for each $w \in W_J$ a commutative diagram

$$\begin{tikzcd} \mathcal{F}_J \arrow[r, "q^{\ell(w)}\eta"] & \mathcal{F}_K \\ \mathcal{F} \arrow[u, "\pi_J"] \arrow[r, "T_{ww^*}"] & \mathcal{F}, \arrow[u, "\pi_K"'] \end{tikzcd} \tag{11}$$

which means that for $w \neq 1$ we have $\pi_K T_{ww^*} = 0$. It now follows from (8) that $\pi_K \Theta_{w_0}^J = \pi_K T_{w^*} = \eta \pi_J$. Therefore, since $\Theta_{w_0}^J(\mathcal{F})$ is simple and $\eta\pi_J(\mathcal{F}) \neq 0$, we see that $\eta\pi_J(\mathcal{F}) \cong \Theta^J_{w_0}(\mathcal{F})$. Finally, since π_J is surjective, we have $\eta(\mathcal{F}_J) \cong \Theta^J_{w_0}(\mathcal{F})$. This completes the proof of Theorem 4.1. $\square$

5. Highest weights

If G is a universal Chevalley group or a twisted subgroup of such a group then there is an algebraic group $G(k)$, the Chevalley group over k, and a Frobenius endomorphism σ of $G(k)$ such that G is the group of fixed points of σ. References for this theory are the original paper [17, 12.4] or the very complete account in [12, Chapter 2].

The simple kG-modules are restrictions of certain simple rational modules $L(\lambda)$ of $G(k)$ ([17, 13.1]), where λ is the highest weight of the module. This connection is well known; for definitions and details we refer to the original sources [16] and [17]. The simple module $L(\lambda)$ is characterized by the property that it has a unique B-stable line, and T acts on this line by the character λ.

Assume that G is a universal Chevalley group over $\mathbf{F}_q$. By Steinberg's theorem [17, 13.3], the simple $G(k)$ modules with highest weights in the set of *q-restricted weights*

$$X_q = \{\lambda = \sum_{i=1}^{\ell} a_i\omega_i \in X_+ \mid 0 \le a_i \le q-1 \ (\forall i)\}$$

remain irreducible upon restriction to G and this gives a complete set of mutually non-isomorphic simple kG-modules.

It is useful to identify the highest weight of the module in Theorem 4.1. We may assume that R is indecomposable, since simple modules for direct products are easily described in terms of the factors.

Let λ_{opp} denote the q-restricted highest weight such that the restriction of the simple $G(k)$-module $L(\lambda_{opp})$ to G is the simple module in Theorem 4.1. The condition in Theorem 4.1 that P_J is the full stabilizer in G of the one-dimensional highest weight space of $L(\lambda_{opp})$ is equivalent to the condition that for $i = 1,\ldots,\ell$, $\langle \lambda_{opp}, \alpha_i^\vee\rangle = 0$ if and only if $i \in J$. The condition in Theorem 4.1 that P_J acts trivially on the highest weight space means that the restriction of λ_{opp} to H is the trivial character, which in turn means that, when λ_{opp} is written as a linear combination of fundamental weights, all of the coefficients must be 0 or $q-1$. These conditions allow us to identify λ_{opp}. The fundamental weights ω_i for the ambient algebraic group are indexed by I, and $\lambda_{opp} = \sum_{i\in I\setminus J}(q-1)\omega_i$.

Consider next a twisted group G, constructed from an overlying universal Chevalley group G^* over $\mathbf{F}_q$, as the group of fixed points of an automorphism induced by an isometry ρ of order e of the Dynkin diagram of G^*. Then $q = q_0^e$ for some prime power q_0. Let $I^* = \{1,\ldots,\ell^*\}$ index the fundamental roots for G^*. Then the index set I for G labels the ρ-orbits on I^*. Let ω_i, $i \in I^*$ be the fundamental weights of the ambient algebraic group. For $J \subseteq I$, let $J^* \subset I^*$ be the union of the orbits in J. Then $\lambda_{opp} = \sum_{i \in I^* \setminus J^*} (q_0 - 1)\omega_i$.

Finally, we consider the case of Suzuki and Ree groups. Here G is the subgroup of fixed points in $G(k)$ of a Steinberg endomorphism τ which induces a length-changing permutation of the Dynkin diagram of $G(k)$ and such that $\tau^2 = \sigma$ is a Frobenius endomorphism of $G(k)$ with respect to a rational structure over a finite field $\mathbf{F}_q$. Then the set I for G indexes the subset of fundamental weights of the ambient algebraic group which are orthogonal to the long simple roots, and for $J \subset I$, we have $\lambda_{opp} = \sum_{i \in I \setminus J} (q - 1)\omega_i$.

Example 5.1. As examples, we can consider the extreme cases. If $J = K = I$, then $L(\lambda_{opp}) \cong k$. If $J = K = \emptyset$, $L(\lambda_{opp})$ is the *Steinberg module* of the group G, which has dimension equal to the the order of U, the p-sylow subgroup of G.

From the discussion above, we see that λ_{opp} has the form $(q-1)\tilde{\omega}$, where $\tilde{\omega}$ is a sum of fundamental weights. If $q = p^t$, then by Steinberg's Tensor Product Theorem, we have an isomorphism

$$L((q-1)\tilde{\omega}) \cong L((p-1)\tilde{\omega}) \otimes L((p-1)\tilde{\omega})^{(p)} \otimes \cdots \otimes L((p-1)\tilde{\omega})^{(p^{t-1})} \tag{12}$$

as modules for the algebraic group. Here, the superscripts indicate twisting by powers of the Frobenius morphism. This twisting changes the isomorphism type of a module, but does not change its dimension. In particular, we have the following numerical result.

Proposition 5.2. *Let the root system R and opposite cotypes J and K be given and let $A(q) = A(q)_{J,K}$ denote the oppositeness incidence matrix for objects of cotypes J and K in the building over $\mathbf{F}_q$, where $q = p^t$. Then* $\operatorname{rank}_p A(q) = (\operatorname{rank}_p A(p))^t$.

Remark 5.3. The proposition states that once $\operatorname{rank}_p A(p)$ is known then we know $\operatorname{rank}_p A(q)$ for all powers q of p. This reduction of the computation to the prime case is significant from the viewpoint of representation theory of algebraic groups, where the Weyl modules with modules highest weight $(p-1)\tilde{\omega}$ are much less complex in structure than those of highest weight $(p^2-1)\tilde{\omega}$, say.

A very useful tool for analyzing Weyl modules is the *Jantzen Sum Formula* [13, II.8.19]: The Weyl module $V(\lambda)$ has a descending filtration, of submodules $V(\lambda)^i$, $i > 0$, such that

$$V(\lambda)^1 = \operatorname{rad} V(\lambda), \quad (\text{so that } V(\lambda)/V(\lambda)^1 \cong L(\lambda))$$

and

$$\sum_{i>0} \operatorname{Ch}(V(\lambda)^i) = -\sum_{\alpha>0} \sum_{\{m: 0 < mp < \langle \lambda+\rho, \alpha^\vee \rangle\}} v_p(mp)\chi(\lambda - mp\alpha) \tag{13}$$

First, we recall the notation in this formula (which is standard, and follows from [13]). The module $V(\lambda)$ is the Weyl module and the module $L(\lambda)$ is its simple quotient. The weight ρ is the half-sum of the positive roots and $v_p(m)$ denotes the exponent of p in the prime factorization of m. Finally, the character $\chi(\mu)$ is the so-called Weyl character; there is a unique weight of the form $\mu' = w(\mu+\rho)-\rho$ in the region $\{\nu : \langle \nu+\rho, \alpha^\vee\rangle \geq 0, \forall \alpha \in R^+\}$, where $w \in W$. Then $\chi(\mu)$ is the $\mathrm{sign}(w)\,\mathrm{Ch}\,V(\mu')$ if μ' is dominant, and zero otherwise. The usefulness of the sum formula is based on the fact that the characters of the Weyl modules themselves are given by Weyl's Character Formula, so that the right hand side can be computed from p, R and λ. In [2] there is a detailed description of a procedure for performing this computation. We shall refer to the quantity in (13) as the Jantzen sum for $V(\lambda)$. As can be seen from the left hand side of (13), the Jantzen sum gives an upper estimate on the composition multiplicities in the radical of the Weyl module $V(\lambda)$ in terms of the composition factors of Weyl modules which have lower highest weights. Sometimes, for weights of a special form, more information can be obtained, using induction and other facts from representation theory. For example when we start with a weight of the form $\lambda = (p-1)\tilde{\omega}$, it may be that the highest weights of the Weyl characters $\chi(\mu)$ in the Jantzen sum are very few in number or all have a similar form, such as $r\tilde{\omega}$ for $r < p-1$. In such cases, it is possible to deduce the character of $L((p-1)\tilde{\omega})$. Numerous examples of this method were worked out in detail in [2].

6. Examples

We consider again some of the examples from the Introduction.

Example 6.1. For type A_ℓ, $J = I\backslash\{i\}$, the p-ranks have been computed in [15]. (In fact it is the p-rank of the complementary relation of nonzero intersection which is computed, which equals one plus the p-rank for zero intersection.) In this case, the simple modules $L((p-1)\omega_i)$ can be found without reference to Weyl modules. Let $S(i(p-1))$ denote the degree $i(p-1)$ homogeneous component of the truncated polynomial ring $k[x_0,\dots,x_\ell]/(x_i^p; 0 \leq i \leq \ell)$, with $G \cong \mathrm{SL}((\ell+1,k)$ acting through linear substitutions. Then it is well known and elementary to show that $S(i(p-1))$ is a simple kG-module. By inspecting the highest weights, we see that $S(i(p-1)) \cong L((p-1)\omega_{\ell+1-i})$, for $i = 1,\dots,$ ℓ.

Example 6.2. For the examples of types B_ℓ, C_ℓ and D_ℓ, with $J = I \setminus \{1\}$ concerning singular points in classical spaces, the p-ranks have been computed in [2] by analysis of the Weyl modules, as outlined above. The Weyl modules in question are $V((p-1)\omega_1)$ and for type C_ℓ they are simple, so work is only needed for types B_ℓ and D_ℓ. The method can also be extended to the classical modules of the classical groups of twisted type, namely the non-split orthogonal groups (type ${}^2D_\ell$) and the unitary groups (type ${}^2A_\ell$). In the latter case one must study the Weyl module $V((p-1)(\omega_1+\omega_\ell))$, in accordance with our discussion in §5.

Example 6.3. We will sketch the computation of the p-rank for Example 2.3. The Jantzen sum for $V((p-1)\omega_1)$ is equal to the following:

$$\begin{aligned}\chi((p-7)\omega_1+3\omega_6)-\chi((p-8)\omega_1+\omega_4+2\omega_6)+\chi((p-9)\omega_1+\omega_3+\omega_6)\\ -\chi((p-10)\omega_1+\omega_2+\omega_5)+\chi((p-11)\omega_1+2\omega_2)\end{aligned} \tag{14}$$

The next step is to study the structure of the Weyl modules appearing in this character, again by using the sum formula.

We observe first that the Jantzen sum for $V((p-11)\omega_1+2\omega_2)$ is zero, so this Weyl module is simple.

Next, the Jantzen sum for $V((p-10)\omega_1+\omega_2+\omega_5)$ is $\chi((p-11)\omega_1+2\omega_2)$. It follows that the radical of $V((p-10)\omega_1+\omega_2+\omega_5)$ is isomorphic to the simple module $V((p-11)\omega_1+2\omega_2)$.

The Jantzen sum for $V((p-9)\omega_1+\omega_3+\omega_6)$ is equal to

$$\chi((p-10)\omega_1+\omega_2+\omega_5)-\chi((p-11)\omega_1+2\omega_2),$$

which by the previous paragraph is equal to $\operatorname{Ch} L((p-10)\omega_1+\omega_2+\omega_5)$. Therefore $\operatorname{rad} V((p-9)\omega_1+\omega_3+\omega_6)\cong L((p-10)\omega_1+\omega_2+\omega_5)$.

The Jantzen sum for $V((p-8)\omega_1+\omega_4+2\omega_6)$ is

$$\chi((p-9)\omega_1+\omega_3+\omega_6)-\chi((p-10)\omega_1+\omega_2+\omega_5)+\chi((p-11)\omega_1+2\omega_2)=\operatorname{Ch} L((p-9)\omega_1+\omega_3+\omega_6),$$

which leads us to conclude that $\operatorname{rad} V((p-8)\omega_1+\omega_4+2\omega_6)\cong L((p-9)\omega_1+\omega_3+\omega_6$.

The Jantzen sum for $V((p-7)\omega_1+3\omega_6)$ is

$$\begin{aligned}\chi((p-8)\omega_1+\omega_4+2\omega_6)-\chi((p-9)\omega_1+\omega_3+\omega_6)+\chi((p-10)\omega_1+\omega_2+\omega_5)\\ -\chi((p-11)\omega_1+2\omega_2)\cong\operatorname{Ch} L((p-8)\omega_1+\omega_4+2\omega_6).\end{aligned} \tag{15}$$

Hence $\operatorname{rad} V((p-7)\omega_1+3\omega_6)\cong L((p-8)\omega_1+\omega_4+2\omega_6)$.

Finally, by (14), we have $\operatorname{rad} V((p-1)\omega_1)\cong L((p-7)\omega_1+3\omega_6)$. Therefore, there is an exact sequence

$$\begin{aligned}0\to V((p-11)\omega_1+2\omega_2)\to V((p-10)\omega_1+\omega_2+\omega_5)\\ \to V((p-9)\omega_1+\omega_3+\omega_6)\to V((p-8)\omega_1+\omega_4+2\omega_6)\\ \to V((p-7)\omega_1+3\omega_6)\to V((p-1)\omega_1)\to L((p-1)\omega_1)\to 0\end{aligned}$$

The dimensions of the $V(\mu)$ are given by Weyl's formula. Hence,

$$\begin{aligned}\dim L((p-1)\omega_1) = \frac{1}{2^7.3.5.11} & p(p+1)(p+3) \\ & \times (3p^8 - 12p^7 + 39p^6 + 320p^5 \\ & \qquad - 550p^4 + 1240p^3 + 2080p^2 - 1920p + 1440)\end{aligned}$$

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