Write your proofs using *complete English sentences* as well as mathematical formulae.
Bonus points may be awarded for particularly well-argued proofs.
In this exam F denotes a field and $\mathbb{R}$ denotes the field of real numbers.

Name: ______________________________

1. Let V be the vector space of real polynomials of degree at most 3.

   (a) (2 points) Show that the mapping $T : V \to V$ given by

   $$T(f(x)) = (1+x)f'(x), \quad \text{for } f(x) \in V,$$

   is a linear mapping. (As usual, $f'(x)$ denotes the derivative of $f(x)$.)

   Solution: Let $f(x)$ and $g(x) \in V$ and $c \in \mathbb{R}$. Then

   $$T(f(x)+cg(x)) = (1+x)(f(x)+cg(x))' = (1+x)f'(x)+c(1+x)g'(x) = T(f(x))+T(g(x)),$$

   which proves that T is a linear mapping.

   (b) (2 points) Compute the matrix $[T]_\beta^\beta$, where $\beta = \{1, x, x^2, x^3\}$ is the standard ordered basis of powers of x.

   Solution:
   We have $T(1) = 0$, $T(x) = 1 + x$, $T(x^2) = (1 + x)2x = 2x + 2x^2$, and $T(x^3) = (1 + x)3x^2 = 3x^2 + 3x^3$. Thus,

   $$[T]_\beta^\beta = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

   (c) (2 points) Determine the rank and nullity of T. (Rank is the dimension of $R(T)$, nullity is the dimension of $N(T)$.)

   Solution:
   The rank of T is equal to the rank of $[T]_\beta^\beta$. Since the first column of $[T]_\beta^\beta$ is zero and the other three columns are linearly independent (there are several ways to check this, e.g. RREF), the rank is 3. Then by the Rank-Nullity Theorem, the nullity is 4-3=1.

2. In R^3 let β be the standard basis and let

$$\beta' = \{ \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix} \}.$$

(a) (4 points) Compute the change-of-basis matrices $[1]_{\beta}^{\beta'}$ and $[1]_{\beta'}^{\beta}$.

Solution: To compute $[1]_{\beta'}^{\beta}$ we must express the vectors of β' in terms of the standard basis, which is easy:

$$[1]_{\beta'}^{\beta} = \begin{pmatrix}1 & 0 & 0\\1 & 1 & 1\\1 & 1 & -1\end{pmatrix}.$$

To find $[1]_{\beta}^{\beta'}$ we must express the standard basis vetors in terms of β'. We have

$$\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} - \begin{pmatrix}0\\1\\1\end{pmatrix},$$

$$\begin{pmatrix}0\\1\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0\\1\\1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0\\1\\-1\end{pmatrix},$$

and

$$\begin{pmatrix}0\\0\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0\\1\\1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}0\\1\\-1\end{pmatrix}.$$

Thus,

$$[1]_{\beta}^{\beta'} = \begin{pmatrix}1 & 0 & 0\\-1 & \frac{1}{2} & \frac{1}{2}\\0 & \frac{1}{2} & -\frac{1}{2}\end{pmatrix}$$

(b) (4 points) Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$. Find the matrix $[L_A]_{\beta'}^{\beta'}$.

Solution: Since β is the standard basis we have $[L_A]_{\beta}^{\beta} = A$. Hence by the change of basis formula, we have

$$\begin{aligned} [L_A]_{\beta'}^{\beta'} &= [1]_{\beta}^{\beta'}[L_A]_{\beta}^{\beta}[1]_{\beta'}^{\beta} \\ &= [1]_{\beta}^{\beta'}A[1]_{\beta'}^{\beta} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ -1 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 1 \\ 0 & -\frac{1}{2} & -\frac{3}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}. \end{aligned}$$

3. (a) (4 points) Prove that there is a unique linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T\left(\begin{pmatrix}1\\1\end{pmatrix}\right) = \begin{pmatrix}\frac{1}{5}\\-1\end{pmatrix}$ and $T\left(\begin{pmatrix}1\\2020\end{pmatrix}\right) = \begin{pmatrix}\frac{-2}{7}\\\ln 2\end{pmatrix}$.

Solution: Since neither of $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\2020\end{pmatrix}$ is a scalar multiple of the other, they are linearly independent, so they form a basis of the 2-dimensional space $\mathbb{R}^2$. By the fundamental Theorem of Linear Algebra, for any vectors w_1 and w_2 in $\mathbb{R}^2$ there exists a unique linear map sending $\begin{pmatrix}1\\1\end{pmatrix}$ to w_1 and $\begin{pmatrix}1\\2020\end{pmatrix}$ to w_2. The problem is just a special case of this with $w_1 = \begin{pmatrix}\frac{1}{5}\\-1\end{pmatrix}$ and $w_2 = \begin{pmatrix}\frac{-2}{7}\\\ln 2\end{pmatrix}$.

(b) (2 points) Is this linear map an isomorphism?

Solution: Yes. The vectors $\begin{pmatrix}\frac{1}{5}\\-1\end{pmatrix}$ and $\begin{pmatrix}\frac{-2}{7}\\\ln 2\end{pmatrix}$ are linearly independent, so they form a basis of $\mathbb{R}^2$. Therefore the map T is surjective, and since in addition the domain and codomain are of the same dimension, T is an isomorphism. (An alternative argument would be to use the Fund. Thm. to define the inverse map of T.)