Write your proofs using *complete English sentences* as well as mathematical formulae.
Work which the grader cannot follow may receive a grade of zero.
In this exam F denotes a field and $\mathbb{R}$ denotes the field of real numbers.

Name: ______________________________

1. (8 points) In $M_{2\times 2}(\mathbb{R})$, consider the set

$$X = \{\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 \\ 4 & -2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\}$$

of four matrices. Find a subset of X which is a basis for the span of X.

Solution: I claim that the first three matrices will work. Note that all the matrices lie in the 3-dimensional subspace of matrices of trace zero. It therefore suffices to show that the first three matrices are linearly independent. Suppose

$$a\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c\begin{pmatrix} 2 & 0 \\ 4 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then comparing entries we get $a + 2c = 0$, $2a + b = 0$, and $b + 4c = 0$, from which it follows that a, b and c are all equal to 0.

2. Let the linear map $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be given by $T(f(x)) = f(x) + f'(x) + f''(x)$.
   (a) (4 points) By computing with the matrix of T or otherwise, prove that T is invertible.
   (b) (4 points) Compute $T^{-1}(x^2 - x + 2)$.

Solution: (a) Using the standard basis β , we get

$$[T]_\beta^\beta = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

which is triangular and therefore easily seen to have determinant 1, hence is invertible.
(b) $T^{-1}(x^2 - x + 2) = x^2 - 3x + 3$. (There are many ways to see this.)

3. (8 points) Determine whether the matrix

$$A = \begin{pmatrix} -1 & -2 & 2 & 2 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \in M_{4\times 4}(\mathbb{R})$$

is diagonalizable or not. If so, find a basis of $\mathbb{R}^4$ consisting of eigenvectors of A.

Solution: The characteristic polynomial is $(-1-t)^3(1-t)$, as one sees from the triangular form of A. The eigenvalues are -1 and 1. A basis of eigenvectors is

$$\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}\}.$$

The first three are eigenvectors of eigenvalue -1 and the last has eigenvalue 1. The matrix is therefore diagonalizable.

4. Give a proof or a counterexample for each of the following statements.
   (a) (2 points) The determinant is a linear map from $M_{n\times n}(F)$ to F.
   (b) (2 points) The union of two subspaces of a vector space is a subspace.
   (c) (2 points) The sum of two surjective linear maps is surjective.
   (d) (2 points) If $T : V \to W$ and $U : W \to Z$ are linear maps such that T is injective and U is surjective, then UT is either injective or surjective.

Solution: (a) False. Consider $n = 2$, $F = \mathbb{R}$, $A = I_2$, $B = -I_2$. Then $\det(A+B) = 0$ while $det(A) + \det(B) = 1 + 1 = 2$,

(b) False, consider the subspaces $x = 0$ and $y = 0$ in $\mathbb{R}^2$. We have elements $(1, 0)$ and $(0, 1)$ in the union such that their sum $(1, 1)$ is in neither subspace, so the union is not closed under addition.

(c) You could use the same example as in (a) !!

(d)Let $T : \mathbb{R}^2 \to \mathbb{R}^3$, $T(a, b) = (a, b, 0)$ and $U : \mathbb{R}^3 \to \mathbb{R}^2$, $U(x, y, z) = (y, z)$. Then it is clear that T is injective and U is surjective, while the map $UT : R^2 \to \mathbb{R}^2$ sends (a, b) to $(b, 0)$ has rank 1, so is neither surjective nor injective.

5. (8 points) Use the Gram-Schmidt procedure to find an orthonomal basis of the subspace of $\mathbb{R}^4$ spanned by $(1, -1, 0, 0)$, $(0, 0, 1, -1)$, $(0, 1, -1, 0)$.