Write your proofs using *complete English sentences* as well as mathematical formulae.
Bonus points may be awarded for particularly well-argued proofs.
In this exam F denotes a field and $\mathbb{R}$ denotes the field of real numbers.

Name: ____________________

1. Give examples of the following.

   (a) (2 points) A linearly dependent set of vectors in $\mathbb{R}^3$ such that at least one of the vectors is not a linear combination of the others.

   Solution: The set $\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}2\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\}$ is linearly dependent, as the second vector is a scalar multiple of the first. The third vector is not a linear combination of the first two because these both have zero in the second row and the third vector does not. An even simpler example is given by a set whose two elements are the zero vector and a nonzero vector.

   (b) (3 points) A vector space that has no finite generating set. Justify your answer.

Solution: The vector space $P(F)$ of polynomials (over any field F) has no finite generating set. If S is a finite set of polynomials then we can consider the maximum degree d of any polynomial in S. From the definition of polynomial addition and scalr multiplication it follows that the degree of every linear combination of polynomilas in S is less than or equal to d. Therefore, since there exist polynomials of degree greater than d, such as x^{d+1}, S cannot generate $P(F)$.

2. (5 points) Determine whether the following set is a basis of $P_2(F)$.

$$\{1 + 2x + x^2, 3 + x^2, x + x^2\}$$

Solution: Since $P_2(F)$ has dimension 3 and this set has size 3, we know from a theorem in the textbook that the set will be a basis as long as it is linearly independent. Suppose we have a, b and $c \in F$ such that

$$a(1 + 2x + x^2) + b(3 + x^2) + c(x + x^2) = 0.$$

Equating the coefficients of the powers of x we obtain the equations:

$$a + 3b = 0, \qquad 2a + c = 0, \qquad a + b + c = 0.$$

substiting for c in the third equation using the second equation, we obtain

$$a + 3b = 0, \qquad a + b - 2a = 0,$$

and it follows that $a = b = 0$, and $c = 0$. Thus the set is linearly independent.

3. Let V be the space of 2×2 matrices with real entries, and let W be the subset defined by

$$W = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(F) : a + b + c = 0\}.$$

(a) (4 points) Prove that W is a subspace of V.

Solution: (Omitted)

(b) (4 points) Find a basis of W.

Solution: Let

$$S = \{m_1 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, m_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, m_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}.$$

The elements of S belong to W. In order to show that S is a basis, we shall use the equation

$$\begin{pmatrix} a & b \\ -a-b & d \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \qquad (1)$$

which clearly holds. If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in W$, then $c = -a - b$, so the equation (1) shows that S generates W. To see if S is linearly independent, suppose we have scalars a, b, d such that $am_1 + bm_2 + dm_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. From (1), we must have $\begin{pmatrix} a & b \\ -a-b & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, from which it folows that $a = b = d = 0$. Therefore S is linearly independent.

(c) (2 points) What is the dimension of W ?

Solution: Since the basis has size 3 the dimesnion is 3.