RANK 3 PERMUTATION MODULES OF THE FINITE CLASSICAL GROUPS

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Abstract. The cross-characteristic permutation modules for the actions of the finite classical groups on singular 1-spaces of their natural modules are studied. The composition factors and submodule lattices are determined.

1. Introduction

We study the permutation modules for the action of the classical groups on the singular 1-spaces of their standard modules. These actions have permutation rank 3 and form the main examples among all rank 3 actions of classical groups, cf. [KaL].

In characteristic zero, the theory is well known and dates back to work of D. G. Higman [Hi] in the sixties. In contrast to this, not much is known about the permutation modules in the natural characteristic, apart from some results on symplectic groups [ZS], [Si]. This paper is concerned with the third alternative, in which the fields of the group and the module have different characteristics. The study of these cross-characteristic permutation modules was first taken up by Liebeck in [L1] and [L2]. His work can be understood in terms of a certain endomorphism Δ which we define below. In characteristic zero, Δ has three distinct integer eigenvalues, whose eigenspaces give the three simple direct summands of the permutation module. One of the eigenvalues has multiplicity one and its eigenspace is the trivial submodule. These three eigenvalues may fail to be distinct modulo ℓ and, accordingly, the ℓ-modular permutation module has a more complicated submodule structure. Liebeck was able to determine this submodule structure in all cases except when the two nontrivial eigenvalues are equal modulo ℓ.

It is these remaining cases which interest us. The symplectic groups (when q is odd and $\ell = 2$) were treated in [LST], so we are left with the various types of orthogonal

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and unitary groups. Drawing upon and extending Liebeck's methods, we prove the following result.

Theorem 1.1. *Let $\mathbb{F}$ be an algebraically closed field of characteristic $\ell > 0$. Let $\mathbf{G}$ be one of the following groups: $\mathbf{GU}(2m, q^2)$ with $m \geq 2$ and $\ell|(q+1)$, $\mathbf{GO}^+(2m, q)$ with $m \geq 3$ and $\ell|(q+1)$, $\mathbf{GO}(2m+1, q)$ with $m \geq 3$, q odd and $\ell = 2$, or $\mathbf{GO}^-(2m, q)$ with $m \geq 3$ and $\ell|(q+1)$. Then the $\mathbb{F}\mathbf{G}$-permutation module $\mathbb{F}^{\mathbf{P}_0}$ of $\mathbf{G}$ on the set $\mathbf{P}_0$ of singular 1-spaces of its natural module has the structure as given in Figures 1 – 8, with dimensions of the simple composition factors as shown. In these figures $\kappa_{j,m} = 1$ if $j \mid m$ and 0 otherwise.*

The conditions on ℓ are precisely those under which the two nontrivial eigenvalues of Δ are equal. (See row $d = c$ of Table 3 in §2.4 below.)

The theorem is stated for the full isometry group of the form but there are also related groups such as the commutator subgroups and the conformal groups which also act on the set of singular 1-spaces. We have attempted to include as much information as possible concerning the structure of the permutation module for these variants of the isometry groups. This information can be found in the more precise statements which make up the proof of the theorem. There is an indeterminacy for $\mathbf{SU}(4, q^2)$ when $\ell = 2$.

$\ell \nmid m:$

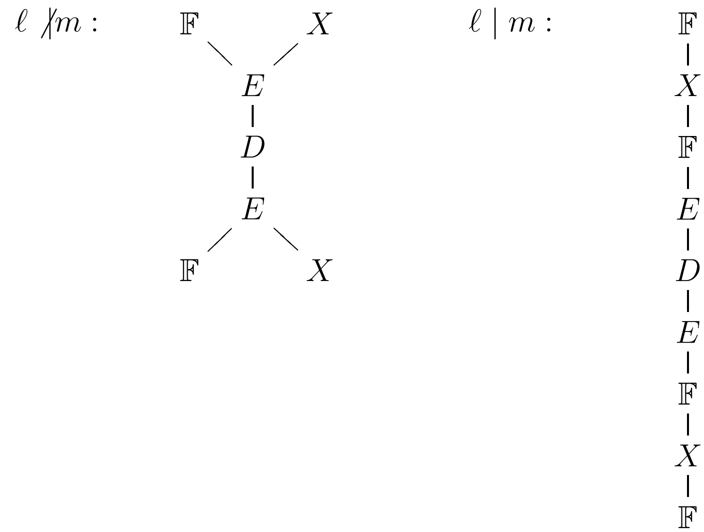
$$\begin{tikzcd}
\mathbb{F} \arrow[d, dash] & X \arrow[d, dash] \arrow[dl, dash] \\
D \arrow[d, dash] \arrow[dr, dash] & Y \arrow[d, dash] \\
\mathbb{F} & X
\end{tikzcd}$$

$\ell \mid m:$

$$\begin{tikzcd}
 & \mathbb{F} \arrow[d, dash] & \\
 & X \arrow[dl, dash] \arrow[ddr, dash] & \\
\mathbb{F} \arrow[d, dash] & & \\
D \arrow[d, dash] & & Y \arrow[ddl, dash] \\
\mathbb{F} \arrow[dr, dash] & & \\
 & X \arrow[d, dash] & \\
 & \mathbb{F} &
\end{tikzcd}$$

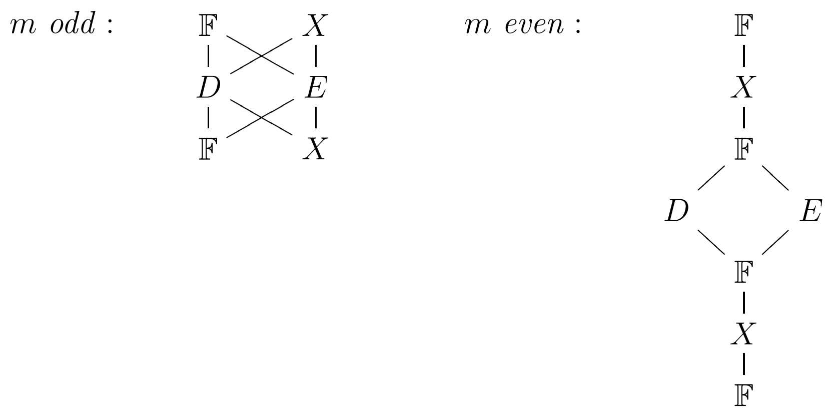
$$\dim Y = \frac{(q^{2m}-1)(q^{2m-1}-q)}{(q+1)^2}, \quad \dim D = \frac{q^{2m}-1}{q+1}, \quad \dim X = \frac{(q^{2m}-1)(q^{2m-1}+1)}{(q^2-1)(q+1)} - 1 - \kappa_{l,m}.$$

FIGURE 1. Submodule structure of $\mathbb{F}^{\mathbf{P}_0}$ for $\mathbf{GU}(2m, q^2)$ when $\ell \mid (q+1)$.



$$\dim X = \frac{(q^{2m+1}+1)(q^{2m}-q^2)}{(q^2-1)(q+1)} - \kappa_{\ell,m}, \quad \dim E = \frac{q^{2m+1}-q}{q+1}, \quad \dim D = \frac{(q^{2m}-1)(q^{2m+1}+1)}{(q+1)^2}.$$

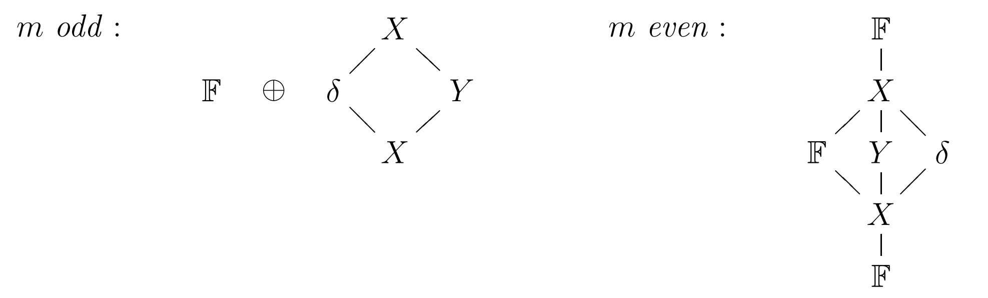
FIGURE 2. Submodule structure of $\mathbb{F}^{\mathbf{P}_0}$ for $\mathbf{GU}(2m+1, q^2)$ when $\ell \mid (q+1)$ and ℓ is odd or $\ell = 2$ and $q \equiv 3 (\mathrm{mod}\ 4)$.



$$\dim X = \frac{(q^{2m+1}+1)(q^{2m}-q^2)}{(q^2-1)(q+1)} - \kappa_{\ell,m}, \dim E = \frac{q^{2m+1}-q}{q+1}, \dim D = \frac{(q^{2m}-1)(q^{2m+1}+q^2+q+1)}{(q+1)^2}.$$

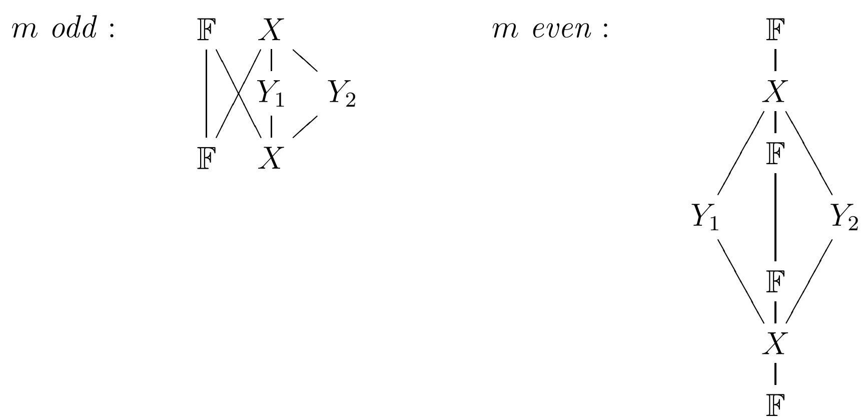
FIGURE 3. Submodule structure of $\mathbb{F}^{\mathbf{P}_0}$ for $\mathbf{GU}(2m+1, q^2)$ when $\ell = 2$ and $q \equiv 1 (\mathrm{mod}\ 4)$.

To illustrate the main idea of the proof we consider the problem of determining the composition factors of a module over a group algebra $\mathbb{F}G$. If we have an $\mathbb{F}G$-filtration of the module, the number of nonzero factors in the filtration is a lower bound for



$$\dim Y = \tfrac{(q^m-1)(q^{m-1}-1)}{q+1}, \quad \dim \delta = 1, \quad \dim X = \tfrac{(q^m-1)(q^{m-1}+q)}{q^2-1} - 1 - \kappa_{2,m}.$$

FIGURE 4. Submodule structure of $\mathbb{F}^{\mathbf{P}_0}$ for $\mathbf{GO}^+(2m,q)$ when $\ell \neq 2$ and $\ell \mid (q+1)$.

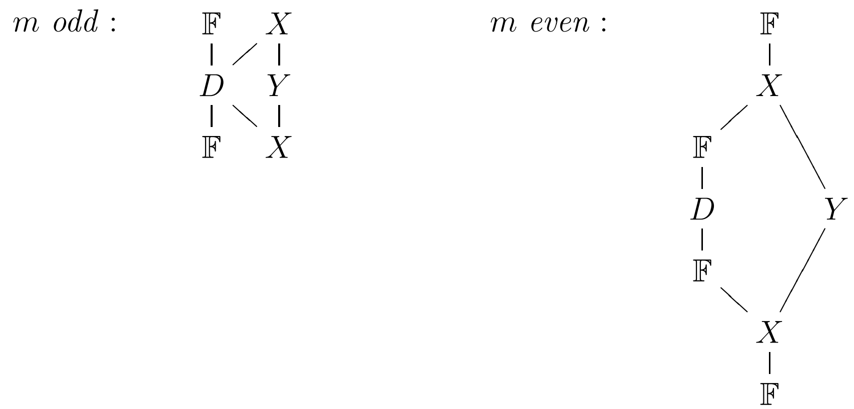


$$\dim Y_1 = \dim Y_2 = \tfrac{(q^m-1)(q^{m-1}-1)}{2(q+1)}, \quad \dim X = \tfrac{(q^m-1)(q^{m-1}+q)}{q^2-1} - 1 - \kappa_{2,m}.$$

FIGURE 5. Submodule structure of $\mathbb{F}^{\mathbf{P}_0}$ for $\mathbf{GO}^+(2m,q)$, q odd, when $\ell = 2$.

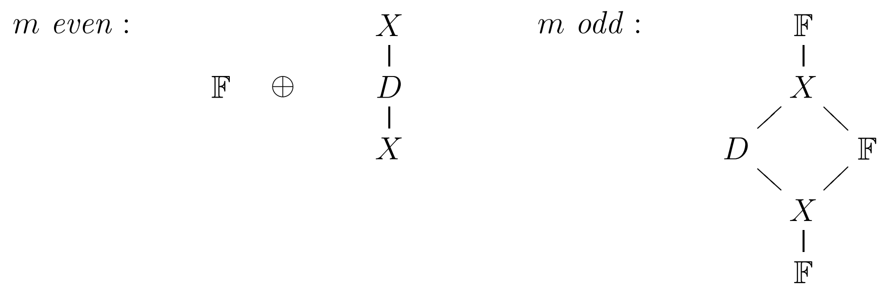
the number of composition factors (counting multiplicities.) On the other hand if, for some subgroup $H \leq G$, we know the $\mathbb{F}H$-composition factors of the module then we can obtain an upper bound on the number of $\mathbb{F}G$-composition factors. If these bounds happen to coincide, our problem will be solved. Indeed, at that point we will know more than just the composition factors, since the filtration tells us something about the submodule structure. Though over-simplified, this is essentially the approach we take.

In §2, after some preliminaries, we make use of endomorphisms and the natural inner product structure of the permutation module to produce submodules. These



$\dim Y = \frac{(q^m-1)(q^m-q)}{2(q+1)}$, $\quad \dim D = \frac{(q^m+1)(q^m+q)}{2(q+1)} - 1$, $\quad \dim X = \frac{q^{2m}-q^2}{q^2-1} - \kappa_{2,m}$.

FIGURE 6. Submodule structure of $\mathbb{F}^{\mathbf{P}_0}$ for $\mathbf{GO}(2m+1, q)$, q odd, when $\ell = 2$.



$\dim D = \frac{(q^m+1)(q^{m-1}+1)}{q+1} - 1$, $\quad \dim X = \frac{(q^m+1)(q^{m-1}-q)}{q^2-1} - 1 + \kappa_{2,m}$.

FIGURE 7. Submodule structure of $\mathbb{F}^{\mathbf{P}_0}$ for $\mathbf{GO}^{-}(2m, q)$ when $\ell \neq 2$ and $\ell \mid (q+1)$.

submodules reflect arithmetical relationships among combinatorial parameters associated with each classical group (Tables 1 and 2). The family of submodules obtained from them by taking sums and intersections plays the role of the filtration in the sketch above.

Continuing with the strategy outlined above, the second stage of the proof involves finding the composition factors of the permutation module for a suitable subgroup. In all cases this subgroup is taken to be the stabilizer $\tilde{P}$ of a maximal totally singular subspace. The action of the largest normal p-subgroup Q of $\tilde{P}$ on the singular points can be understood in terms of the geometry of the form. This is a typical situation where the standard results of Clifford theory [Dade] apply and we are thus able to *m even* :

$$\begin{tikzcd}[column sep=small, row sep=small]
\mathbb{F} \arrow[d, dash] \arrow[dr, dash] & X \arrow[d, dash] \arrow[dl, dash] \\
D_1 \arrow[d, dash] \arrow[dr, dash] & D_2 \arrow[d, dash] \arrow[dl, dash] \\
\mathbb{F} & X
\end{tikzcd}$$

m odd :

$$\begin{tikzcd}[column sep=small, row sep=small]
& \mathbb{F} \arrow[d, dash] & \\
& X \arrow[d, dash] & \\
& \mathbb{F} \arrow[dl, dash] \arrow[dr, dash] & \\
D_1 \arrow[dr, dash] & & D_2 \arrow[dl, dash] \\
& \mathbb{F} \arrow[d, dash] & \\
& X \arrow[d, dash] & \\
& \mathbb{F} &
\end{tikzcd}$$

$$\dim D_1 = \dim D_2 = \tfrac{(q^m+1)(q^{m-1}+1)}{2(q+1)} - 1, \quad \dim X = \tfrac{(q^m+1)(q^{m-1}-q)}{q^2-1} - 1 + \kappa_{2,m}.$$

FIGURE 8. Submodule structure of $\mathbb{F}^{\mathbf{P}_0}$ for $\mathbf{GO}^-(2m,q)$, q odd, when $\ell = 2$.

determine almost exactly the decomposition of the permutation module for $\tilde{P}$. Unlike the first part of the proof, in which we are able to treat all groups simultaneously, the differences in the structure of Q and some exceptional phenomena in small cases necessitate seperate calculations for each group. As can be seen in Figures 1 – 8, these variations are reflected in the submodule lattices.

Those results which can be treated uniformly are given in §3 together with other general lemmas. The detailed calculations for each series of groups are given in §4 (unitary groups in even dimension), §5 (unitary groups in odd dimension), §6 (split orthogonal groups in even dimension), §7 (orthogonal groups in odd dimension) and §8 (non-split orthogonal groups in even dimension). In each case, after obtaining the decomposition of the permutation module for $\widetilde{P}$, the decomposition for G is not immediate as in the sketch and some further arguments are needed, including, in the hardest cases, results from [T] on dual pairs. For the unitary groups in odd dimension, to find the decomposition with respect to G we also need to invoke some techniques of Harish-Chandra theory. In all cases, we are able to proceed further and give a complete description of the submodule lattice of the permutation module.

The $\mathbb{F}_2$-permutation modules for groups of odd characteristic have been studied in small ranks by Brouwer et. al. [BWH] in connection with the codes associated with generalized quadrangles. These are examples of the modules considered here and our results will include solutions of the conjectures posed there. See Remark 8.12 below.

In the course of our work we are also able to settle some questions concerning representations of small degree. We prove the sharpness of the lower bound given by Hoffman [Ho] on the smallest degree of nontrivial representations of $\Omega^+(2n,q)$ in characteristic $\ell|(q+1)$, cf. Corollary 6.7. We also prove that the lower bound on the "second smallest" degree of $\mathbf{SU}(n,q^2)$ given in [GMST] is sharp in characteristic $\ell|(q+1)$, cf. Corollaries 4.8 and 5.10.

2. The permutation module on singular points

2.1. **Notation and Background.** Let $q=p^t$ be a prime power. Let V be either a vector space over $\mathbb{F}_q$, endowed with a nondegenerate alternating bilinear form or a quadratic form, or a vector space over $\mathbb{F}_{q^2}$ carrying a nondegenerate hermitian form. Our convention is that all hermitian forms are linear on the second component.

Let the dimension of V be $2m$ or $2m+1$ and let $\mathbf{G}$ be the group of linear automorphisms of V which preserve the given form. Thus $\mathbf{G}$ is isomorphic to one of the groups $\mathbf{Sp}(2m,q)$, $\mathbf{GO}(2m+1,q)$, $\mathbf{GO}^+(2m,q)$, $\mathbf{GO}^-(2m,q)$, $\mathbf{GU}(2m,q^2)$ or $\mathbf{GU}(2m+1,q^2)$.

When dealing with quadratic forms, the term singular subspace will refer to the quadratic form, so that a maximal singular subspace is one of dimension equal to the Witt index.

We begin by summarizing some standard theory from [Hi]. For any subspace W of V, let $\mathbf{P}(W)$, resp. $\mathbf{P}_0(W)$ denote the set of all 1-spaces, resp. all singular 1-spaces, that are contained in W. Let $\mathbb{Z}^{\mathbf{P}_0}$ be the permutation module for the action of $\mathbf{G}$ on $\mathbf{P}_0=\mathbf{P}_0(V)$. To avoid possible confusion later, we point out that the scalar matrices in $\mathbf{G}$ act trivially, so it is really the projective group which acts faithfully. However, it is more convenient to consider the modules as $\mathbb{F}\mathbf{G}$-modules since elements of $\mathbf{G}$ are matrices.

For $x\in\mathbf{P}_0$, we denote by $\Phi(x)$ the set of elements of $\mathbf{P}_0\setminus\{x\}$ which are orthogonal to x and by $\Delta(x)$ the set of those which are not.

For any subset S of $\mathbf{P}_0$ we will denote by $[S]$ the element $\sum_{x\in S}x$ of $\mathbb{Z}^{\mathbf{P}_0}$, and we write $\mathbf{1}=\mathbf{1}_{\mathbf{P}_0}$ for the element $[\mathbf{P}_0]$. By abuse of notation, we can also apply the above notation to a homogeneous subset S of singular vectors of V, i.e. one invariant under multiplication by nonzero scalars, by identifying S with the set of 1-spaces it contains.

There is a natural inner product on $\mathbb{Z}^{\mathbf{P}_0}$, with $\mathbf{P}_0$ as an orthonormal basis, which we shall denote by $\langle z,w\rangle$.

For $x\in\mathbf{P}_0$, we define

$$a=|\Delta(x)|,\quad r=|\Delta(x)\cap\Delta(y)|\ (y\in\Delta(x)),\quad s=|\Delta(x)\cap\Delta(y)|\ (y\in\Phi(x)) \tag{1}$$

These numbers do not depend on x.

Next we consider the map

$$\Delta:\mathbb{Z}^{\mathbf{P}_0}\to\mathbb{Z}^{\mathbf{P}_0},\qquad x\mapsto[\Delta(x)]. \tag{2}$$

It is not difficult to show using the observation that $\mathbf{1}$ is an eigenvector of Δ with eigenvalue a and the definitions that Δ satisfies the equation

$$(\Delta - a)(\Delta^2 - (r-s)\Delta - (a-s)) = 0\,. \tag{3}$$

It is shown in [Hi] that the two roots other than a are also integers, and as $a-s>0$, they are different from a and exactly one of them is negative. We denote them by c and d, with $c>0$, $d<0$. We also note that $a-s$ is a power of q in all our examples.

The eigenspaces of Δ in $\mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{Z}^{\mathbf{P}_0}$ are the three irreducible constituents of this $\mathbb{Q}\mathbf{G}$-permutation module. The a-eigenspace affords the trivial module. Let φ be the character afforded by the d-eigenspace and ψ the character afforded by the c-eigenspace. The degrees can easily be found using the fact Δ has trace zero, together with the other obvious fact that the multiplicities add up to $|\mathbf{P}_0|$. For convenience, we have listed these combinatorial data in Tables 1 and 2.

$\mathbf{G}$	$\mathbf{GO}^+(2m,q)$	$\mathbf{GO}^-(2m,q)$	$\mathbf{GO}(2m+1,q)$
$\lvert\mathbf{P}_0\rvert$	$\frac{(q^m-1)(q^{m-1}+1)}{q-1}$	$\frac{(q^m+1)(q^{m-1}-1)}{q-1}$	$\frac{q^{2m}-1}{q-1}$
$\varphi(1)$	$\frac{q(q^m-1)(q^{m-2}+1)}{q^2-1}$	$\frac{q^2(q^{m-1}+1)(q^{m-1}-1)}{q^2-1}$	$\frac{q(q^m-1)(q^{m-1}+1)}{2(q-1)}$
$\psi(1)$	$\frac{q^2(q^{m-1}+1)(q^{m-1}-1)}{q^2-1}$	$\frac{q(q^m+1)(q^{m-2}-1)}{q^2-1}$	$\frac{q(q^m+1)(q^{m-1}-1)}{2(q-1)}$
$\lvert\Delta(x)\rvert$	q^{2m-2}	q^{2m-2}	q^{2m-1}
$\lvert\Phi(x)\rvert$	$\frac{q(q^{m-1}-1)(q^{m-2}+1)}{q-1}$	$\frac{q(q^{m-1}+1)(q^{m-2}-1)}{q-1}$	$\frac{q(q^{2m-2}-1)}{q-1}$
r	$q^{m-2}(q-1)(q^{m-1}-1)$	$q^{m-2}(q-1)(q^{m-1}+1)$	$q^{2m-2}(q-1)$
s	$q^{2m-3}(q-1)$	$q^{2m-3}(q-1)$	$q^{2m-2}(q-1)$
d, c	$-q^{m-1}$, q^{m-2}	$-q^{m-2}$, q^{m-1}	$-q^{m-1}$,q^{m-1}

TABLE 1. Combinatorial parameters

Let $\mathbb{F}$ be any field and let ℓ be its characteristic. We will keep the same notations for the elements and induced maps of the $\mathbb{F}\mathbf{G}$-permutation module $\mathbb{F}^{\mathbf{P}_0}=\mathbb{F}\otimes_{\mathbb{Z}}\mathbb{Z}^{\mathbf{P}_0}$ as we used for $\mathbb{Z}^{\mathbf{P}_0}$. The symmetric bilinear form induced by the inner product will still be non-singular and will also be denoted by $\langle z,w\rangle$. For any subset A of $\mathbb{F}^{\mathbf{P}_0}$ we shall write $A^{\perp}$ for the subspace of elements orthogonal to A with respect to this form.

2.2. **The modules U and $\mathcal{C}$.** Let e be the number of singular 1-subspaces in a maximal (totally) singular subspace W which are not orthogonal to a fixed singular 1-space outside W. This does not depend on W or the 1-space chosen. Now e is simply the number of 1-spaces in W which lie outside a given subspace of codimension 1 and so it is a power of q.

$\mathbf{G}$	$\mathbf{GU}(2m,q^2)$	$\mathbf{GU}(2m+1,q^2)$	$\mathbf{Sp}(2m,q)$
$\lvert\mathbf{P}_0\rvert$	$\frac{(q^{2m}-1)(q^{2m-1}+1)}{q^2-1}$	$\frac{(q^{2m}-1)(q^{2m+1}+1)}{q^2-1}$	$\frac{q^{2m}-1}{q-1}$
$\varphi(1)$	$\frac{q^2(q^{2m}-1)(q^{2m-3}+1)}{(q+1)(q^2-1)}$	$\frac{q^3(q^{2m}-1)(q^{2m-1}+1)}{(q+1)(q^2-1)}$	$\frac{q(q^m-1)(q^{m-1}+1)}{2(q-1)}$
$\psi(1)$	$\frac{q^3(q^{2m-2}-1)(q^{2m-1}+1)}{(q+1)(q^2-1)}$	$\frac{q^2(q^{2m+1}+1)(q^{2m-2}-1)}{(q+1)(q^2-1)}$	$\frac{q(q^m+1)(q^{m-1}-1)}{2(q-1)}$
$\lvert\Delta(x)\rvert$	q^{4m-3}	q^{4m-1}	q^{2m-1}
$\lvert\Phi(x)\rvert$	$\frac{q^2(q^{2m-2}-1)(q^{2m-3}+1)}{q^2-1}$	$\frac{q^2(q^{2m-1}+1)(q^{2m-2}-1)}{q^2-1}$	$\frac{q(q^{2m-2}-1)}{q-1}$
r	$q^{2m-3}(q^{2m}-q^{2m-2}-q+1)$	$q^{2m-2}(q^{2m+1}-q^{2m-1}+q-1)$	$q^{2m-2}(q-1)$
s	$q^{4m-5}(q^2-1)$	$q^{4m-3}(q^2-1)$	$q^{2m-2}(q-1)$
$d,\ c$	$-q^{2m-2},\ q^{2m-3}$	$-q^{2m-2}, q^{2m-1}$	$-q^{m-1},\ q^{m-1}$

TABLE 2. Combinatorial parameters (cont.)

Since the relation of non-orthogonality is symmetric, it follows that the map Δ is self-adjoint with respect to the natural form on $\mathbb{F}^{\mathbf{P}_0}$. As $\mathrm{End}_{\mathbb{F}\mathbf{G}}(\mathbb{F}^{\mathbf{P}_0})$ has a basis consisting of Δ, the identity map and the map $\mathbf{1}$ sending every element of $\mathbf{P}_0$ to $\mathbf{1}$, it follows that all $\mathbb{F}\mathbf{G}$-endomorphisms are self-adjoint.

Define $f : \mathbb{F}^{\mathbf{P}_0} \to \mathbb{F}^{\mathbf{P}_0}$ by $f(\omega) = e\omega + [\Delta(\omega)]$. This is clearly a map of $\mathbb{F}\mathbf{G}$-modules.

Lemma 2.1. *Let W be a maximal singular subspace and let $x \in \mathbf{P}_0$. Then*

(i) $\langle [W], f(x)\rangle = e$.
(ii) $f([W]) = e\mathbf{1}$.
(iii) $\langle f(x), \mathbf{1}\rangle = e + |\Delta(x)|$.
(iv) $e = -d$, *where d is the unique negative eigenvalue of Δ.*

Proof. If $x \subset W$, then no element of $\Delta(x)$ lies in W, so the bilinear product has value e, while if x does not lie in W, the value of the bilinear product is the number of 1-spaces of W which lie in $\Delta(x)$, which is e. This proves (i). Part (ii) is immediate by the self-adjoint property of f. Part (iii) is clear. By (ii), f has a nonzero kernel, proving (iv). □

As f is self-adjoint, we may define a new symmetric bilinear form on $\mathbb{F}^{\mathbf{P}_0}$ by the formula

$$\langle x, y\rangle_f = \langle f(x), y\rangle = \langle x, f(y)\rangle. \tag{4}$$

The radical of the new form is $\mathrm{Ker} f$. Let U be the image of f. We therefore have a non-singular $\mathbf{G}$-invariant symmetric bilinear form on U defined by

$$[f(x), f(y)] = \langle x, y\rangle_f. \tag{5}$$

In particular we have proved the following fact.

Lemma 2.2. *The $\mathbb{F}\mathbf{G}$-module U is isomorphic to its dual.* □

Let $\mathcal{C}$ be the submodule of $\mathbb{F}^{\mathbf{P}_0}$ spanned by the characteristic vectors $[W]$ of maximal singular subspaces of V. We think of $\mathcal{C}$ as the image of the natural incidence map $\mathbb{F}^{\Omega_{max}} \to \mathbb{F}^{\mathbf{P}_0}$, where Ω_{max} is the set of maximal singular subspaces of V. Let $\mathcal{C}' \subseteq \mathcal{C}$ denote the submodule spanned by the differences $[W] - [W']$ of generators of $\mathcal{C}$. Let $U' \subseteq U$ be the submodule generated by the elements $f(x) - f(y)$, for x, $y \in \mathbf{P}_0$. (Liebeck [L1] uses the term *graph submodule* for U'.) It is clear from the definitions that $\mathcal{C}'$ and U' are nonzero $\mathbb{F}\mathbf{G}$-submodules of $\mathbb{F}^{\mathbf{P}_0}$. By Lemma 2.1(i), we have

$$\langle \mathcal{C}, U' \rangle = 0, \qquad \langle \mathcal{C}', U \rangle = 0. \tag{6}$$

Lemma 2.3. (i) $\dim_{\mathbb{F}} \mathcal{C}/\mathcal{C} \cap \mathcal{C}^{\perp} \geq 2$.

(ii) *If $\mathbf{G}$ has a subgroup H which acts transitively on the set of maximal singular subspaces of V, then $\dim_{\mathbb{F}} \mathrm{Hom}_{\mathbb{F}H}(\mathcal{C}, \mathbb{F}) \leq 1$.*

(iii) *If the subgroup H of (ii) is perfect and if $\mathcal{C}/\mathcal{C} \cap \mathcal{C}^{\perp}$ has a one-dimensional $\mathbb{F}H$-composition factor then the inequality in (i) is strict.*

Proof. For each dimension r from 0 to the dimension of a maximal singular subspace, one can find two maximal singular subspaces which have an r-dimensional intersection. In particular, by taking $r = 0$ and $r = 1$, we see that the bilinear product in $\mathcal{C}/\mathcal{C} \cap \mathcal{C}^{\perp}$ can take the values 0 or 1 on different pairs of nonzero elements, which proves (i). (We are assuming that nonzero singular subspaces exist, i.e we exclude the case of 2-dimensional anisotropic quadratic modules.) In (ii) the fact that $\mathcal{C}$ is a homomorphic image of the permutation module on maximal singular subspaces implies by Frobenius reciprocity that $\dim_{\mathbb{F}} \mathrm{Hom}_{\mathbb{F}H}(\mathcal{C}, \mathbb{F}) \leq 1$. Finally, to prove (iii), let H be the perfect subgroup given and suppose for a contradiction that the dimension is equal to 2. Then since H is perfect, it must act trivially on $\mathcal{C}/\mathcal{C} \cap \mathcal{C}^{\perp}$. But this contradicts (ii). □

2.3. **Cross-characteristic permutation modules.** In this subsection, we assume that d is invertible in $\mathbb{F}$. Since in our examples $-d$ is a power of q, the lemmas of this subsection will apply to our examples whenever q is not a power of ℓ.

Lemma 2.4. *Assume that d is a unit in $\mathbb{F}$.*

(i) *$\mathcal{C}/\mathcal{C}'$ and U/U' are one-dimensional trivial modules. Also, for H as in part (ii) of Lemma 2.3, we have $\mathrm{Hom}_{\mathbb{F}H}(\mathcal{C}, \mathbb{F}) \cong \mathbb{F}$.*

(ii) *$\mathbf{1} \in U$ and $\mathbf{1}$ is singular with respect to the form $[-,-]$ on U if and only if ℓ divides the number of singular 1-spaces in a maximal singular subspace of V.*

Proof. The first part of (i) follows from Lemma 2.1(i) and (6), and then the second part follows from Lemma 2.3(ii). By Lemma 2.1(ii) we have $\mathbf{1} = (-d)^{-1} f[W] \in U$ and from the definitions, we have $[\mathbf{1}, \mathbf{1}] = (-d)^{-1}\langle \mathbf{1}, [W] \rangle$, proving (ii). □

Let W be a maximal totally singular subspace and $\mathbf{G}_W$ its stabilizer in $\mathbf{G}$. The group $\mathbf{G}_W$ acts as $\mathbf{GL}(W)$ on W, by Witt's Lemma. The induced action on $\mathbf{P}(W)$ is doubly transitive, so let its character be $1+\tau$. Assume W has dimension m over $\mathbb{F}_q$. The structure of the mod ℓ permutation module $\mathbb{F}^{\mathbf{P}(W)}$ is well known (cf. [Mo] for $m \geq 3$, [Bu] for $m = 2$).

Lemma 2.5. *If $\ell \nmid \frac{q^m-1}{q-1}$ then the $\mathbb{F}\mathbf{GL}(W)$-permutation module $\mathbb{F}^{\mathbf{P}(W)}$ is the direct sum of $\mathbb{F}$ and a simple module. If $\ell | \frac{q^m-1}{q-1}$ then $\mathbb{F}^{\mathbf{P}(W)}$ is uniserial with three composition factors, the top and bottom ones being isomorphic to $\mathbb{F}$. Further, the restriction of the simple constituents to $\mathbf{SL}(W)$ remain simple except when $m = 2$ and $\ell = 2$, in which case the nontrivial simple component splits into two simple $\mathbf{SL}(W)$-modules of dimension $(q-1)/2$.* □

Lemma 2.6. (i) *Assume that d is a unit in $\mathbb{F}$. Then as an $\mathbb{F}\mathbf{G}_W$-submodule, U has a direct summand isomorphic to $\mathbb{F}^{\mathbf{P}(W)}$.*
(ii) *The character τ is a constituent of $\varphi|_{\mathbf{G}_W}$ and of $\psi|_{\mathbf{G}_W}$.*

Proof. Let $\omega \in \mathbf{P}(W) \subseteq \mathbf{P}_0(V)$. Then $\Delta(\omega)$ is disjoint from $\mathbf{P}(W)$ so it is clear that the elements $f(\omega) = e\omega + [\Delta(\omega)]$, $\omega \in \mathbf{P}(W)$ are linearly independent and span an $\mathbb{F}\mathbf{G}_W$-submodule of U isomorphic to $\mathbb{F}^{\mathbf{P}(W)}$. The restriction to U of the natural projection from $\mathbf{P}_0(V)$ onto $\mathbf{P}(W)$ gives the $\mathbb{F}\mathbf{G}_W$-splitting.

Assume now that we are in characteristic zero. By definition, $\mathrm{Ker} f$ is the d-eigenspace of Δ, which affords φ, so U affords $1+\psi$, proving that $\psi|_{\mathbf{G}_W}$ has τ as a constituent. We can similarly consider the endomorphism $f_c(\omega) = [\Delta(\omega)] - c\omega$ and see, by the same argument as above, that its image affords $1+\varphi$ and its restriction to $\mathbf{G}_W$ contains $1+\tau$. □

Remark 2.7. Let ρ be the permutation character of $\mathbf{G}$ acting on $\mathbf{P}_0(V)$ and let Q denote the largest normal p-subgroup of $\mathbf{G}_W$. Since $\mathbf{G}_W$ has two orbits on $\mathbf{P}_0(V)$, $\rho|_{\mathbf{G}_W}$ has two trivial constituents. Together with Lemma 2.6(ii), this tells us that the Q-trivial part of $\rho|_{\mathbf{G}_W}$ contains at least $2.1_{\mathbf{G}_W} + 2\tau$.

2.4. **The case $c = d$.** In [L1] a *graph submodule* is defined for each of the eigenvalues c and d. When $c \neq d$, consideration of the properties of these two submodules suffices to determine the submodule lattice of $\mathbb{F}^{\mathbf{P}_0}$, and these lattices are described in [L1], [L2]. The module U' we have defined above is the graph submodule for d. In the case $c = d$ of interest to us, this will be the unique graph submodule.

In this subsection we will be dealing with three conditions which depend on ℓ, m and the geometry of V. These conditions and when they are satisfied are in Table 3.

In the case $c = d$ of a unique graph submodule, the important minimality properties of graph submodules proved in [L1, Theorem 2.6] and [L2, Theorems 1.1, 2.1] can be stated as the following very useful lemma.

Lemma 2.8. *Every $\mathbb{F}\mathbf{G}$-submodule of $\mathbb{F}^{\mathbf{P}_0}$ which is not contained in $\mathbb{F}\mathbf{1}$ contains U'.* □

$\mathbf{G}$	$\mathbf{GO}^+(2m,q)$	$\mathbf{GO}^-(2m,q)$	$\mathbf{GO}(2m+1,q)$
$d=c$	$\ell \mid (q+1)$	$\ell \mid (q+1)$	$\ell = 2$
$U \subseteq U^\perp$, $d=c$	$\ell = 2$	$\ell = 2$	always
$[\mathbf{1},\mathbf{1}]=0$, $d=c$	m even	m odd	m even
$\mathbf{G}$	$\mathbf{GU}(2m,q^2)$	$\mathbf{GU}(2m+1,q^2)$	$\mathbf{Sp}(2m,q)$
$d=c$	$\ell \mid (q+1)$	$\ell \mid (q+1)$	$\ell = 2$
$U \subseteq U^\perp$, $d=c$	always	always	always
$[\mathbf{1},\mathbf{1}]=0$, $d=c$	$\ell \mid m$	$\ell \mid m$	m even

TABLE 3. Conditions on ℓ

Remark 2.9. We remark that, [L1, Theorem 2.6] is stated for the group $\mathbf{PSU}(2m,q^2)$, for which $m=2$, $\ell=2$ is an exception to this statement. For this group the module $U'/\mathbb{F}\mathbf{1}$ splits into two simple modules. However, the module is simple over $\mathbf{G}=\mathbf{GU}(2m,q^2)$, for which the above minimal property of U' always holds.

Corollary 2.10. *Assume $c=d$.*

(i) *If $[\mathbf{1},\mathbf{1}]\neq 0$ then $U=U'\oplus\mathbb{F}\mathbf{1}$ and U' is simple.*
(ii) *If $[\mathbf{1},\mathbf{1}]=0$ then U is uniserial, with composition series $\mathbb{F}\mathbf{1}\subset U'\subset U$, with $U/U'\cong\mathbb{F}$.*

Proof. Both parts are seen by combining Lemma 2.8 with Lemma 2.4(ii). □

Notation 2.11. *Let X denote the nontrivial composition factor of U'.*

Remark 2.12. By Lemma 2.8 (or direct computation), we have $U'\subseteq {U'}^\perp$. We observe that, as a consequence of Lemma 2.8, Corollary 2.10 and the self-duality of $\mathbb{F}^{\mathbf{P}_0}$, we will know the entire submodule structure of $\mathbb{F}^{\mathbf{P}_0}$ if we know that of ${U'}^\perp/U'$.

Lemma 2.13. *Suppose that $c=d$. Then*

(i) *X occurs as a composition factor in $\mathbb{F}^{\mathbf{P}_0}$ with multiplicity ≥ 2.*
(ii) *If $U\subseteq U^\perp$ then the composition multiplicity in $\mathbb{F}^{\mathbf{P}_0}$ of $\mathbb{F}$ is ≥ 4 if $[\mathbf{1},\mathbf{1}]=0$ and ≥ 2 if $[\mathbf{1},\mathbf{1}]\neq 0$.*
(iii) *If $U\subseteq U^\perp$ then we have*

$$U'\subseteq U\subseteq U^\perp\subseteq {U'}^\perp,\quad U/U'\cong\mathbb{F},\quad {U'}^\perp/U^\perp\cong\mathbb{F}. \tag{7}$$

Proof. Since $\mathbb{F}^{\mathbf{P}_0}/{U'}^\perp$ is isomorphic to the dual of U', and the composition factors of U' are self-dual, part (i) follows. Similarly the hypothesis of (ii) gives us the filtration $U\subseteq U^\perp\subseteq\mathbb{F}^{\mathbf{P}_0}$, with $\mathbb{F}^{\mathbf{P}_0}/U^\perp\cong\mathrm{Hom}_{\mathbb{F}}(U,\mathbb{F})\cong U$, since U is self-dual. Therefore, each composition factor of U occurs in $\mathbb{F}^{\mathbf{P}_0}$ with at least twice the multiplicity that it has in U. The inclusions in (iii) are obvious and the two isomorphisms follow from Lemma 2.4(i) and the self-duality of ${U'}^\perp/U'$. □

We note that in most cases, we have $U \subseteq U^{\perp}$. (See Table 3.)

We end this subsection by recalling a general fact. If L is a $\mathbb{Z}\mathbf{G}$-lattice and S is a $\mathbb{Q}\mathbf{G}$-submodule of $\mathbb{Q} \otimes_{\mathbb{Z}} L$, then $S \cap L$ is both a pure sublattice of L and an $\mathbb{Z}$-form of S, so $\mathbb{F} \otimes_{\mathbb{Z}} L$ contains the submodule $\mathbb{F} \otimes_{\mathbb{Z}} (S \cap L)$ which is a mod ℓ reduction of S. Therefore, the minimal property of U' implies the following result.

Corollary 2.14. *The mod ℓ reduction of any $\mathbb{Z}\mathbf{G}$-lattice affording φ or ψ has the composition factors of U' among its composition factors.* □

3. Further Preliminaries

From now on we take $\mathbb{F}$ to be an algebraically closed field of characteristic ℓ, where $\ell \neq p$. Here we consider for each series of groups the mod ℓ decomposition of the permutation character and the submodule structure of $\mathbb{F}^{\mathbf{P}_0}$. We shall assume that ℓ is such that $c = d$, since this is the case which has not been treated in the existing literature.

To be precise, we work with an ℓ-modular system $(R, K, \mathbb{F})$. Here, R is a complete discrete valuation ring of characteristic zero having $\mathbb{F}$ as its residue field and K is its field of fractions. For any character χ of $K\mathbf{G}$, we will denote by $\overline{\chi}$ the associated ℓ-modular Brauer character and for any $\mathbb{F}\mathbf{G}$-module M we denote by $\beta(M)$ the Brauer character it affords. In this paper we will always identify Brauer characters with the corresponding element of the Grothendieck group of $\mathbb{F}\mathbf{G}$. If α and β are Brauer characters, we will say that β *contains* α if $\beta - \alpha$ is a nonnegative integral combination of irreducible Brauer characters.

Let W be a maximal singular subspace and $\mathbf{G}_W$ its stabilizer. Our next result will provide us with a simple way to recognize the composition factors of $U'|_{\mathbf{G}_W}$ from among those of the restrictions of $\overline{\varphi}|_{\mathbf{G}_W}$ and $\overline{\psi}|_{\mathbf{G}_W}$.

Let $1 + \tau$ be the permutation character of $\mathbf{G}_W$ acting on $\mathbf{P}(W)$. We know by Lemma 2.6 that $\overline{\tau}$ is properly contained in $\beta(U')|_{\mathbf{G}_W}$. Hence, by Corollary 2.14, the nonzero Brauer character $\beta(U')|_{\mathbf{G}_W} - \overline{\tau}$ is contained in both $\overline{\varphi} - \overline{\tau}$ and $\overline{\psi} - \overline{\tau}$. We formulate this in the following lemma, which will be used repeatedly in our calculations.

Lemma 3.1. (i) *$\overline{\psi} - \overline{\tau}$ and $\overline{\varphi} - \overline{\tau}$ have a common nontrivial simple $\mathbf{G}_W$-constituent. If there is a unique such contituent σ, then $\beta(U')|_{\mathbf{G}_W} = \overline{\tau} + \sigma$.*

(ii) *If a nontrivial simple constituent σ_1 of $\overline{\psi} - \overline{\tau}$ and one σ_2 of $\overline{\varphi} - \overline{\tau}$ have the same degree and are the unique pair with this property, then $\sigma_1 = \sigma_2$ is the unique common simple constituent.* □

We will frequently use the following two standard facts from Clifford theory:

Lemma 3.2. [Dade] *Let G be a finite group and H a normal subgroup of G. Assume that N is an irreducible $\mathbb{F}G$-module such that $N|_H$ is irreducible, and V is* *an irreducible $\mathbb{F}(G/H)$-module inflated to G. Then the $\mathbb{F}G$-module $N \otimes V$ is also irreducible.* □

Lemma 3.3. *Let P be a finite group with a normal subgroup Z. Let L be a simple $\mathbb{F}Z$-module and $R = \mathrm{Stab}_P(L)$. Assume that M is a simple $\mathbb{F}R$-module such that $\mathrm{Hom}_Z(L, M|_Z) \neq 0$. Then the induced module $\mathrm{Ind}_R^P(M)$ is irreducible.* □

4. The unitary groups in even dimensions

4.1. **Character restriction.** Let $m \geq 2$, $V = \mathbb{F}_{q^2}^{2m}$ be endowed with a non-degenerate hermitian form with Gram matrix $\begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix}$ in a basis $(e_1, \dots, e_m, f_1, \dots, f_m)$. Here and below, I_m denotes the identity $m \times m$-matrix. Let $\tilde{G} = \mathbf{GU}(2m, q^2)$ be the group of all linear transformations of V that preserve the hermitian form, and let $G = \mathbf{SU}(2m, q^2) = \tilde{G} \cap SL(V)$. Fix $W' := \langle e_1, \dots, e_m \rangle$, $W = \langle f_1, \dots, f_m \rangle$, $\tilde{P} := \mathrm{Stab}_{\tilde{G}}(W') = Q : \tilde{L}$, $P := \tilde{P} \cap G = Q : L$, where $Q := O_p(P)$ (the largest normal p-subgroup of P). Let

$$\tilde{L} = \left\{ [A, 0] := \begin{pmatrix} {}^tA^{-1q} & 0 \\ 0 & A \end{pmatrix} \mid A \in \mathbf{GL}(W) \right\},$$

where X^q means the matrix obtained from X by raising each matrix entry to the q-th power, and let

$$L = \left\{ [A, 0] \in \tilde{L} \mid \det(A)^{q-1} = 1 \right\} .$$

Fix $\theta \in \mathbb{F}_{q^2}$ with $\theta^{q-1} = -1$, a primitive p^{th} root ϵ of 1 in $\mathbb{C}$, and let $H_m(q) := \{X \in M_m(\mathbb{F}_{q^2}), X = {}^tX^q\}$. Then

$$Q = \left\{ [I, X] := \begin{pmatrix} I_m & \theta X \\ 0 & I_m \end{pmatrix} \mid X \in H_m(q) \right\},$$

and any linear character of Q is of the form

$$\lambda_B \ : \ \begin{pmatrix} I_m & \theta X \\ 0 & I_m \end{pmatrix} \mapsto \epsilon^{\mathrm{tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mathrm{Tr}(BX))}$$

for some $B \in H_m(q)$. If a Q-invariant subspace M of a $\mathbb{F}\tilde{P}$-module affords the Brauer Q-character λ_B, then for any $a := [A, 0] \in \tilde{L}$, $a(M)$ affords the Q-character $a \circ \lambda_B : x \mapsto \lambda_B(a \circ x)$, where $a \circ x = a^{-1}xa$ for $x \in Q$, and

$$(a \circ \lambda_B)(x) = \lambda_B(a \circ x) = \lambda_{AB\,{}^tA^q}(x). \tag{8}$$

Let $\mathcal{O}_1$ be the set of all λ_B with $\mathrm{rank}(B) = 1$ and $\mathcal{O}_2$ the set of all λ_B with $\mathrm{rank}(B) = 2$. Then $|\mathcal{O}_1| = (q^{2m} - 1)/(q + 1)$, $|\mathcal{O}_2| = (q^{2m} - 1)(q^{2m-1} - q)/(q^2 - 1)(q + 1)$. The subgroup L acts transitively on $\mathcal{O}_1$, and on $\mathcal{O}_2$ if $m \geq 3$, or if $m = 2$ and q is even. If $m = 2$ and q is odd, then $\mathcal{O}_2$ splits into two L-orbits, $\mathcal{O}_2^+$ consisting of all λ_B

with $\det(B)$ a nonzero square in $\mathbb{F}_q$ and $\mathcal{O}_2^-$ consisting of those λ_B with $\det(B)$ a non-square. In any event, $\tilde{L}$ acts transitively on $\mathcal{O}_2$.

Let ρ be the permutation character of $\tilde{G}$ on $\mathbf{P}_0(V)$. Clearly, Q acts trivially on $\mathbf{P}_0(W')$. Next, for any $l := \langle f\rangle \in \mathbf{P}(W)$, Q acts transitively on the set of q^{2m-1} singular 1-spaces $\langle f+u\rangle$ with $u \in W'$. Denoting $Q_l := \mathrm{Stab}_Q(l)$, we then have

$$\rho|_Q = \frac{q^{2m}-1}{q^2-1}\cdot 1_Q + \sum_{l\in\mathbf{P}(W)} \mathrm{Ind}_{Q_l}^Q(1_{Q_l}). \tag{9}$$

In particular, the multiplicity of 1_Q in $\rho|_Q$ is $2(q^{2m}-1)/(q^2-1)$. Fix $l = \langle f_1\rangle$ for the moment. Then one checks that λ_B occurs in $\mathrm{Ind}_{Q_l}^Q(1_{Q_l})$ exactly when Q_l is contained in $\mathrm{Ker}(\lambda_B)$, and this happens exactly for $q-1$ characters λ_B with $\mathrm{rank}(B) = 1$ and $q^{2m-1}-q$ characters λ_B with $\mathrm{rank}(B) = 2$. Thus the only characters of Q that occur in $\rho|_Q$ are the trivial one and the ones from $\mathcal{O}_1$ and $\mathcal{O}_2$. Since $\tilde{G}$ acts on $\mathbf{P}_0(V)$ and $\tilde{P}$ acts transitively on $\mathcal{O}_1$ and $\mathcal{O}_2$, the multiplicity of λ_B in $\rho|_Q$ is the same for all λ_B in each of $\mathcal{O}_1$ and $\mathcal{O}_2$, say a for $\mathcal{O}_1$ and b for $\mathcal{O}_2$. Then $a = (q-1)|\mathbf{P}(W)|/|\mathcal{O}_1| = 1$, $b = (q^{2m-1}-q)|\mathbf{P}(W)|/|\mathcal{O}_2| = q+1$. Thus,

$$\rho|_Q = \frac{2(q^{2m}-1)}{q^2-1}\cdot 1_Q + \sum_{\lambda_B\in\mathcal{O}_1}\lambda_B + (q+1)\sum_{\lambda_B\in\mathcal{O}_2}\lambda_B\ . \tag{10}$$

Remark 2.7 tells us that the Q-trivial component of $\rho|_{\tilde{P}}$ affords the character $2.1_{\tilde{P}} + 2\tau$. Since $\tilde{L}$ and L act transitively on $\mathcal{O}_1$, the orbit $\mathcal{O}_1$ on $\rho|_Q$ gives rise to an irreducible $\tilde{P}$-character ζ of degree $(q^{2m}-1)/(q+1)$, with $\zeta|_Q = \sum_{\lambda_B\in\mathcal{O}_1}\lambda_B$, and $\zeta|_P$ is irreducible.

Next, we fix $B = \begin{pmatrix} B_0 & 0\\ 0 & 0\end{pmatrix}$ with $B_0 = \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}$. According to (8), the stabilizer $\tilde{S} := \{a \in \tilde{L} \mid a\circ\lambda_B = \lambda_B\}$ equals

$$\left\{[A,0] \mid A = \begin{pmatrix} X & Y\\ 0 & Z\end{pmatrix} \in \mathbf{GL}(m,q^2), X \in \mathbf{GL}(2,q^2), XB_0\,{}^tX^q = B_0\right\}. \tag{11}$$

Since the matrix B_0^2 is scalar, $XB_0\,{}^tX^q = B_0$ implies ${}^tX^qB_0X = B_0$. Therefore, in the notation of (11), the map $[A,0] \mapsto {}^tX^{-q}$ is a surjective homomorphism $\pi : \tilde{S} \to \mathbf{GU}(2,q^2)$, the second group here being the unitary group corresponding to the hermitian form with Gram matrix B_0. Let $\tilde{R} := Q : \tilde{S} = \mathrm{Stab}_{\tilde{P}}(\lambda_B)$, $R := G\cap\tilde{R}$ and let K be the kernel of λ_B. For $x \in Q$, $a \in \tilde{S}$, we have $\lambda_B(x^{-1}a^{-1}xa) = (a\circ\lambda_B)(x)/\lambda_B(x) = 1$, i.e. $[x,a] \in K$. Thus $K \lhd \tilde{S}$ and $\tilde{S}$ centralizes Q modulo K. Hence, $\tilde{R}/K \simeq \mathrm{Z}_p\times\tilde{S}$, the first group being Q/K. Combining this isomorphism with π, we see that $\tilde{R}$ maps onto $\mathrm{Z}_p\times\mathbf{GU}(2,q^2)$. Let μ_i be the character of $\mathrm{Z}_p\times\mathbf{GU}(2,q^2)$, equal λ_B on Q, and equal the trivial character, resp. the Steinberg character St, of $\mathbf{GU}(2,q^2)$ if $i = 0$, resp. if $i = 1$. Here, the Steinberg character St of $\mathbf{GU}(2,q^2)$

is (the permutation character on the set of singular 1-spaces of the natural module) -1; in particular, its degree is q. We will denote the pullback of μ_i to $\tilde{R}$ by the same symbol μ_i.

Proposition 4.1. *In the above notation let $i = 0, 1$ and $\sigma_i = \mathrm{Ind}_{\tilde{R}}^{\tilde{P}}(\mu_i)$. Then $(\rho|_{\tilde{P}}, \sigma_i)_{\tilde{P}} > 0$ and $(\rho|_P, \mathrm{Ind}_R^P(\mu_i|_R))_P > 0$ for $i = 0, 1$. Furthermore, σ_i is irreducible over $\tilde{P}$, and $\mathrm{Ind}_R^P(\mu_i|_P)$ is irreducible over P.*

Proof. The proofs are the same for $\tilde{P}$ and for P, so we will give the details for $\tilde{P}$. Denoting $\tilde{P}_1 := \mathrm{Stab}_{\tilde{G}}(\langle f_1 \rangle)$ and $P_1 = G \cap \tilde{P}_1$, we have $\rho = \mathrm{Ind}_{\tilde{P}_1}^{\tilde{G}}(1_{\tilde{P}_1})$. Then by Frobenius reciprocity and Mackey's formula,

$$(\rho|_{\tilde{P}}, \mathrm{Ind}_{\tilde{R}}^{\tilde{P}}(\mu_i))_{\tilde{P}} = (\rho, \mathrm{Ind}_{\tilde{R}}^{\tilde{G}}(\mu_i))_{\tilde{G}} = (\mathrm{Ind}_{\tilde{P}_1}^{\tilde{G}}(1_{\tilde{P}_1}), \mathrm{Ind}_{\tilde{R}}^{\tilde{G}}(\mu_i))_{\tilde{G}} =$$

$$= (1_{\tilde{P}_1}, (\mathrm{Ind}_{\tilde{R}}^{\tilde{G}}(\mu_i))|_{\tilde{P}_1})_{\tilde{P}_1} = (1_{\tilde{P}_1}, \sum_{t \in \tilde{P}_1 \backslash \tilde{G} / \tilde{R}} \mathrm{Ind}_{\tilde{R}^t \cap \tilde{P}_1}^{\tilde{P}_1}(\mu_i^t|_{\tilde{R}^t \cap \tilde{P}_1}))_{\tilde{P}_1} \geq$$

$$\geq (1_{\tilde{P}_1}, \mathrm{Ind}_{\tilde{R} \cap \tilde{P}_1}^{\tilde{P}_1}(\mu_i|_{\tilde{R} \cap \tilde{P}_1}))_{\tilde{P}_1} = (1_{\tilde{R} \cap \tilde{P}_1}, \mu_i|_{\tilde{R} \cap \tilde{P}_1})_{\tilde{R} \cap \tilde{P}_1}.$$

(Here $\tilde{R}^t = t\tilde{R}t^{-1}$ and $\mu_i^t(x) = \mu_i(txt^{-1})$.) So it suffices to show that $\mu_i|_{\tilde{R} \cap \tilde{P}_1}$ contains $1_{\tilde{R} \cap \tilde{P}_1}$. The statement is now obvious if $i = 0$, so we will assume $i = 1$. Observe that

$$\tilde{R} \cap \tilde{P}_1 = (Q \cap P_1) \cdot (\tilde{S} \cap \tilde{P}_1). \tag{12}$$

(Indeed, assume that $g = ax \in \tilde{R} \cap \tilde{P}_1$ for $x \in Q$ and $a \in \tilde{S}$. Then $ax(f_1) = \alpha f_1$ for some $\alpha \in \mathbb{F}_{q^2}$. Also, $x(f_1) = f_1 + u$ for some $u \in W'$. Now $\alpha f_1 = ax(f_1) = a(f_1) + a(u)$ with $a(f_1) \in W$ and $a(u) \in W'$. It follows that $a(u) = 0$ and $a(f_1) = \alpha f_1$. Thus $u = 0$, i.e. $x \in Q \cap P_1$, and $a \in \tilde{S} \cap \tilde{P}_1$, as stated.)

Now if $x = [I, X] \in Q \cap P_1$, then $\mathrm{Tr}(BX) = 0$, whence $\lambda_B(x) = 1$, i.e. $Q \cap P_1 \lhd K$. The factorization (12) then implies that $\mu_1|_{\tilde{R} \cap \tilde{P}_1}$ is a character of $\tilde{S} \cap \tilde{P}_1$. By the definition of μ_1, $\mu_1|_{\tilde{R} \cap \tilde{P}_1}$ can be viewed as the restriction of the Steinberg character St of $H := \mathbf{GU}(2, q^2)$ to the subgroup

$$T := \pi(\tilde{S} \cap \tilde{P}_1) = \left\{ \begin{pmatrix} x & y \\ 0 & x^{-q} \end{pmatrix} \mid x \in \mathbb{F}_{q^2}^\times, y \in \mathbb{F}_{q^2}, x^q y + x y^q = 0 \right\}.$$

Notice that $St = \mathrm{Ind}_T^H(1_T) - 1_H$. Hence

$$(St|_T, 1_T)_T = (\mathrm{Ind}_T^H(1_T)|_T - 1_T, 1_T)_T = (\mathrm{Ind}_T^H(1_T), \mathrm{Ind}_T^H(1_T))_H - (1_T, 1_T)_T = 2 - 1 = 1.$$

Consequently, $\mu_1|_{\tilde{R} \cap \tilde{P}_1}$ contains $1_{\tilde{R} \cap \tilde{P}_1}$ (with multiplicity 1), as desired.

Applying Lemma 3.3 to $(\tilde{P}, Q, \tilde{R}, \lambda_B, \mu_i)$ in place of (P, Z, R, L, M), we see that σ_i is irreducible. $\square$

Clearly, $1 = (\sigma_i, \sigma_i)_{\tilde{P}} = (\sigma_i|_{\tilde{R}}, \mu_i)_{\tilde{R}}$. In particular, $\sigma_0|_Q$ contains $\mu_0|_Q = \lambda_B$ and $\sigma_1|_Q$ contains $\mu_1|_Q = q\lambda_B$. It follows that

$$\sigma_0|_Q = \sum_{\lambda_B \in \mathcal{O}_2} \lambda_B \ , \sigma_1|_Q = q \sum_{\lambda_B \in \mathcal{O}_2} \lambda_B \ .$$

Proposition 4.2. *The permutation character ρ of $\tilde{G} = \mathbf{GU}(2m, q^2)$ on $\mathbf{P}_0(V)$ decomposes into irreducible Q- and $\tilde{P}$-constituents as follows:*

$$\begin{array}{rclcclccl} \rho|_Q & = & \dfrac{2(q^{2m}-1)}{q^2-1} \cdot 1_Q & + & \sum_{\lambda_B \in \mathcal{O}_1} \lambda_B & + & (q+1)\sum_{\lambda_B \in \mathcal{O}_2} \lambda_B \ , \\ \rho|_{\tilde{P}} & = & 2 \cdot (1_{\tilde{P}} + \tau) & + & \zeta & + & (\sigma_0 + \sigma_1) \ , \end{array}$$

where the constituents in the same columns of the two decompositions correspond to each other. All the above $\tilde{P}$-constituents remain irreducible over P, except for the case where $m = 2$, q is odd, and $\sigma_i|_P$ decomposes into two irreducible components, with $i = 0, 1$.

Proof. The decomposition for $\rho|_Q$ has already been established in (10). Next, we have shown that $\rho|_{\tilde{P}}$ contains the right-hand side of the decomposition, as σ_i is irreducible and $(\rho|_{\tilde{P}}, \sigma_i)_{\tilde{P}} > 0$ by Proposition 4.1. Comparing the restriction to Q we get the decomposition. We already mentioned that $\tau|_P$ and $\zeta|_P$ are irreducible. If $m > 2$ or if $(m, p) = (2, 2)$, then L acts transitively on $\mathcal{O}_2$, whence $\sigma_i|_P = \mathrm{Ind}_R^P(\mu|_R)$ is irreducible by Proposition 4.1. Suppose $m = 2$ and $p > 2$. Then $\mathcal{O}_2$ splits into two L-orbits, $\mathcal{O}_2^+$ and $\mathcal{O}_2^-$. By Proposition 4.1, $\mathrm{Ind}_R^P(\mu_i|_P)$ is still irreducible, but $\sigma_i|_P$ is the sum of two P-characters of this kind, one for $B \in \mathcal{O}_2^+$ and another for $B \in \mathcal{O}_2^-$. ☐

From Section 2.1 and Tables 1 and 2 we have $\rho = 1 + \varphi + \psi$, where $\varphi, \psi \in \mathrm{Irr}(\tilde{G})$, $\varphi(1) = (q^{2m}-1)(q^{2m-1}+q^2)/(q^2-1)(q+1)$, $\psi(1) = (q^{2m}-q^2)(q^{2m}+q)/(q^2-1)(q+1)$, and φ, ψ are irreducible over G.

Corollary 4.3. *Assume that $(m, q) \neq (2, 2)$. Then $\varphi|_{\tilde{P}} = 1_{\tilde{P}} + \tau + \zeta + \sigma_0$ and $\psi|_{\tilde{P}} = \tau + \sigma_1$.*

Proof. We will make use of Proposition 4.2. Since $(m, q) \neq (2, 2)$, $\sigma_1(1) > \varphi(1)$, whence σ_1 is a constituent of $\psi|_{\tilde{P}}$. So is τ, by Lemma 2.6. The corollary follows by degrees. ☐

4.2. **Reduction mod ℓ.** Next we study the mod ℓ decomposition of the above characters of $\tilde{P}$ and P. Since we will need the result in the case $\ell|(q+1)$, we formulate the result only for that case. The decomposition of $\overline{\tau}$ has already been given in Lemma 2.5.

Proposition 4.4. *Assume that $\ell|(q+1)$.*

(i) *$\overline{\zeta}$ is irreducible, both over $\tilde{P}$ and P.*

(ii) $\overline{\sigma}_0$ *is irreducible over* $\tilde{P}$. *Next,* $\overline{\sigma}_0$ *is also irreducible over* P, *except for the case* $m = 2$ *and* $p > 2$, *when it splits into two irreducible constituents.*

(iii) $\overline{\sigma}_1 = \sigma_{01} + \sigma_{11}$, *where* $\sigma_{01}, \sigma_{11} \in \mathrm{IBr}_\ell(\tilde{P})$ *and* $\sigma_{01}(1) = |\mathcal{O}_2|$, $\sigma_{11}(1) = (q-1)|\mathcal{O}_2|$. *Furthermore, for* $j = 0, 1$, σ_{j1} *is also irreducible over* P, *except for the case* $m = 2$ *and* $p > 2$. *If* $m = 2$ *and* $p > 2$, *then* $\sigma_{01}|_P$ *splits into two irreducible constituents of the same degree, and* $\sigma_{11}|_P$ *is the sum of* s *irreducible constituents, with* $s = 2$ *if* $\ell \neq 2$ *or if* $\ell = 2$ *but* $q \equiv 1 (\bmod 4)$, *and* $s = 4$ *if* $\ell = 2$ *and* $q \equiv 3 (\bmod 4)$.

Proof. (i) Since Q is an elementary abelian p-group and $\tilde{L}$, L act transitively on $\mathcal{O}_1$, $\overline{\zeta}$ is irreducible (over both $\tilde{P}$ and P) for any $\ell \neq p$. The same argument applies to (ii).

(iii) Since $\ell | (q+1)$, $\overline{\mu}_1 = \mu_{01} + \mu_{11}$, where μ_{j1} is an irreducible Brauer character of degree 1 for $j = 0$ and $q-1$ for $j = 1$. By Lemma 3.3, $\sigma_{j1} := \mathrm{Ind}_{\tilde{R}}^{\tilde{P}}(\mu_{j1})$ is irreducible, and $\overline{\sigma}_1 = \sigma_{01} + \sigma_{11}$.

Assume $m > 2$. Then $\pi(R) = \mathbf{GU}(2, q^2)$, whence $\mu_{j1}|_R$ is irreducible. Also, L is transitive on $\mathcal{O}_2$, so $\sigma_{j1}|_P$ is irreducible. The same is true when $m = 2$ and $p = 2$ since $\mu_{j1}|_{\mathbf{SU}(2,q^2)}$ is irreducible.

Assume $m = 2$ and $p > 2$. Then $\mathcal{O}_2$ splits into two L-orbits, so $\sigma_{01}|_P$ is a sum of two Brauer characters of degree $|\mathcal{O}_2|/2$. The same argument shows that $\sigma_{11}|_P$ has s irreducible constituents with $s \geq 2$. If $\ell \neq 2$ then $\mu_{11}|_{\mathbf{SU}(2,q^2)}$ is irreducible, whence $s = 2$. If $\ell = 2$ then $\mu_{11}|_{\mathbf{SU}(2,q^2)}$ is a sum of two irreducibles, so $2 \leq s \leq 4$. In fact $s = 4$ if $q \equiv 3 (\bmod 4)$, as $\pi(S) \leq Z(\mathbf{GU}(2, q^2)) \cdot \mathbf{SU}(2, q^2)$ in this case. If $q \equiv 1 (\bmod 4)$, then $\mathbf{GU}(2, q^2) = Z(\mathbf{GU}(2, q^2)) \cdot \pi(S)$, so $\mu_{11}|_R$ is irreducible and $s = 2$. □

We can now give the decomposition of $\overline{\varphi}$ and $\overline{\psi}$ into irreducible Brauer characters of $\tilde{G}$.

Corollary 4.5. (i) $\beta(U') = \begin{cases} \beta(X) \text{ if } \ell \not| m \\ 1 + \beta(X) \text{ if } \ell | m \end{cases}$, $\beta(U')|_{\tilde{P}} = \overline{\tau} + \overline{\sigma}_0$, *and* X *is simple.*

(ii) $\overline{\psi} = \beta(U') + \beta(Y)$, *where* Y *is simple and* $\beta(Y)|_{\tilde{P}} = \sigma_{11}$.

(iii) $\overline{\varphi} = 1 + \beta(U') + \beta(D)$, *where* D *is simple and* $\beta(D)|_{\tilde{P}} = \zeta$.

(iv) *The simple* $\tilde{G}$*-modules* X, Y *and* D *remain simple over* G *except possibly in the case* $m = 2$ *and* $\ell = 2$ *(and* q *is odd). In the exceptional case,* X *splits into two simple modules and we know only that* Y *has one, two or four simple summands.*

Proof. The first equation in (i) is simply Corollary 2.10. The second follows from Lemma 3.1 since $\overline{\sigma}_0$ is simple for $\tilde{P}$ by Proposition 4.4. Then since $(\overline{\psi} - \beta(U'))|_{\tilde{P}} = \sigma_{11}$ is simple, by Proposition 4.4, we have (ii). We know from the fact that $U \subset U^\perp$ (Table 3) that $\overline{\rho} - 2\beta(U')$ has at least 2 trivial constituents, so $\overline{\varphi} - \beta(U')$ has at least one. Then the restriction to $\tilde{P}$ of $\overline{\varphi} - \beta(U') - 1$ is ζ, which is simple. This proves (iii). Part (iv) follows from the corresponding statements about the restrictions to P in Proposition 4.4, except when $m = 2$ and q is odd.

Assume $m = 2$ and q is odd. Then it is known, cf. [G1], that if $\ell \neq 2$ all irreducible ℓ-modular composition factors of $\overline{\rho}$ restrict irreducibly from $\mathbf{GU}(4, q^2)$ to $\mathbf{SU}(4, q^2)$. If $\ell = 2$, then we are in the exceptional case mentioned in Remark 2.9. The possible multiplicities for $Y|_G$ are limited by the last statement of Proposition 4.4. $\square$

So we leave open the question of whether the 2-modular irreducible module Y of $\mathbf{GU}(4, q^2)$, q odd, is irreducible for $\mathbf{SU}(4, q^2)$. Its Brauer character is unipotent, labeled by the partition $(2, 1, 1)$.

We now turn to the problem of finding the $\mathbb{F}\mathbf{G}$-submodule lattice which, in view of Remark 2.12 and Corollary 2.10, is given by the following result.

Lemma 4.6. *Assume that $\ell \mid (q + 1)$. We have ${U'}^{\perp}/U' \cong \mathcal{C}/U' \oplus Y$ as $\mathbb{F}\mathbf{G}$-modules and $\mathcal{C}/U'$ is a self-dual uniserial $\mathbb{F}\mathbf{G}$-module whose composition factors are $\mathbb{F}$, D, $\mathbb{F}$.*

Proof. By Lemma 2.8, we have $U' \subseteq \mathcal{C} \cap \mathcal{C}^{\perp} \subseteq \mathcal{C} \subseteq {U'}^{\perp}$. Taking the composition factors of U' into account, we see from Corollary 4.5 that the composition factors of ${U'}^{\perp}/U'$ are $\mathbb{F}$ (twice), D and Z.

Since over $\mathbb{Q}$, we know that $\mathcal{C}_{\mathbb{Q}}$ has character $1 + \varphi$, it follows that the composition factors of $\mathcal{C}$ are a subset of those of $1 + \overline{\varphi}$. Thus, the composition factors of $\mathcal{C}/U'$ are at most $\mathbb{F}$ (twice) and D and so the same holds for its quotient $\mathcal{C}/\mathcal{C} \cap \mathcal{C}^{\perp}$. We claim that the self-dual module $\mathcal{C}/\mathcal{C} \cap \mathcal{C}^{\perp}$ is uniserial with all three possible composition factors $\mathbb{F}$, D, $\mathbb{F}$. Assuming this claim, it follows that $\mathcal{C}/U'$ maps isomorphically to $\mathcal{C}/\mathcal{C} \cap \mathcal{C}^{\perp}$, so has the structure stated. Further, the kernel of the map from ${U'}^{\perp}/U'$ to ${U'}^{\perp}/\mathcal{C} \cap \mathcal{C}^{\perp}$ is simple and isomorphic to Y. Since Y occurs with multiplicity one in ${U'}^{\perp}/U'$, it follows that this kernel is in fact a direct summand, and the proof will be complete. It remains to establish the claim. First, we have $\dim_{\mathbb{F}}(\mathcal{C}/\mathcal{C} \cap \mathcal{C}^{\perp}) \geq 3$, by Lemma 2.3 (iii), since $\mathbf{SU}(V)$ is perfect. Therefore, D must be a composition factor. Next, since D is a simple module for $\tilde{P}$, we have

$$\mathrm{Hom}_{\mathbb{F}\mathbf{G}}(\mathbb{F}^{\Omega_{max}}, D) = \mathrm{Hom}_{\mathbb{F}\tilde{P}}(\mathbb{F}, D) = 0, \tag{13}$$

so it follows that

$$\mathrm{Hom}_{\mathbb{F}\mathbf{G}}(\mathcal{C}/\mathcal{C} \cap \mathcal{C}^{\perp}, D) = 0. \tag{14}$$

We know the composition factors are D and at most two trivial modules. Therefore (14) implies that in every composition series, there must be a trivial composition factor which occurs above D, and by self-duality, there must also be one which occurs below D. This establishes the claim. $\square$

Fig. 1 may help to visualize the module structure.

Remark 4.7. The case $m = 2$ arises in the study of the incidence maps of the hermitian generalized quadrangles. We will say more about this later since the discussion also involves the permutation module on singular points of a 6-dimensional orthogonal space of Witt index 2, which we study in Section 8.

4.3. **The second smallest degree of cross-characteristic representations.** Let $\kappa_{\ell,m}$ equal 1 if $\ell|m$ and 0 otherwise. Then Corollary 4.5 implies that the simple G-module X has dimension

$$\varphi(1) - \frac{q^{2m}-1}{q+1} - 1 - \kappa_{\ell,m} = \frac{(q^{2m}-1)(q^{2m-1}+1)}{(q^2-1)(q+1)} - 1 - \kappa_{\ell,m}$$

if $m \geq 3$. On the other hand, if $q = 2$ (and $\ell = 3$) then $\dim(Y) = |\mathcal{O}_2| = (q^{2m} - 1)(q^{2m-1} - q)/(q^2-1)(q+1)$. Combining these observations with [GMST, Theorem 2.7], we obtain the following result.

Corollary 4.8. *Let $m \geq 3$ and $\ell|(q+1)$. Assume M is a nontrivial irreducible $\mathbf{SU}(2m, q^2)$-module in characteristic ℓ. Then either M is a Weil module, or*

$$\dim(M) \geq \begin{cases} \dfrac{(q^{2m}-1)(q^{2m-1}+1)}{(q^2-1)(q+1)} - \kappa_{\ell,m} - 1, & \text{if } q > 2, \\ \dfrac{(q^{2m}-1)(q^{2m-1}-q)}{(q^2-1)(q+1)}, & \text{if } q = 2, \end{cases}$$

and this bound is sharp. □

If $m = 2$ and $\ell|(q+1)$, one has to replace the bound in Corollary 4.8 by $(q^2 + 1)(q^2 - q + 1)/\gcd(2, q-1) - \kappa_{\ell,2}$ and again this bound is sharp (cf. [HM]); notice that this bound is equal to the dimension of any irreducible $\mathbf{SU}(4, q^2)$-constituent inside X.

5. The unitary groups in odd dimensions

5.1. **Character restriction.** Let $m \geq 2$, $V = \mathbb{F}_{q^2}^{2m+1}$ be endowed with a nondegenerate hermitian form with Gram matrix $\begin{pmatrix} 0 & I_m & 0 \\ I_m & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ in a basis $(e_1, \ldots, e_m, f_1, \ldots, f_m, g)$. Let $\tilde{G} = \mathbf{GU}(V) = \mathbf{GU}(2m+1, q^2)$ be the corresponding general unitary group, and let $G = \mathbf{SU}(2m+1, q^2) = \tilde{G} \cap SL(V)$. Fix $W' := \langle e_1, \ldots, e_m \rangle$, $W = \langle f_1, \ldots, f_m \rangle$, $\tilde{P} := \mathrm{Stab}_{\tilde{G}}(W') = Q : \tilde{L}$, $P := \tilde{P} \cap G = Q : L$, where $Q := O_p(P)$ and

$$L = \left\{ [A, 0] := \begin{pmatrix} {}^t\!A^{-q} & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & \det(A)^{q-1} \end{pmatrix} \mid A \in \mathbf{GL}(W) \right\}.$$

As in §4, fix $\theta \in \mathbb{F}_{q^2}$ with $\theta^{q-1} = -1$, a primitive p^{th} root ϵ of 1 in $\mathbb{C}$, and let $H_m(q) := \{X \in M_m(\mathbb{F}_{q^2}), X = {}^tX^q\}$. Then

$$Q = \left\{ [I, X, a] := \begin{pmatrix} I_m & \theta X & a \\ 0 & I_m & 0 \\ 0 & -{}^ta^q & 1 \end{pmatrix} \mid \begin{array}{c} X \in M_m(\mathbb{F}_{q^2}), a \in \mathbb{F}_{q^2}^m, \\ X - {}^tX^q + \theta^{-1}a \cdot {}^ta^q = 0 \end{array} \right\}.$$

The multiplication in Q is as follows: $[X, a] \cdot [Y, b] = [X + Y - \theta^{-1}a \cdot {}^tb^q, a + b]$. In particular, $Z := Z(Q) = \{[I, X, 0] \mid X \in H_m(q)\}$. If we identify $\mathrm{Stab}_G(\langle g \rangle)$ with the group $\mathbf{GU}(2m, q^2)$ considered in §4, then Z plays the role of the group Q in §4. Any linear character (over $\mathbb{C}$ or over $\mathbb{F}$) of Z is of the form

$$\lambda_B \;:\; [I, X, 0] \mapsto \epsilon^{\mathrm{tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mathrm{Tr}(BX))}$$

for some $B \in H_m(q)$. For any $a := [A, 0] \in \tilde{L}$, the Z-character $a \circ \lambda_B : x \mapsto \lambda_B(a \circ x)$, where $a \circ x = a^{-1}xa$ for $x \in Z$, is given by (8). As in §4, let $\mathcal{O}_1$ be the set of all λ_B with $\mathrm{rank}(B) = 1$ and $\mathcal{O}_2$ the set of all λ_B with $\mathrm{rank}(B) = 2$. Then $|\mathcal{O}_1| = (q^{2m}-1)/(q+1)$, $|\mathcal{O}_2| = (q^{2m} - 1)(q^{2m-1} - q)/(q^2 - 1)(q + 1)$. The subgroup L acts transitively on both $\mathcal{O}_1$ and $\mathcal{O}_2$.

Let ρ be the permutation character of $\tilde{G}$ on $\mathbf{P}_0(V)$. For any $f \in W \setminus \{0\}$, Z acts transitively on the set Ω_{g+f} of q^{2m-1} singular 1-spaces $\langle g + f + u \rangle$ with $u \in W'$. The stabilizer in Z of $\langle g + xf + yu \rangle \in \Omega_{g+xf}$ (where $x, y \in \mathbb{F}_{q^2}$ and $x \neq 0$) is independent from x and equal to the stabilizer Z_l of $l := \langle f \rangle$ in Z. Also, the permutation character of Z on the singular 1-spaces inside $g^\perp$ is given by (9). Hence

$$\rho|_Z = \frac{q^{2m} - 1}{q^2 - 1} \cdot 1_Z + q^2 \sum_{l \in \mathbf{P}(W)} \mathrm{Ind}_{Z_l}^Z(1_{Z_l}). \tag{15}$$

The calculation with $\mathrm{Ind}_{Z_l}^Z(1_{Z_l})$ in §4 shows that

$$\rho|_Z = \frac{(q^2 + 1)(q^{2m} - 1)}{q^2 - 1} \cdot 1_Z + q^2 \sum_{\lambda_B \in \mathcal{O}_1} \lambda_B + q^2(q + 1) \sum_{\lambda_B \in \mathcal{O}_2} \lambda_B\ . \tag{16}$$

Recall that $\mathbf{P}(W') = \mathbf{P}_0(W') \subset \mathbf{P}_0(V)$, and Q fixes $\mathbf{P}(W')$ pointwise. Next, Z fixes each vector $[\Omega_l] := \sum_{\omega \in \Omega_l} \omega$ with $l \in \mathbf{P}(W)$ and each vector $[\Omega_{g+f}] := \sum_{\omega \in \Omega_{g+f}} \omega$ with $0 \neq f \in W$. Observe that for each $l \in \mathbf{P}(W)$, Q/Z permutes the q^2 vectors $\{[\Omega_l], [\Omega_{g+f}] \mid f \in l\}$ transitively. Thus the multiplicity of 1_Q in $\rho|_Q$ is $2(q^{2m}-1)/(q^2-1)$, and $\rho|_Q$ also contains $q^{2m}-1$ nontrivial linear characters of Q/Z. Since Q/Z has exactly $q^{2m} - 1$ nontrivial linear characters and L permutes them transitively, all of them have to occur in $\rho|_Q$. According to [GMST, Lemma 12.6], for each λ_B with $\mathrm{rank}(B) = i$ there is a complex irreducible character Λ_B of degree q^i of Q such that $\Lambda_B|_Z = q^i\lambda_B$, and conversely any irreducible character of Q containing λ_B when restricted to Z has degree q^i. Such a character is not uniquely determined by λ_B. But it will become clear later that we can choose Λ_B for each $\lambda_B \in \mathcal{O}_1$ and for each $\lambda_B \in \mathcal{O}_2$ in such a way that

$$\rho|_Q = \frac{2(q^{2m}-1)}{q^2-1} \cdot 1_Q + \sum_{1_Q \neq \gamma \in \mathrm{Irr}(Q/Z)} \gamma + q \sum_{\lambda_B \in \mathcal{O}_1} \Lambda_B + (q+1) \sum_{\lambda_B \in \mathcal{O}_2} \Lambda_B \ . \tag{17}$$

Remark 2.7 tells us that the Q-trivial component of $\rho|_{\tilde{P}}$ affords the character $2.1_{\tilde{P}} + 2\tau$.

From Section 2.1 and Tables 1 and 2 we have $\rho = 1 + \varphi + \psi$, where $\varphi, \psi \in \mathrm{Irr}(\tilde{G})$, $\psi(1) = (q^{2m+1}+1)(q^{2m}-q^2)/(q^2-1)(q+1)$, $\varphi(1) = (q^{2m+1}+q^2)(q^{2m+1}-q)/(q^2-1)(q+1)$, and φ, ψ are irreducible over G. According to [GMST, Theorem 2.6], $\psi|_Z$ has to afford some, and therefore all, characters $\lambda_B \in \mathcal{O}_2$. Since $\psi(1) - q^2|\mathcal{O}_2| = (q^{2m}-q^2)/(q^2-1)$ is less than $|\mathcal{O}_1|$ and $|\mathrm{Irr}(Q/Z)| - 1$, it follows that

$$\psi|_Q = \frac{q^{2m}-q^2}{q^2-1} \cdot 1_Q + \sum_{\lambda_B \in \mathcal{O}_2} \Lambda_B \ , \tag{18}$$

where the part $\frac{q^{2m}-q^2}{q^2-1} \cdot 1_Q$ yields the P-character τ. Since the multiplicity of each $\lambda_B \in \mathcal{O}_2$ in $\psi|_Z$ is q^2, the λ_B-homogeneous component for Z in ψ yields a (uniquely determined) irreducible character Λ_B of Q of degree q^2.

Fix $B = \begin{pmatrix} B_2 & 0 \\ 0 & 0 \end{pmatrix}$ (of rank 2) with $B_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. According to (8), the stabilizer $\tilde{S} := \{a \in \tilde{L} \mid a \circ \lambda_B = \lambda_B\}$ equals $S \times T$, where

$$S = \left\{ [A, 0] \mid A = \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix} \in \mathbf{GL}(m, q^2), X \in \mathbf{GL}(2, q^2), X B_2 {}^t X^q = B_2 \right\}. \tag{19}$$

and

$$T = \{\mathrm{diag}(I_{2m}, a) \mid a \in \mathbb{F}_{q^2}, a^{q+1} = 1\} \simeq \mathrm{Z}_{q+1} \ .$$

Let $\tilde{R} := \mathrm{Stab}_{\tilde{P}}(\lambda_B)$ and $R := Q : S = \tilde{R} \cap P$. Since $Z \lhd R$ and the λ_B-homogeneous component for Z in ψ yields the irreducible character Λ_B of Q, we conclude that it also yields a (uniquely determined) irreducible character ω of $\tilde{R}$ of degree q^2.

In the notation of (19) we see that the map $[A, 0] \mapsto X$ is a surjective homomorphism $\pi : S \to \mathbf{GU}(2, q^2)$. We extend π to $\tilde{S} = S \times T$ by letting π act trivially on T. Let μ_0 be the trivial character and let μ_1 be the Steinberg character of $\mathbf{GU}(2, q^2)$, pulled back to $\tilde{S}$ using π and then inflated to $\tilde{R}$. In particular, $\mu_1(1) = q$.

Proposition 5.1. *In the above notation let $i = 0, 1$ and $\sigma_i = \mathrm{Ind}_{\tilde{R}}^{\tilde{P}}(\omega\mu_i)$. Then σ_i is irreducible (both over P and $\tilde{P}$) and $(\rho|_{\tilde{P}}, \sigma_i)_{\tilde{P}} > 0$ for $i = 0, 1$.*

Proof. The proof is the same for both P and $\tilde{P}$, so we will give the details for P. To ease the notation in this proof, we will denote $\omega|_R$ and $\mu_i|_R$ by ω and μ_i. By Lemma 3.2, $\omega\mu_i$ is irreducible. Applying Lemma 3.3 to $(P, Z, R, \lambda_B, \omega\mu_i)$ in place of

(P, Z, R, L, M), we see that $\sigma_i|_P = \mathrm{Ind}_R^P(\omega\mu_i)$ is irreducible. It remains to prove that $(\rho|_P, \sigma_i|_P)_P > 0$.

This is clear for $i = 0$ as

$$(\psi|_P, \sigma_0)_P = (\psi|_P, \mathrm{Ind}_R^P(\omega))_P = (\psi|_R, \omega)_R > 0.$$

Denoting $P_1 := \mathrm{Stab}_G(\langle f_1\rangle)$ we have $\rho|_G = \mathrm{Ind}_{P_1}^G(1_{P_1})$. By Frobenius reciprocity and Mackey's formula

$$(\mathrm{Ind}_R^P(\omega\mu_i), \rho|_P)_P = (\mathrm{Ind}_R^G(\omega\mu_i), \rho|_G)_G = (\mathrm{Ind}_R^G(\omega\mu_i), \mathrm{Ind}_{P_1}^G(1_{P_1}))_G =$$

$$= (\omega\mu_i, (\mathrm{Ind}_{P_1}^G(1_{P_1}))|_R)_R = (\omega\mu_i, \sum_{t\in R\backslash G/P_1} \mathrm{Ind}_{P_1^t\cap R}^R(1|_{P_1^t\cap R}))_R =$$

$$= \sum_{t\in R\backslash G/P_1} (\omega\mu_i|_{P_1^t\cap R}, 1|_{P_1^t\cap R})_{P_1^t\cap R}.$$

Since the statement is true for $i = 0$, it follows that there is $t \in R\backslash G/P_1$ such that $\omega|_{P_1^t\cap R}$ contains the trivial character. Each double coset RyP_1 corresponds to an R-orbit $\Omega(y)$ on G/P_1 which we identify with $\mathbf{P}_0(V)$. We claim that $\Omega(t)$ contains $\langle f_1\rangle$. Indeed, assume the contrary. Since R contains Q, we may assume that $\Omega(t) = w^R$, where either $w \in W'$, or $w = f + u$ with $u \in W'$ and $f \in W\backslash f_1^R$. Changing t to a suitable representative of the double coset, we may assume $t(f_1) = w$. Fix $b \in \mathbb{F}_{q^2}$ such that $\mathrm{tr}_{\mathbb{F}_{q^2}/\mathbb{F}_p}(b) = 1$. In the first case consider $x := [I, X, 0]$ with $X = \begin{pmatrix} 0 & b & 0 & \dots & 0 \\ b^q & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ & & & \dots & \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$. Then $\mathrm{Tr}(BX) = b+b^q$ and so $\lambda_B(x) = \epsilon \neq 1$. In the second case we have two subcases: $f \notin \langle f_1, f_2\rangle$ and $f \in \langle f_1, f_2\rangle$. If $f \notin \langle f_1, f_2\rangle$ then some element r of R sends f to f_3, and we may change t to rt, f to f_3 and choose the same x as in the first case. Suppose $f \in \langle f_1, f_2\rangle$. Since $f \notin f_1^R$, $f = z(f_1 + yf_2)$ for some nonzero $y, z \in \mathbb{F}_{q^2}$ and $y + y^q \neq 0$. Then there is $c \in \mathbb{F}_q$ such that $\mathrm{tr}_{\mathbb{F}_q/\mathbb{F}_p}(c(y + y^q)) = 1$. In this case consider $x := [I, X, 0]$ with $X = \begin{pmatrix} -y^{q+1}c & y^q c & 0 & \dots & 0 \\ yc & -c & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ & & & \dots & \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$. Then $\mathrm{Tr}(BX) = c(y + y^q)$ and so $\lambda_B(x) = \epsilon \neq 1$. Notice that in all cases $x \in Z$ and x fixes $w = t(f_1)$. Thus $x \in P_1^t \cap R$, but $\omega(x) = q^2\lambda_B(x) = \epsilon\omega(1)$, contrary to the condition that $\omega|_{P_1^t\cap R}$ contains $1|_{P_1^t\cap R}$.

We have shown that $t(f_1)$ belongs to the R-orbit of f_1, i.e. $RtP_1 = RP_1$. Hence we may choose $t = 1$ and conclude that $\omega|_{P_1\cap R}$ contains $1|_{P_1\cap R}$.

Similarly to (12) we have $R \cap P_1 = (Q \cap P_1) \cdot (S \cap P_1)$. Next, $\mu_1|_{R \cap P_1}$ is actually a character of $S \cap P_1$. By the definition of μ_1, $\mu_1|_{S \cap P_1}$ can be viewed as the restriction of the Steinberg character St of $H := \mathbf{GU}(2, q^2)$ to the subgroup

$$T := \pi(S \cap P_1) = \left\{ \begin{pmatrix} x & y \\ 0 & x^{-q} \end{pmatrix} \mid x \in \mathbb{F}_{q^2}^{\times}, y \in \mathbb{F}_{q^2}, x^q y + x y^q = 0 \right\}.$$

Notice that $St = \mathrm{Ind}_T^H(1_T) - 1_H$. Hence,

$$(St|_T, 1_T)_T = (\mathrm{Ind}_T^H(1_T)|_T - 1_T, 1_T)_T = (\mathrm{Ind}_T^H(1_T), \mathrm{Ind}_T^H(1_T))_H - (1_T, 1_T)_T = 2 - 1 = 1.$$

Consequently, $\mu_1|_{R \cap P_1}$ contains $1_{R \cap P_1}$.

It now immediately follows that $\omega\mu_1|_{R \cap P_1}$ contains $1_{R \cap P_1}$. Using Mackey's formula again, we see that $(\rho|_P, \sigma_1|_P)_P > 0$, as stated. $\square$

Next we let $B = \begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix}$ (of rank 1) with $B_1 = 1$. Then the stabilizer $\tilde{S} := \{a \in \tilde{L} \mid a \circ \lambda_B = \lambda_B\}$ equals $S \times T$, with $T \simeq \mathrm{Z}_{q+1}$ as above and

$$S = \left\{ [A, 0] \mid A = \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix} \in \mathbf{GL}(m, q^2), X \in \mathbf{GL}(1, q^2), X B_1 {}^t X^q = B_1 \right\}. \quad (20)$$

Let $\tilde{R} := \mathrm{Stab}_{\tilde{P}}(\lambda_B)$ and $R = Q : S = \tilde{R} \cap P$. Recall (cf. for instance [TZ1]) that $\tilde{G}$ has a complex irreducible Weil character $\zeta_{2m+1,q}^{0,0}$ of degree $(q^{2m+1} - q)/(q+1)$, and it is known (see for instance [GMST]) that $\zeta_{2m+1,q}^{0,0}|_Z = q \sum_{\lambda_B \in \mathcal{O}_1} \lambda_B$. Let $\zeta := \zeta_{2m+1,q}^{0,0}|_P$. Clearly, the λ_B-homogeneous component for Z in ζ yields a (uniquely determined) irreducible character Λ_B of Q, and a (uniquely determined) irreducible character ξ of $\tilde{R}$ of degree q. One can check that $\xi|_{\tilde{S}} = \sum_{i=1}^{q} \upsilon_i$ for some linear characters of S. For the next proposition we will view υ_i^{-1} as characters of $\tilde{R}$.

Proposition 5.2. *In the above notation let $1 \leq i \leq q$ and $\zeta_i = \mathrm{Ind}_{\tilde{R}}^{\tilde{P}}(\xi\upsilon_i^{-1})$. Then ζ_i is irreducible (both over P and $\tilde{P}$), and $(\rho|_{\tilde{P}}, \zeta_i)_{\tilde{P}} > 0$.*

Proof. The proof is the same for both P and $\tilde{P}$, so we will give the details for P. To ease the notation in this proof, we will denote $\xi|_R$ and $\upsilon_i|_R$ by ω and υ_i. By Lemma 3.2, $\omega\upsilon_i^{-1}$ is irreducible, and by Lemma 3.3, $\zeta_i|_P = \mathrm{Ind}_R^P(\xi\upsilon_i^{-1})$ is irreducible. It remains to prove that $(\rho|_P, \zeta_i|_P)_P > 0$. Let $P_1 = \mathrm{Stab}_G(\langle f_1 \rangle)$. As in the proof of Proposition 4.1, we have

$$(\rho|_P, \mathrm{Ind}_R^P(\xi\upsilon_i^{-1}))_P \geq (1|_{R \cap P_1}, \xi\upsilon_i|_{R \cap P_1})_{R \cap P_1},$$

hence it suffices to prove that $\xi\upsilon_i^{-1}|_{R \cap P_1}$ contains $1|_{R \cap P_1}$.

We have mentioned above that $\xi|_Q = \Lambda_B$ is irreducible. Notice that $Q/\mathrm{Ker}(\lambda_B) \simeq Q_1 \times Q_2$, where Q_1 is an extraspecial p-group of order pq^2 and Q_2 is elementary abelian of order q^{2m-2}. Also, $\Lambda_B|_{Q_1}$ is irreducible, so Λ_B can be viewed as the outer tensor product of $\Lambda_B|_{Q_1}$ with a linear character λ' of Q_2. But one can check that

S acts transitively on the nontrivial linear characters of Q_2, since $m \geq 2$. Now Λ_B extends to the character ξ of $R = Q : S$. Hence λ' must be trivial. This implies that ξ is trivial on $Q \cap P_1$.

Similarly to (12) we have $R \cap P_1 = (Q \cap P_1) \cdot (S \cap P_1)$, and notice that $S = S \cap P_1$. Thus $\xi|_{R \cap P_1}$ (and v_i^{-1}) are actually characters of S. By the definition of v_i, $\xi|_S$ contains v_i, whence $\xi v_i^{-1}|_S$ contains $1|_S$, as desired. □

Since $(\omega(\mu_0 + \mu_1))|_Q = (q+1)\Lambda_B$ and $(\sum_{i=1}^{q} \xi v_i^{-1})|_Q = q\Lambda_B$, we have justified the terms $(q+1)\sum_{\lambda_B \in \mathcal{O}_2} \Lambda_B$ and $q \sum_{\lambda_B \in \mathcal{O}_1} \Lambda_B$ in (17).

Since $\tilde{L}$ and L act transitively on $\mathrm{Irr}(Q/Z) \setminus \{1\}$, the sum $\sum_{1_Q \neq \gamma \in \mathrm{Irr}(Q/Z)} \gamma$ in $\rho|_Q$ gives rise to an irreducible $\tilde{P}$-character κ of degree $q^{2m}-1$, with $\kappa|_Q = \sum_{1_Q \neq \gamma \in \mathrm{Irr}(Q/Z)} \gamma$, and $\kappa|_P$ is irreducible.

Proposition 5.3. *The permutation character ρ of $\tilde{G} = \mathbf{GU}(2m+1, q^2)$ on $\mathbf{P}_0(V)$ decomposes into irreducible Q- and $\tilde{P}$-constituents as follows:*

$$\begin{array}{lllll} \rho|_Q & = \dfrac{2(q^{2m}-1)}{q^2-1} \cdot 1_Q & + \sum_{1 \neq \gamma \in \mathrm{Irr}(Q/Z)} \gamma & + q \sum_{\lambda_B \in \mathcal{O}_1} \Lambda_B & + (q+1) \sum_{\lambda_B \in \mathcal{O}_2} \Lambda_B \ , \\ \rho|_{\tilde{P}} & = 2 \cdot (1_{\tilde{P}} + \tau) & + \kappa & + \sum_{i=1}^{q} \zeta_i & + (\sigma_0 + \sigma_1) \ , \end{array}$$

where the constituents in the same columns of the two decompositions correspond to each other. All the above $\tilde{P}$-constituents remain irreducible over P.

Proof. The decomposition for $\rho|_Q$ has already been established in (17). The decompositions for $\rho|_P$ and $\rho|_{\tilde{P}}$ follow from the above discussion and Clifford's Theorem. □

Corollary 5.4. *ψ and φ restrict to $\tilde{P}$ as follows: $\psi|_{\tilde{P}} = \tau + \sigma_0$, and $\varphi|_{\tilde{P}} = 1_{\tilde{P}} + \tau + \kappa + \sum_{i=1}^{q} \zeta_i + \sigma_1$.*

Proof. The decomposition for $\psi|_P$ follows from (18) and Proposition 5.1. The decomposition for $\varphi|_P$ follows from Proposition 5.3 and the formula $\varphi = \rho - 1_{\tilde{G}} - \psi$. □

5.2. **Reduction mod ℓ.** Next we study the mod ℓ decomposition of the above characters of $\tilde{P}$ and P. Since we will need the result in the case $\ell|(q+1)$, we formulate the result only for that case. The decomposition of τ has already been given in Lemma 2.5.

Proposition 5.5. *Assume that $\ell|(q+1)$.*

(i) *The characters $\overline{\kappa}$, $\overline{\zeta}_i$, $\overline{\sigma}_0$ are irreducible, both over $\tilde{P}$ and P.*

(ii) *$\overline{\sigma}_1 = \sigma_{01} + \sigma_{11}$, where $\sigma_{01}, \sigma_{11} \in \mathrm{IBr}_\ell(\tilde{P})$ and $\sigma_{01}(1) = q^2|\mathcal{O}_2|$, $\sigma_{11}(1) = q^2(q-1)|\mathcal{O}_2|$. Furthermore, for $j = 0, 1$, σ_{j1} is also irreducible over P.*

Proof. Notice that $L \simeq \mathbf{GL}(m, q^2)$, and it is transitive on $\mathcal{O}_1$, $\mathcal{O}_2$, and on $\mathrm{Irr}(Q/Z) \setminus \{1_Q\}$. Now we can apply Proposition 5.3 and argue as in the proof of Proposition 4.4. □

We can now give the decomposition of $\overline{\varphi}$ and $\overline{\psi}$ into irreducible Brauer characters of G and $\tilde{G}$.

Corollary 5.6. (i) $\beta(U') = \begin{cases} \beta(X) \text{ if } \ell \nmid m \\ 1+\beta(X) \text{ if } \ell | m \end{cases}$, $\beta(U')|_{\tilde{P}} = \overline{\tau} + \overline{\sigma}_0$, *and* X *is simple.*

(ii) $\overline{\psi} = \beta(U')$.

(iii) $\overline{\varphi} = 1 + \beta(U') + \beta(S)$ *for some* $\tilde{G}$*-module* S *with* $\beta(S)|_{\tilde{P}} = \kappa + \sigma_{11} + \sum_{i=1}^{q} \zeta_i$. *(The precise composition factors of* S *will be given later.)*

Proof. The first equation in (i) is simply Corollary 2.10. The second equation in (i) follows from Lemma 3.1 and Proposition 5.5. Part (ii) is immediate from (i) and Corollary 2.14. To prove (iii), we note that, by Table 3, $\overline{\rho} - 2\beta(U')$ has at least two trivial constituents, so we see from (ii) that $\overline{\varphi} - \beta(U')$ has a trivial constituent. $\square$

In part (iii) of Corollary 5.6, we know that κ is not the restriction of a Brauer character of $\tilde{G}$, since it has Z in its kernel. So the remaining problem here is to determine whether the characters $\overline{\zeta_i}$ extend to $\tilde{G}$. At least one of them extends to $\tilde{G}$, as follows from the following result of [T]. It is well known that $\overline{\zeta^{0,0}_{2m+1,q}}$ is irreducible, of degree $(q^{2m+1} - q)/(q+1)$.

Proposition 5.7. [T] *Assume* $\ell | (q+1)$. *Then* $\overline{\varphi}$ *contains* $\overline{\zeta}^{0,0}_{2m+1,q}$ *with multiplicity at least* 1 *if* $\ell = 2$ *and* $q \equiv 1 (\bmod 4)$, *and at least* 2 *otherwise.* $\square$

To prove that the lower bound for the multiplicity of $\overline{\zeta}^{0,0}_{2m+1,q}$ in $\overline{\varphi}$ actually gives the exact value of it, we need the next proposition. (The approach taken follows detailed suggestions of the referee, to whom we are grateful.) It is well-known that unipotent characters of $\mathbf{GU}(n, q^2)$ are labeled by partitions of n; in particular, $\zeta^{0,0}_{2m+1,q}$, φ, and ψ correspond to $(n-1, 1)$, $(n-2, 1^2)$, and $(n-2, 2)$, respectively, with $n = 2m+1$. According to [G2], the ℓ-modular decomposition matrix for the unipotent characters is lower unitriangular if the partition labels are ordered lexicographically. We will denote the complex character and the Brauer character labeled by the partition λ by χ_λ and ϕ_λ, respectively. Following [HM], we say that a matrix A is *approximated* by a matrix B (of the same size) if the columns of B are non-negative integral linear combinations of the columns of A.

Proposition 5.8. *Suppose* $(\ell, q) = 1$ *and the* ℓ*-modular decomposition matrix for the unipotent characters of* $\mathbf{GU}(3, q^2)$ *is approximated by*

λ			
(3)	1	0	0
$(2,1)$	0	1	0
(1^3)	1	α	1

for some integer α. Then for $n = 2m+1 \geq 5$, part of the ℓ-modular decomposition matrix for the unipotent characters of $\mathbf{GU}(n, q^2)$ is approximated by

$$\begin{array}{c|cccc} \lambda & & & & \\ \hline (n) & 1 & 0 & 0 & 0 \\ (n-1,1) & 0 & 1 & 0 & 0 \\ (n-2,2) & m-1 & 0 & 1 & 0 \\ (n-2,1^2) & m & \alpha & 1 & 1 \end{array}.$$

Proof. By virtue of [HM, Propositions 6, 8], it suffices to prove the second column of the decomposition matrix. We induct on $m \geq 2$ (and observe that the partition $(n-2,2)$ occurs only when $n > 3$). Consider the Levi subgroup $L_1 = \mathbf{GU}(n-2, q^2) \times \mathbb{Z}_{q^2-1}$ of $\tilde{G} = \mathbf{GU}(n, q^2)$ and the parabolic subgroup $P_1 = Q_1 L_1$ (that fixes a singular 1-space of $\mathbb{F}_{q^2}^n$). Abusing the notation, for a partition μ of $n-2$ we will denote the unipotent character $\chi_\mu \otimes 1$ of L_1 also by χ_μ, and do the same for Brauer characters. Consider the character Φ of the principal indecomposable module corresponding to the Brauer character $\phi_{(n-3,1)}$ of L_1, and its Harish-Chandra induction $R_{L_1}^{\tilde{G}}(\Phi)$. By induction hypothesis,

$$\Phi = \chi_{(n-3,1)} + \gamma \cdot \chi_{(n-4,1^2)} + \sum_\nu a_\nu \chi_\nu + \text{(non-unipotent characters)},$$

where $0 \leq \gamma \leq \alpha$, $a_\nu \geq 0$, and the summation runs over partitions $\nu = (\nu_1, \nu_2, \dots)$ of $n-2$ with $\nu_1 \leq n-5$. We need to find the multiplicity m_λ of χ_λ in $R_{L_1}^{\tilde{G}}(\Phi)$ for $\lambda \in \{(n), (n-1,1), (n-2,2), (n-2,1^2)\}$. Notice that the Harish-Chandra induction respects Lusztig series; in particular it sends non-unipotent characters to non-unipotent characters. Next, χ_λ is a constituent of $R_{L_1}^{\tilde{G}}(\chi_\mu)$ precisely when λ can be obtained from μ by adding a 2-hook, cf. the proof of [HM, Lemma 11]. It follows that $m_{(n)} = m_{(n-2,2)} = 0$. Furthermore, $m_{(n-1,1)} = (R_{L_1}^{\tilde{G}}(\chi_{(n-3,1)}), \chi_{(n-1,1)}) = 1$ and $m_{(n-2,1^2)} = \gamma \cdot (R_{L_1}^{\tilde{G}}(\chi_{(n-4,1^2)}), \chi_{(n-2,1^2)}) = \gamma$, where the last equality in each chain of equalities follows from the computation in the proof of [HM, Proposition 8]. Thus the second column of the matrix follows. $\square$

Corollary 5.9. *Assume $\ell | (q+1)$. Then, both over $\tilde{G}$ and G, $\beta(S)$ is the sum of an irreducible Brauer character and $s \cdot \overline{\zeta}_{2m+1,q}^{0,0}$, with $s = 1$ if $\ell = 2$ and $q \equiv 1 (\mathrm{mod}\, 4)$, and $s = 2$ otherwise.*

Proof. The smallest possible value of the parameter α in Proposition 5.8 was conjectured in [G1] and determined in [E], [Hiss], [OW]. Consequently, one can take $\alpha = 1$ if $\ell = 2$ and $q \equiv 1 (\mathrm{mod}\, 4)$, and $\alpha = 2$ otherwise. Combined with Propositions 5.7, 5.8, this yields the desired statement over $\tilde{G}$. It remains to show that $\pi := \beta(S) - s \cdot \overline{\zeta}_{2m+1,q}^{0,0}$ is irreducible over G. The claim is obvious if $(m, q) = (2, 2)$, since in this case $\tilde{G} = G \times \mathbb{Z}_3$. Assume $(m, q) \neq (2, 2)$. Since $\sigma_{11}|_P$ is irreducible by

Proposition 5.5, at least one of the irreducible constituents of $\pi|_G$, say π_0, contains σ_{11} when restricted to P. But the assumptions $m \geq 2$ and $(m,q) \neq (2,2)$ imply that $\sigma_{11}(1) > \pi(1)/2$. It follows that $\pi_0 = \pi$, as stated. □

5.3. **The second smallest degree of cross-characteristic representations.** Let $\kappa_{\ell,m}$ be equal to 1 if ℓ divides $\frac{q^{2m}-1}{q^2-1}$ and 0 otherwise. Then Corollary 5.6 implies that the simple G-module X has dimension

$$\psi(1) - \kappa_{\ell,m} = \frac{(q^{2m+1}+1)(q^{2m}-q^2)}{(q^2-1)(q+1)} - \kappa_{\ell,m}. \tag{21}$$

This is under the assumption that $\ell|(q+1)$. In the complementary case $\ell \not|(q+1)$ it can also be seen, from [L1, pp. 327-328], that there is a simple composition factor of $\overline{\psi}$ whose dimension is (21). Combining this with [GMST, Theorem 2.7], we obtain the following result.

Corollary 5.10. *Let $m \geq 2$. Assume M is a nontrivial irreducible* $\mathbf{SU}(2m+1,q^2)$-*module in characteristic $\ell \not| q$. Then either M is a Weil module, or*

$$\dim(M) \geq \begin{cases} \dfrac{(q^{2m+1}+1)(q^{2m}-q^2)}{(q^2-1)(q+1)} - \kappa_{\ell,m}\,, & \text{if } m > 2,\\[2ex] \dfrac{(q^{2m+1}+1)(q^{2m}-q^2)}{(q^2-1)(q+1)} - 1, & \text{if } m = 2. \end{cases}$$

Moreover, this bound is sharp if $m > 2$ or if $m = \ell = 2$. □

5.4. **Submodule structure.** To determine the submodule structure of $\mathbb{F}^{\mathbf{P}_0}$ we will also need some facts concerning the Harish-Chandra restriction of some modules and characters. We will keep the notation P_1, Q_1, L_1 introduced in the proof of Proposition 5.8 for $\tilde{G} = \mathbf{GU}(n,q^2)$, and let K denote the subgroup $\mathbf{GU}(n-2,q^2)$ of L_1.

Lemma 5.11. *Assume $n \geq 5$. The group K has three orbits on the set of Q_1-orbits on $\mathbf{P}_0$. Two of these orbits are fixed points and the third is isomorphic as a K-set with the set of singular 1-spaces in the natural $n-2$-dimensional module for K over $\mathbb{F}_{q^2}$.*

Proof. Consider a basis $(e_1, \ldots, e_n)$ of $V = \mathbb{F}_{q^2}^n$ such that the Hermitian scalar product (e_i, e_j) equals $\delta_{i,n+1-j}$. Then we can take $P_1 = Stab_{\tilde{G}}(w)$, where $w := \langle e_1 \rangle_{\mathbb{F}_{q^2}}$. Let $U_1 := \langle e_2, \ldots, e_{n-1} \rangle_{\mathbb{F}_{q^2}}$, $U_2 := \langle e_1, \ldots, e_{n-1} \rangle_{\mathbb{F}_{q^2}}$. Clearly, P_1 fixes the partition $\mathbf{P}_0(V) = \{w\} \cup M \cup N$, where $M := \mathbf{P}_0(U_2) \setminus \{w\}$ and $N := \mathbf{P}_0(V) \setminus \mathbf{P}_0(U_2)$. Obviously, $\{w\}$ is a fixed point for K.

Any point in M is of the form $\langle u + xe_1 \rangle_{\mathbb{F}_{q^2}}$ with $x \in \mathbb{F}_{q^2}$ and $0 \neq u \in U_1$, $(u,u) = 0$. Fix such a u. It is easy to see that the Q_1-orbit of $\langle u \rangle_{\mathbb{F}_{q^2}}$ consists of q^2 points

$\langle u + xe_1\rangle_{\mathbb{F}_{q^2}}$, $x \in \mathbb{F}_{q^2}$. Thus, the set of Q_1-orbits in M is isomorphic as a K-set with $\mathbf{P}_0(U_1)$.

Any point in N can be written uniquely in the form $\langle e_n + u + xe_1\rangle_{\mathbb{F}_{q^2}}$ with $x \in \mathbb{F}_{q^2}$, $u \in U_1$, and $x + x^q + (u, u) = 0$; in particular, $|N| = q^{2n-3} = |Q_1|$. Observe that the stabilizer in Q_1 of the point $\langle e_n\rangle_{\mathbb{F}_{q^2}}$ is trivial. Hence Q_1 acts transitively on N. This single remaining Q_1-orbit must also be a fixed point in the action of K on the set of Q_1-orbits on $\mathbf{P}_0$.

□

Let ρ_n denote the permutation character of $\mathbf{GU}(n, q^2)$ acting on $\mathbf{P}_0$, and let F_n denote the permutation module over $\mathbb{F}$ of the same action. If x is any module or (ordinary or Brauer) character of $\mathbf{GU}(n, q^2)$, the corresponding module or character K given by the Q_1-trivial component will be denoted by x^{Q_1}. The following is now immediate from Lemma 5.11.

Corollary 5.12. *For $n = 2m+1 \geq 5$, we have $\rho_n^{Q_1} = 2+\rho_{n-2}$ and $F_n^{Q_1} \cong \mathbb{F}\oplus\mathbb{F}\oplus F_{n-2}$.* □

Since we will need to consider the groups $\mathbf{GU}(n, q^2)$ for various values of $n = 2m+1$ at the same time, we will add subscripts to our standard notations when needed. Thus, for example, ϕ_7 and U'_7 are the character and the module for $\mathbf{GU}(7, q^2)$ previously denoted by ϕ and U' respectively. We shall also need to extend our notation to the case $n = 3$. Here we denote the character of the doubly transitive permutation module by $\rho_3 = 1 + \phi_3$ and make the convention that $\psi_3 = 0$. We next consider the Harish-Chandra restriction of F_n to K in more detail. Let $E = E_n$ denote the simple module affording $\overline{\zeta}_{n,q}^{0,0}$ and let $D = D_n$ be the other simple module given by Corollary 5.9.

Lemma 5.13. *Let $n = 2m + 1 \geq 5$.*

(i) ${\psi_n}^{Q_1} = 1 + \psi_{n-2}$.
(ii) ${\phi_n}^{Q_1} = 1 + \phi_{n-2}$.
(iii) $E_n^{Q_1} = E_{n-2}$ *and* $D_n^{Q_1} = D_{n-2}$.

Proof. By Corollary 5.12, we have $\psi_n^{Q_1} + \phi_n^{Q_1} = 2 + \psi_{n-2} + \phi_{n-2}$. By Frobenius reciprocity it is clear that $\psi_n^{Q_1}$ and $\phi_n^{Q_1}$ have 1 as a constituent. First we consider the case $n = 5$. By Corollary 5.6(iii), we know that $\overline{\phi}_5$ contains $\overline{\psi}_5$. Therefore the complex character $\phi_5^{Q_1}$ must contain ϕ_3. It now follows that $\phi_5^{Q_1} = 1+\phi_3$ and $\psi_5^{Q_1} = 1$, proving (i) and (ii) in the case $n = 5$. Assume $n \geq 7$. The label of ψ_n is $(n-2, 2)$, that of ψ_{n-2} is $(n-4, 2)$. Therefore, since the latter is obtained by removing a 2-hook from $(n-2, 2)$, it follows that ψ_{n-2} is a constituent of $\psi_n^{Q_1}$. We also know that $\overline{\psi}_n$ is contained in $\overline{\phi}_n$, by Corollary 5.6(iii). This means that the ordinary character $\phi_n^{Q_1}$ has a nontrivial constituent, which must be ϕ_{n-2}. From these facts we conclude that

(i) and (ii) hold. To prove (iii), note that by Corollary 5.6

$$\overline{\phi}_n - \overline{\psi}_n - 1 = s\beta(E_n) + \beta(D_n),$$

where $s = 1$ or 2, by Corollary 5.9. Since we have shown that $\phi_n^{Q_1} - \psi_n^{Q_1} = \phi_{n-2} - \psi_{n-2}$, we have

$$s\beta(E_n)^{Q_1} + \beta(D_n)^{Q_1} = s\beta(E_{n-2}) + \beta(D_{n-2}). \tag{22}$$

Recall that $\beta(E_n) = \overline{\zeta}_{n,q}^{0,0}$, where $\zeta_{n,q}^{0,0}$ has label $(n-1,1)$. The arguments in the last four sentences of the proof of Proposition 5.8 imply that $(\zeta_{n,q}^{0,0})^{Q_1} = \zeta_{n-2,q}^{0,0}$, and (iii) follows. □

We now turn to the problem of determining the submodule structure. We will employ an inductive argument which rests ultimately on the knowledge of the submodule structure of the permutation module in the case $n = 3$. This information is a combination of results of the papers [E], [G1], [OW], [Hiss] and [KK]. A definitive statement of the submodule structure is given in [Hiss, Theorem 4.1]. Under our standing assumption that $\ell \mid (q+1)$ there are two cases, just as for the decomposition. If ℓ is odd or if $\ell = 2$ and $q \equiv 3 (\mathrm{mod}\ 4)$, then F_3 is uniserial with radical series $\mathbb{F}$, E_3, D_3, E_3, $\mathbb{F}$. If $\ell = 2$ and $q \equiv 1(\mathrm{mod}\ 4)$, then F_3 has radical length 3 with layers $\mathbb{F}$, $D_3 \oplus E_3$, $\mathbb{F}$.

We are now ready to describe the submodule structure of F_n. By Lemma 2.8, it suffices to describe $(U_n')^\perp/U_n'$. In accordance with our convention that $\psi_3 = 0$, we have $U_3' = \{0\}$.

Proposition 5.14. *Assume $n = 2m+1 \geq 3$*

(i) *If ℓ is odd or if $\ell = 2$ and $q \equiv 3(\mathrm{mod}\ 4)$, then $(U_n')^\perp/U_n'$ is uniserial with radical series $\mathbb{F}$, E_n, D_n, E_n, $\mathbb{F}$.*
(ii) *If $\ell = 2$ and $q \equiv 1(\mathrm{mod}\ 4)$, then $(U_n')^\perp/U_n'$ has radical length 3 with layers $\mathbb{F}$, $D_n \oplus E_n$, $\mathbb{F}$.*
(iii) *For $n = 2m+1 \geq 5$, in all cases we have $\mathcal{C}_n = (U_n')^\perp$.*

Proof. Our main task is to prove (i). We argue by induction, starting with the case $n = 3$, for which (i) is precisely the description of the known structure of F_3. Suppose for a contradiction that (i) is false for some $n > 3$, but true for $n-2$. Let P_1, Q_1, L_1 and K be the subgroups of $\tilde{G} = \mathbf{GU}(n, q^2)$ considered earlier. We begin by looking at $U_n^\perp/U_n$, which has composition factors E_n (twice) and D_n. It follows from Lemma 5.13(iii) that $(U_n^\perp/U_n)^{Q_1}$ has composition factors E_{n-2} (twice), and D_{n-2}. Since this module has no trivial quotients, we see from Corollary 5.12 that it is isomorphic to a subquotient of F_{n-2}. Since (i) holds for $n-2$, we see that the only subquotient of F_{n-2} which has these composition factors is uniserial with series E_{n-2}, D_{n-2}, E_{n-2}. It follows that $U_n^\perp/U_n$ is uniserial with series E_n, D_n, E_n. Since (i) is false for n, the module $(U_n')^\perp/U_n'$ is not uniserial so by self-duality, it must have an $\mathbb{F}\tilde{G}$-submodule isomorphic to E_n. Let N be the preimage of this submodule in $(U'_n)^\perp$. Then we have

$$0 \neq \mathrm{Hom}_{\mathbb{F}\tilde{G}}(N, F_n) \cong \mathrm{Hom}_{\mathbb{F}P_1}(N, \mathbb{F}) \cong \mathrm{Hom}_{\mathbb{F}L_1}(N^{Q_1}, \mathbb{F}). \tag{23}$$

Therefore N^{Q_1} has an $\mathbb{F}K$-submodule N_0 such that $N^{Q_1}/N_0 \cong \mathbb{F}$. From Lemma 5.13 and Corollary 5.6(ii), we have

$$\beta(N_0) = \beta(U'_{n-2}) + \beta(E_{n-2}).$$

Let N'_0 be the largest submodule of N_0 with no trivial quotients. Then by Corollary 5.12, $N'_0 \subseteq F_{n-2}$. Also, N'_0 is not trivial, since it has a composition factor E_{n-2}. Therefore by Lemma 2.8, we have $U'_{n-2} \subset N'_0$ and, by comparing Brauer characters, we see that $N'_0 = N_0$. Since $F_{n-2}/(U'_{n-2})^\perp$ has no composition factors isomorphic to E_{n-2}, we see that $N_0/U'_{n-2} \cong E_{n-2}$ is a submodule of $(U'_{n-2})^\perp/U'_{n-2}$, contradicting the fact that the latter module has trivial socle.

To prove (ii) we note that the argument in (i) showing that $(U'_n)^\perp/U'_n$ has no submodule E_n works equally well in this case and, furthermore, applies also to D_n. Therefore, the socle of $(U'_n)^\perp/U'_n$ is trivial and, by self-duality, so is the head. The heart of the module then has non-isomorphic composition factors D_n and E_n and so, by self-duality, is isomorphic to their direct sum.

Finally we prove (iii) for $n \geq 5$. (The assumption is only to avoid defining $\mathcal{C}_3$.) By Lemma 2.8, we have $U'_n \subseteq \mathcal{C}_n \cap \mathcal{C}_n^\perp$. Therefore, by Lemma 2.3(i) $\mathcal{C}_n/U'_n$ is a nontrivial submodule of $(U'_n)^\perp/U'_n$ and by Lemma 2.4(i) it has a submodule of codimension one. Inspection of the submodule structure proved in (i) and (ii) shows that only $(U'_n)^\perp/U'_n$ has these properties. Thus by Lemma 2.8, $\mathcal{C}_n = (U'_n)^\perp$. □

The submodule structures are depicted in Figures 2 and 3.

6. The split orthogonal groups in even dimensions

6.1. **Character restriction.** Let $m \geq 3$, and let $V = \mathbb{F}_q^{2m}$ be endowed with a non-degenerate (split) orthogonal form $\mathfrak{q}$. Since the main results can be checked easily for $(m,q) = (3,2)$, we will henceforth assume that $(m,q) \neq (3,2)$. We assume that the associated bilinear form has Gram matrix $\begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix}$ in a basis $(e_1, \ldots, e_m, f_1, \ldots, f_m)$. Also, we may assume that $\mathfrak{q}$ vanishes on all e_i and f_i. Let $\tilde{G} = \mathbf{GO}^+(2m,q)$ be the group of all linear transformations of V that preserve $\mathfrak{q}$, and let $G = \Omega^+(2m,q)$. (See [KL, pp. 29, 30] for the definition of this subgroup of $\tilde{G}$.) Fix $W' := \langle e_1, \ldots, e_m \rangle$, $W = \langle f_1, \ldots, f_m \rangle$, $\tilde{P} := \mathrm{Stab}_{\tilde{G}}(W') = Q : \tilde{L}$, $P := \tilde{P} \cap G = Q : L$. Here

$$Q := O_p(P) = \left\{ [I,X] := \begin{pmatrix} I_m & X \\ 0 & I_m \end{pmatrix} \mid X \in M_m(\mathbb{F}_q), X + {}^tX = 0, \mathrm{diag}(X) = 0 \right\}$$

(where diag(X) is the main diagonal of X). Fix a primitive p^{th} root ϵ of 1 in $\mathbb{C}$. If q is odd, let $H_m(q) := \{X \in M_m(\mathbb{F}_q) \mid X + {}^tX = 0\}$. If q is even, let $H_m(q)$ be the quotient space of $M_m(\mathbb{F}_q)$ by the subspace of symmetric matrices. Then any linear character (over $\mathbb{C}$ or over $\mathbb{F}$) of Q is of the form $\lambda_B \; : \; [I, X] \mapsto \epsilon^{\mathrm{tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mathrm{Tr}(BX))}$ for some $B \in H_m(q)$. Furthermore,

$$\tilde{L} = \left\{[A, 0] := \mathrm{diag}({}^tA^{-1}, A) \mid A \in \mathbf{GL}(W)\right\},\ L = \left\{[A, 0] \in \tilde{L} \mid \det(A) \in \mathbb{F}_q^{\times 2}\right\}\ .$$

Notice that G has two orbits on the maximal totally singular subspaces of V. Clearly, if a subspace M of a $\tilde{P}$-module affords the Q-character λ_B, then for any $a := [A, 0] \in \tilde{L}$, $a(M)$ affords the Q-character $a \circ \lambda_B : x \mapsto \lambda_B(a \circ x)$, where $a \circ x = a^{-1}xa$ for $x \in Q$, and

$$(a \circ \lambda_B)(x) = \lambda_B(a \circ x) = \lambda_{AB\,{}^tA}(x). \tag{24}$$

If q is odd, let $\mathcal{O}_1$ be the set of all λ_B with $\mathrm{rank}(B) = 2$. If q is even, let $\mathcal{O}_1$ be the set of all λ_B, where B is chosen so that $\mathrm{rank}(B - {}^tB) = 2$. Then $|\mathcal{O}_1| = (q^m - 1)(q^{m-1} - 1)/(q^2 - 1)$. Since $m > 2$, the subgroup L acts transitively on $\mathcal{O}_1$.

Let ρ be the permutation character of $\tilde{G}$ on $\mathbf{P}_0(V)$. Clearly, Q acts trivially on $\mathbf{P}_0(W')$. Next, for any $l := \langle f \rangle \in \mathbf{P}(W)$, Q acts transitively on the set Ω_l of q^{m-1} singular 1-spaces $\langle f + u \rangle$ with $u \in W'$. Denoting $Q_l := \mathrm{Stab}_Q(l)$, we then have

$$\rho|_Q = \frac{q^m - 1}{q - 1} \cdot 1_Q + \sum_{l \in \mathbf{P}(W)} \mathrm{Ind}_{Q_l}^{Q}(1_{Q_l}). \tag{25}$$

In particular, the multiplicity of 1_Q in $\rho|_Q$ is $2(q^m - 1)/(q - 1)$. Fix $l = \langle f_1 \rangle$ for the moment. Then one checks that λ_B occurs in $\mathrm{Ind}_{Q_l}^{Q}(1_{Q_l})$ exactly when Q_l is contained in $\mathrm{Ker}(\lambda_B)$, and this happens exactly for $q^{m-1} - 1$ characters λ_B with $\lambda_B \in \mathcal{O}_1$. Thus the only characters of Q that occur in $\rho|_Q$ are the trivial one and the ones from $\mathcal{O}_1$. Since $\tilde{G}$ acts on $\mathbf{P}_0(V)$ and $\tilde{P}$ acts transitively on $\mathcal{O}_1$, the multiplicity of λ_B in $\rho|_Q$ is the same for all $\lambda_B \in \mathcal{O}_1$. It follows that

$$\rho|_Q = \frac{2(q^m - 1)}{q - 1} \cdot 1_Q + (q + 1) \sum_{\lambda_B \in \mathcal{O}_1} \lambda_B\ . \tag{26}$$

Remark 2.7 tells us that the Q-trivial component of $\rho|_{\tilde{P}}$ affords the character $2.1_{\tilde{P}} + 2\tau$.

Next, we let $B_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and fix $B = \mathrm{diag}(B_0, \underbrace{0, \ldots, 0}_{m-2})$ for q odd and $B = \mathrm{diag}(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \underbrace{0, \ldots, 0}_{m-2})$. According to (24), the stabilizer $\tilde{S} := \{a \in \tilde{L} \mid a \circ \lambda_B =$

$\lambda_B\}$ equals

$$\left\{[A,0] \mid A = \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix} \in \mathbf{GL}(m,q), X \in \mathbf{GL}(2,q), XB_0\,{}^tX = B_0\right\}. \tag{27}$$

By the choice of B_0, $XB_0\,{}^tX = B_0$ is equivalent to $\det(X) = 1$. Therefore, in the notation of (27), the map $[A,0] \mapsto X$ is a surjective homomorphism $\pi : \tilde{S} \to \mathbf{SL}(2,q)$. Let $\tilde{R} := Q : \tilde{S} = \mathrm{Stab}_{\tilde{P}}(\lambda_B)$ and let K be the kernel of λ_B. For $x \in Q$, $a \in \tilde{S}$, we have $\lambda_B(x^{-1}a^{-1}xa) = (a \circ \lambda_B)(x)/\lambda_B(x) = 1$, i.e. $[x,a] \in K$. Thus $K \lhd \tilde{S}$ and $\tilde{S}$ centralizes Q modulo K. Hence, $\tilde{R}/K \simeq \mathrm{Z}_p \times \tilde{S}$, the first group being Q/K. Combining this isomorphism with π, we see that $\tilde{R}$ maps onto $\mathrm{Z}_p \times \mathbf{SL}(2,q)$. Let μ_i be the character of $\mathrm{Z}_p \times \mathbf{SL}(2,q)$, equal λ_B on Q, and equal the trivial character, resp. the Steinberg character St, of $\mathbf{SL}(2,q)$ if $i=0$, resp. if $i=1$. Here, the Steinberg character St of $\mathbf{SL}(2,q)$ is (the permutation character on the set of 1-spaces of the natural module) -1; in particular, its degree is q. We will denote the pullback of μ_i to $\tilde{R}$ by the same symbol μ_i.

Proposition 6.1. *In the above notation let $i = 0,1$ and $\sigma_i = \mathrm{Ind}_{\tilde{R}}^{\tilde{P}}(\mu_i)$. Then $(\rho|_{\tilde{P}}, \sigma_i)_{\tilde{P}} > 0$ and $(\rho|_P, \mathrm{Ind}_R^P(\mu_i|_R))_P > 0$ for $i = 0,1$. Furthermore, σ_i is irreducible over $\tilde{P}$ and over P.*

Proof. The proofs are the same for $\tilde{P}$ and for P, so we will give the details for $\tilde{P}$. Denoting $\tilde{P}_1 := \mathrm{Stab}_{\tilde{G}}(\langle f_1\rangle)$ and $P_1 = G \cap \tilde{P}_1$, we have $\rho = \mathrm{Ind}_{\tilde{P}_1}^{\tilde{G}}(1_{\tilde{P}_1})$. As in the proof of Proposition 4.1, it suffices to show that $\mu_i|_{\tilde{R}\cap\tilde{P}_1}$ contains $1_{\tilde{R}\cap\tilde{P}_1}$. The statement is now obvious if $i=0$, so we will assume $i=1$. Similarly to (12), we also have

$$\tilde{R} \cap \tilde{P}_1 = (Q \cap P_1) \cdot (\tilde{S} \cap \tilde{P}_1). \tag{28}$$

Now if $x = [I, X] \in Q \cap P_1$, then $\mathrm{Tr}(BX) = 0$, whence $\lambda_B(x) = 1$, i.e. $Q \cap P_1 \lhd K$. The factorization (28) then implies that $\mu_1|_{\tilde{R}\cap\tilde{P}_1}$ is a character of $\tilde{S} \cap \tilde{P}_1$. By the definition of μ_1, $\mu_1|_{\tilde{R}\cap\tilde{P}_1}$ can be viewed as the restriction of the Steinberg character St of $H := \mathbf{SL}(2,q)$ to the subgroup

$$T := \pi(\tilde{S} \cap \tilde{P}_1) = \left\{\begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} \mid x \in \mathbb{F}_q^\times, y \in \mathbb{F}_{q^2}\right\}.$$

Notice that $St = \mathrm{Ind}_T^H(1_T) - 1_H$. Hence

$$(St|_T, 1_T)_T = (\mathrm{Ind}_T^H(1_T)|_T - 1_T, 1_T)_T = (\mathrm{Ind}_T^H(1_T), \mathrm{Ind}_T^H(1_T))_H - (1_T, 1_T)_T = 2-1 = 1.$$

Consequently, $\mu_1|_{\tilde{R}\cap\tilde{P}_1}$ contains $1_{\tilde{R}\cap\tilde{P}_1}$ (with multiplicity 1), as desired.

Applying Lemma 3.3 to $(\tilde{P}, Q, \tilde{R}, \lambda_B, \mu_i)$ in place of (P, Z, R, L, M), we see that σ_i is irreducible. $\square$

Clearly, $1 = (\sigma_i, \sigma_i)_{\tilde{P}} = (\sigma_i|_{\tilde{R}}, \mu_i)_{\tilde{R}}$. In particular, $\sigma_0|_Q$ contains $\mu_0|_Q = \lambda_B$ and $\sigma_1|_Q$ contains $\mu_1|_Q = q\lambda_B$. It follows that

$$\sigma_0|_Q = \sum_{\lambda_B \in \mathcal{O}_1} \lambda_B \ , \sigma_1|_Q = q \sum_{\lambda_B \in \mathcal{O}_1} \lambda_B \ .$$

Proposition 6.2. *The permutation character ρ of $\tilde{G} = \mathbf{GO}^+(2m, q)$ on $\mathbf{P}_0(V)$ decomposes into irreducible Q- and $\tilde{P}$-constituents as follows:*

$$\begin{array}{rcl} \rho|_Q & = & \dfrac{2(q^m-1)}{q-1} \cdot 1_Q \quad + \quad (q+1) \sum_{\lambda_B \in \mathcal{O}_1} \lambda_B \ , \\ \rho|_{\tilde{P}} & = & 2 \cdot (1_{\tilde{P}} + \tau) \qquad\quad + \quad (\sigma_0 + \sigma_1) \ , \end{array}$$

where the constituents in the same columns of the two decompositions correspond to each other. All the above $\tilde{P}$-constituents remain irreducible over P.

Proof. The decomposition for $\rho|_Q$ has already been established in (26). Next, we have shown that $\rho|_{\tilde{P}}$ contains the right-hand side of the decomposition, as σ_i is irreducible and $(\rho|_{\tilde{P}}, \sigma_i)_{\tilde{P}} > 0$ by Proposition 6.1. Comparing the restriction to Q we get the decomposition. □

From Section 2.1 and Tables 1 and 2 we have $\rho = 1 + \varphi + \psi$, where $\varphi, \psi \in \mathrm{Irr}(\tilde{G})$, $\varphi(1) = (q^m-1)(q^{m-1}+q)/(q^2-1)$, $\psi(1) = (q^{2m}-q^2)/(q^2-1)$, and φ, ψ are irreducible over G.

Corollary 6.3. *Under the above notation, $\varphi|_{\tilde{P}} = 1_{\tilde{P}} + \tau + \sigma_0$ and $\psi|_{\tilde{P}} = \tau + \sigma_1$.*

Proof. We will make use of Proposition 6.2. Since $m \geq 3$ and $(m, q) \neq (3, 2)$, $\sigma_1(1) > \varphi(1)$, whence σ_1 is a constituent of $\psi|_{\tilde{P}}$. So is τ by Lemma 2.6. The corollary follows by degrees. □

6.2. **Reduction mod ℓ.** Next we study the mod ℓ decomposition of the above characters of $\tilde{P}$ and P. Since we will need the result in the case $\ell|(q+1)$, we formulate the result only for that case. The decomposition of τ has already been given in Lemma 2.5.

Proposition 6.4. *Assume that $\ell|(q+1)$.*

(i) *$\overline{\sigma}_0$ is irreducible over $\tilde{P}$ and over P.*

(ii) *If $\ell \neq 2$, $\overline{\sigma}_1 = \sigma_{11} + \overline{\sigma}_0$, where $\sigma_{11} \in \mathrm{IBr}_\ell(\tilde{P})$, $\sigma_{11}(1) = (q-1)|\mathcal{O}_1|$, and σ_{11} is irreducible over P. If $\ell = 2$ then $\overline{\sigma}_1 = \sigma_{11}^+ + \sigma_{11}^- + \overline{\sigma}_0$, where $\sigma_{11}^\pm \in \mathrm{IBr}_\ell(\tilde{P})$, $\sigma_{11}^\pm(1) = (q-1)|\mathcal{O}_1|/2$, and $\sigma_{11}^\pm$ is irreducible over P.*

Proof. (i) Since Q is an elementary abelian p-group and $\tilde{L}$, L act transitively on $\mathcal{O}_1$, $\overline{\sigma}_0$ is irreducible (over both $\tilde{P}$ and P) for any $\ell \neq p$.

(ii) Assume $\ell \neq 2$. Since $\ell|(q+1)$, $\overline{\mu}_1 = \mu_{11} + \overline{\mu}_0$, where μ_{11} is an irreducible Brauer character of degree $q-1$. (Recall that μ_1 is a character of $\mathrm{Z}_p \times \mathbf{SL}(2, q)$ inflated to $\tilde{R}$.) By Lemma 3.3, $\sigma_{11} := \mathrm{Ind}_{\tilde{R}}^{\tilde{P}}(\mu_{11})$ is irreducible, and $\overline{\sigma}_1 = \sigma_{11} + \overline{\sigma}_0$.

Assume $\ell = 2$ (in particular q is odd). Then $\overline{\mu}_1 = \mu_{11}^+ + \mu_{11}^- + \overline{\mu}_0$, where $\mu_{11}^{\pm}$ are irreducible Brauer characters of degree $(q-1)/2$. Now the same argument as above applies. □

We can now give the decomposition of $\overline{\varphi}$ and $\overline{\psi}$ into irreducible Brauer characters of G and $\tilde{G}$. Let $\mathbf{CO}^+(2m,q)$ be the subgroup of $\mathbf{GL}(V)$ consisting of those transformations which preserve the form up to nonzero scalar multiplication. This group acts on $\mathbf{P}_0$ and contains $\mathbf{GO}^+(2m,q)$ as a normal subgroup.

Corollary 6.5. *Assume $\ell|(q+1)$, $m \geq 3$, $(m,q) \neq (3,2)$.*

(i) $\beta(U') = \begin{cases} \beta(X) \text{ if } m \text{ is odd} \\ 1+\beta(X) \text{ if } m \text{ is even} \end{cases}$, $\beta(U')|_{\tilde{P}} = \overline{\tau} + \overline{\sigma}_0$, *and X is simple.*

(ii) $\overline{\varphi} = \delta + \beta(U')$, *where δ is one-dimensional (trivial when $\ell = 2$). Furthermore, if q is odd then δ is trivial on $\mathbf{SO}^+(2m,q)$.*

(iii) $\overline{\psi} = \beta(U') + \beta(Y)$, *where the module Y has the following properties. If $\ell \neq 2$ then Y is simple over $G = \Omega^+(2m,q)$ and $\tilde{G} = \mathbf{GO}^+(2m,q)$, and $\beta(Y)|_{\tilde{P}} = \sigma_{11}$. If $\ell = 2$ then $\beta(Y)|_{\tilde{P}} = \sigma_{11}^+ + \sigma_{11}^-$ and Y is simple for the group $\mathbf{CO}^+(2m,q)$ but decomposes as a direct sum of two simple modules for G and $\tilde{G}$.*

Proof. The first equation in (i) is simply Corollary 2.10. The second follows from Lemma 3.1 since σ_0 is simple for $\tilde{P}$ by Proposition 6.4. Next $\delta := \overline{\varphi} - \beta(U')$ is one-dimensional. Since G is perfect, δ is trivial on G; in particular, δ is trivial if $\ell = 2$ as $\tilde{G}/G$ is a 2-group. Now assume q is odd. By Proposition 6.2, δ is trivial on $\tilde{P}$. Hence δ is trivial on $G\tilde{P} = \mathbf{SO}^+(2m,q)$, so (ii) holds. Next we consider composition factors of $\overline{\rho}$ over $\hat{G} := \mathbf{CO}^+(2m,q)$. By virtue of (i) and (ii) we may assume $\ell = 2$. Direct computation shows that instead of (27) the stabilizer of λ_B in $\hat{P} := \mathrm{Stab}_{\hat{G}}(W')$ is $Q : \hat{S}$, with

$$\hat{S} = \left\{ \mathrm{diag}(\lambda\,{}^t\!A^{-1}, A) \mid A = \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix} \in \mathbf{GL}(m,q), X \in \mathbf{GL}(2,q), \lambda = \det(X) \right\}.$$

Since the Brauer characters μ_{11}^+ and μ_{11}^- mentioned in the proof of Proposition 6.4 fuse under $\hat{S}$, σ_{11}^+ and σ_{11}^- fuse under $\hat{P}$, and the claims follow. The fact that Y splits into a direct sum of two irreducibles over G and $\tilde{G}$ when $\ell = 2$ is proved in [T]. □

The character δ will be made precise in Lemma 6.6 below. To the end of this section, we define $\mathbf{SO}^+(2m,q)$ to be $G = \Omega^+(2m,q)$ if q is even. By the *determinant character* we mean the unique linear character of $\mathbf{G} = \mathbf{GO}^+(2m,q)$ with kernel $\mathbf{SO}^+(2m,q)$ (for any q).

Recall that $\mathcal{C}$ is the subspace of $\mathbb{F}^{\mathbf{P}_0}$ spanned by the characteristic vectors of maximal singular subspaces. By Witt's lemma, $\mathbf{G}$ acts transitively on the set of maximal singular subspaces, but under the action of $\mathbf{SO}^+(2m,q)$ the set falls into two orbits. Accordingly, we can define $\mathcal{C}_1$ and $\mathcal{C}_2$ as the $\mathbb{F}\mathbf{SO}^+(2m,q)$-submodules of $\mathcal{C}$ spanned by the characteristic vectors of maximal singular subspaces in the respective orbits. Clearly $\mathcal{C}$ is the sum of $\mathcal{C}_1$ and $\mathcal{C}_2$.

In combination with Corollary 2.10 and Remark 2.12 the $\mathbb{F}\mathbf{G}$-submodule lattice of $\mathbb{F}^{\mathbf{P}_0}$ is given by the following result.

Lemma 6.6. *Assume that $\ell \mid (q+1)$.*

(i) *The summands $\mathcal{C}_1/U'$ and $\mathcal{C}_2/U'$ are one-dimensional, trivial for $\mathbf{SO}^+(2m,q)$ and interchanged by $\mathbf{G}$.*
(ii) *The character δ in Proposition 6.5 is the determinant character.*
(iii) *${U'}^{\perp}/U' \cong \mathcal{C}_1/U' \oplus \mathcal{C}_2/U' \oplus Y$ as $\mathbb{F}\mathbf{G}$-modules where Y is irreducible if $\ell \neq 2$, while if $\ell = 2$ we have $Y = Y_1 \oplus Y_2$, a direct sum of two simple modules, conjugate by an element of the conformal group.*

Proof. We know from Corollary 6.5 that the composition factors of the $\mathbb{F}\mathbf{SO}^+(2m,q)$-module ${U'}^{\perp}/U'$ are $\mathbb{F}$ (with multiplicity 2) and those of Y as described in the statement. We also know by Lemma 2.8 that $U' \subseteq \mathcal{C} \cap \mathcal{C}^{\perp}$ and by Lemma 2.3(i) that $\dim_{\mathbb{F}} \mathcal{C}/U' \geq 2$. Since the dimension of $\mathcal{C}$ is bounded above by the corresponding dimension $1+\varphi(1)$ in characteristic zero, and since $2+\dim_{\mathbb{F}} U' = 1+\varphi(1)$ by our character calculations, it follows that $\mathcal{C}/U' \cong \mathbb{F} \oplus \mathbb{F}$ as $\mathbb{F}\mathbf{SO}^+(2m,q)$-modules. Since $\mathcal{C}_1$ and $\mathcal{C}_2$ are images of transitive permutation modules for $\mathbb{F}\mathbf{SO}^+(2m,q)$, they each have at most one trivial homomorphic image. Therefore, $\mathcal{C}_1/U'$ and $\mathcal{C}_2/U'$ are both proper submodules of their sum $\mathcal{C}/U'$, proving (i). Part (ii) is immediate from the fact that $\mathbf{G}$ interchanges $\mathcal{C}_1/U'$ with $\mathcal{C}_2/U'$. By the self-duality of ${U'}^{\perp}/U'$ and the fact that Y has no trivial composition factors, we obtain ${U'}^{\perp}/U' \cong \mathcal{C}/U' \oplus Y$. The remaining statements about Y are from Proposition 6.5(iii). □

The submodule structures we have obtained are depicted in Figures 4 and 5.

6.3. **The smallest degree of cross-characteristic representations.** Again, let $\kappa_{2,m}$ be equal 1 if $2|m$ and 0 otherwise. Then Corollary 6.5 implies that the simple G-module X has dimension

$$\varphi(1) - 1 - \kappa_{2,m} = \frac{(q^m-1)(q^{m-1}+q)}{q^2-1} - 1 - \kappa_{2,m}.$$

On the other hand, if $q \leq 3$ (and $\ell|(q+1)$) then the dimension of any simple G-constituent of Y is $|\mathcal{O}_1| = (q^m-1)(q^{m-1}-1)/(q^2-1)$. Combining these observations with [Ho], we obtain the following result.

Corollary 6.7. *Let $m \geq 3$ and $\ell|(q+1)$. Assume $(m,q) \neq (3,2)$, $(3,3)$, $(4,2)$. Then the smallest degree of nontrivial irreducible projective representations of $\Omega^+(2m,q)$*

is

$$\begin{cases} \dfrac{(q^m-1)(q^{m-1}+q)}{q^2-1} - \kappa_{2,m} - 1, & \text{if } q > 3, \\ \dfrac{(q^m-1)(q^{m-1}-1)}{q^2-1}, & \text{if } q \leq 3, \end{cases}.$$

□

7. The orthogonal groups in odd dimensions

7.1. **Character restriction.** Let $m \geq 3$, q odd, and let $V = \mathbb{F}_q^{2m+1}$ be endowed with a nondegenerate orthogonal form $\mathfrak{q}$ with Gram matrix $\begin{pmatrix} 0 & I_m & 0 \\ I_m & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ in a basis $(e_1, \dots, e_m, f_1, \dots, f_m, g)$. Let $\tilde{G} = \mathbf{SO}(2m+1, q)$ be the group of all linear transformations of V that preserve $\mathfrak{q}$ with determinant 1; notice that $\mathbf{GO}(2m+1, q) = \mathrm{Z}_2 \times \tilde{G}$, and let $G = \Omega(2m+1, q)$. Fix $W' := \langle e_1, \dots, e_m \rangle$, $W = \langle f_1, \dots, f_m \rangle$ and let $\tilde{P} := \mathrm{Stab}_{\tilde{G}}(W') = Q : \tilde{L}$, $P := \tilde{P} \cap G = Q : L$, where $Q := O_p(P)$. We have

$$Q = \left\{ [I, X, a] := \begin{pmatrix} I_m & X & a \\ 0 & I_m & 0 \\ 0 & -{}^t a & 1 \end{pmatrix} \mid X \in M_m(\mathbb{F}_q), a \in \mathbb{F}_q^m, X + {}^t X + a \cdot {}^t a = 0 \right\}.$$

As in §6, let $H_m(q) := \{X \in M_m(\mathbb{F}_q) \mid X + {}^t X = 0\}$, and fix a primitive p^{th} root ϵ of 1 in $\mathbb{C}$. Then $Q' = Z(Q) = \{[I, X, 0] \mid X \in H_m(q)\}$ is elementary abelian of order $q^{m(m-1)/2}$. Any linear character (over $\mathbb{C}$ or over $\mathbb{F}$) of $Z(Q)$ is of the form $\lambda_B : [I, X, 0] \mapsto \epsilon^{\mathrm{tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mathrm{Tr}(BX))}$ for some $B \in H_m(q)$. Furthermore,

$$\tilde{L} = \left\{ [A, 0] := \mathrm{diag}({}^t A^{-1}, A) \mid A \in \mathbf{GL}(W) \right\}, \ L = \left\{ [A, 0] \in \tilde{L} \mid \det(A) \in \mathbb{F}_q^{\times 2} \right\}.$$

Let $\mathcal{O}_1$ be the set of all λ_B with $\mathrm{rank}(B) = 2$. Then $|\mathcal{O}_1| = (q^m-1)(q^{m-1}-1)/(q^2-1)$. Since $m > 2$, the subgroup L acts transitively on $\mathcal{O}_1$. For the rest of this section, we choose $B = \mathrm{diag}(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \underbrace{0, \dots, 0}_{m-2})$. Let $N := \{[I, X, 0] \in Q \mid \mathrm{Tr}(BX) = 0\}$. Then $N < Z(Q)$, so $N \lhd Q$, and $N \leq \mathrm{Ker}(\lambda_B)$. One can check that $Q/N = \tilde{Q}_1 \times Q_2$, with $\tilde{Q}_1$ a group of symplectic type of order q^{1+2}, and Q_2 is elementary abelian of order q^{m-2}. In fact, we can write

$$\tilde{Q}_1 = \left\{ [I, X, a]N \mid [I, X, a] \in Q, {}^t a = (a_1, a_2, 0, \dots, 0) \right\},$$
$$Q_2 = \left\{ [I, 0, b]N \mid [I, 0, b] \in Q, {}^t b = (0, 0, b_3, \dots, b_m) \right\}.$$

Let $K := \mathrm{Ker}(\lambda_B)$. Then $Q/K = Q_1 \times Q_2$, with Q_1 an extraspecial group of order pq^2. So λ_B gives rise to a unique irreducible character ω of degree q of Q that restricts to $Z(Q)$ as $q\lambda_B$ and trivial on Q_2. This is also true for any $\lambda_{B'} \in \mathcal{O}_1$.

Let ρ be the permutation character of $\tilde{G}$ on $\mathbf{P}_0(V)$. Let V_1 be the hyperplane of V spanned by $e_1, \dots, e_m, f_1, \dots, f_m$. Since $(V_1, Z(Q))$ plays the role of (V, Q) in §6, the $Z(Q)$-permutation character on $\mathbf{P}_0(V_1)$ is $\frac{2(q^m-1)}{q-1} \cdot 1_{Z(Q)} + (q+1)\sum_{\lambda_B \in \mathcal{O}_1} \lambda_B$. Also from §6 we see that for each $l = \langle f \rangle \in \mathbf{P}(W)$, $Z(Q)$ acts transitively on the set Ω_l of q^{m-1} singular 1-spaces $\langle f+u \rangle$ with $u \in W'$. Next, for any $f \in W \setminus \{0\}$, $Z(Q)$ acts transitively on the set Ω'_f of q^{m-1} singular 1-spaces $\langle g+f+u \rangle$ with $u \in W'$, with the permutation character $\mathrm{Ind}^{Z(Q)}_{Z(Q)_{f+u}}(1_{Z(Q)_{f+u}})$, where $Z(Q)_{f+u} := \mathrm{Stab}_{Z(Q)}(f+u)$. Setting $l := \langle f \rangle_{\mathbb{F}_q}$, we see that $Z(Q)_{f+u}$ plays the role of Q_l in §6; in particular, $\mathrm{Ind}^{Z(Q)}_{Z(Q)_{f+u}}(1_{Z(Q)_{f+u}})$ consists of $1_{Z(Q)}$ and $q^{m-1}-1$ distinct characters $\lambda_B \in \mathcal{O}_1$. Thus the only characters of $Z(Q)$ that occur in $\rho|_{Z(Q)}$ are the trivial one and the ones from $\mathcal{O}_1$. Since $\tilde{G}$ acts on $\mathbf{P}_0(V)$ and $\tilde{P}$ acts transitively on $\mathcal{O}_1$, the multiplicity of λ_B in $\rho|_{Z(Q)}$ is the same for all $\lambda_B \in \mathcal{O}_1$. It follows that

$$\rho|_{Z(Q)} = (q+1) \cdot \frac{q^m-1}{q-1} \cdot 1_{Z(Q)} + q(q+1) \sum_{\lambda_B \in \mathcal{O}_1} \lambda_B \ . \tag{29}$$

Each orbit Ω_l, resp. Ω'_f, gives a fixed point, call it $[\Omega_l]$, resp. $[\Omega'_f]$, for $Z(Q)$ on the permutation module with character ρ. Now $Q/Z(Q)$ acts on the $Z(Q)$-fixed points, and observe that this action is nontrivial: if we choose $X = \mathrm{diag}(-\frac{1}{2}, 0, \dots, 0)$ and ${}^t a = (1, 0, \dots, 0)$, then the element $[I, X, a] \in Q$ maps $[\Omega_{\langle f_1 \rangle}]$ to $[\Omega'_{-f_1}]$. Also, $Q/Z(Q)$ is elementary abelian of order q^m. It follows that $\rho|_Q$ contains nontrivial linear characters of $Q/Z(Q)$. But L acts transitively on nontrivial linear characters of $Q/Z(Q)$, so $\rho|_Q$ contains all $q^m - 1$ nontrivial linear characters of $Q/Z(Q)$. By Remark 2.7, $\rho|_Q$ contains 1_Q with multiplicity at least $2(q^m-1)/(q-1)$. Together with (29), this discussion shows that the linear Q-characters in $\rho|_Q$ constitute

$$\frac{2(q^m-1)}{q-1} \cdot 1_Q + \sum_{1_Q \neq \nu \in \mathrm{Irr}(Q/Z(Q))} \nu \ . \tag{30}$$

Remark 2.7 now tells us that the Q-trivial component of $\rho|_{\tilde{P}}$ affords the character $2.1_{\tilde{P}} + 2\tau$. We have seen that the Q-component listed in (30) gives rise to the $\tilde{P}$-character

$$2 \cdot 1_{\tilde{P}} + 2\tau + \varsigma \ . \tag{31}$$

Next we analyze the nonlinear Q-characters inside $\rho|_Q$. For the chosen B, the stabilizer $\tilde{S} := \{a \in \tilde{L} \mid a \circ \lambda_B = \lambda_B\}$ equals

$$\left\{ [A, 0] \mid A = \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix} \in \mathbf{GL}(m,q), X \in \mathbf{Sp}(2,q) \right\}. \tag{32}$$

In the notation of (32), the map $[A, 0] \mapsto X$ is a surjective homomorphism $\pi : \tilde{S} \to \mathbf{Sp}(2, q)$. Let $\tilde{R} := Q : \tilde{S} = \mathrm{Stab}_{\tilde{P}}(\lambda_B)$ and $R := \tilde{R} \cap P$. Recall (see e.g. [TZ2]) that $\mathbf{Sp}(2, q)$ has two irreducible *Weil* characters ξ_1, ξ_2 of degree $(q+1)/2$, and two irreducible *Weil* characters η_1, η_2 of degree $(q-1)/2$. Using π, we inflate ξ_i to $\tilde{S}$, and then to an $\tilde{R}$-character that we also denote by ξ_i, $i = 1, 2$.

Also recall that $K = \mathrm{Ker}(\lambda_B)$ and $Q/K = Q_1 \times Q_2$. For $x \in Z(Q)$, $a \in \tilde{S}$, we have $\lambda_B(x^{-1}a^{-1}xa) = (a \circ \lambda_B)(x)/\lambda_B(x) = 1$, i.e. $[x, a] \in K$. Thus $K \lhd \tilde{R}$, and $\tilde{R}$ centralizes $Z(Q_1)$ in $\tilde{R}/K$. One can check that $\tilde{S}$ normalizes Q_2, so $Q_2 \lhd \tilde{R}/K$. Next, $\mathrm{Ker}(\pi)$ centralizes Q_1 modulo Q_2, and $\tilde{S}$ acts on Q_1 (modulo Q_2) as the group $\mathbf{Sp}(2, q)$ of (certain) outer automorphisms of Q_1 that act trivially on $Z(Q_1)$. Consequently, the quotient of $\tilde{R}$ by $\langle K, Q_2, \mathrm{Ker}(\pi)\rangle$ is $Q_1 : \mathbf{Sp}(2, q)$. It is known that ω extends to an irreducible character (of degree q) of $Q_1 : \mathbf{Sp}(2, q)$. We will inflate this character to $\tilde{R}$ and denote it also by ω. Notice that $\omega|_{\mathbf{Sp}(2,q)}$ equals $\xi_1 + \eta_1$ or $\xi_2 + \eta_2$.

By Lemma 3.2, $\omega\xi_i$ is irreducible over $\tilde{R}$ and also over R for $i = 1, 2$. Notice that $\omega\xi_1 \neq \omega\xi_2$ (indeed, if $x \in \mathbf{Sp}(2, q)$ is of order p then $|(\omega\xi_1 - \omega\xi_2)(x)| = q$).

Proposition 7.1. *In the above notation let $i = 1, 2$ and $\sigma_i = \mathrm{Ind}_{\tilde{R}}^{\tilde{P}}(\omega\xi_i)$. Then $(\rho|_{\tilde{P}}, \sigma_i)_{\tilde{P}} > 0$ and $(\rho|_P, \mathrm{Ind}_R^P(\omega\xi_i|_R))_P > 0$ for $i = 1, 2$. Furthermore, σ_i is irreducible over $\tilde{P}$ and over P.*

Proof. The proofs are the same for $\tilde{P}$ and for P, so we will give the details for $\tilde{P}$. Denoting $\tilde{P}_1 := \mathrm{Stab}_{\tilde{G}}(\langle f_1\rangle)$ and $P_1 = G \cap \tilde{P}_1$, we have $\rho = \mathrm{Ind}_{\tilde{P}_1}^{\tilde{G}}(1_{\tilde{P}_1})$. As in the proof of Proposition 4.1, it suffices to show that $\omega\xi_i|_{\tilde{R}\cap\tilde{P}_1}$ contains $1_{\tilde{R}\cap\tilde{P}_1}$. Similarly to (12), we also have

$$\tilde{R} \cap \tilde{P}_1 = (Q \cap P_1) \cdot (\tilde{S} \cap \tilde{P}_1). \tag{33}$$

For short we denote $R_1 := \tilde{R} \cap \tilde{P}_1$.

Observe that if $[I, X, a] \in R_1$, then the first column of X and the first entry of a are 0. It follows that $Z(Q) \cap R_1 \leq K = \mathrm{Ker}(\lambda_B)$. By definition, $\omega\xi_i$ is trivial on K, Q_2, and $\mathrm{Ker}(\pi)$. Consequently, modulo $\mathrm{Ker}(\omega\xi_i)$, $Q \cap P_1$ is $Q_3 \simeq (\mathbb{F}_q, +)$ and $\tilde{S} \cap \tilde{P}_1$ is $S_3 = \{\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{F}_q^\times, b \in \mathbb{F}_q\}$. By (33), R_1 modulo $\mathrm{Ker}(\omega\xi_i)$ is just $Q_3 : S_3$ with S_3 acting on Q_3 via $scs^{-1} = ac$ for $s = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in S_3$ and $c \in Q_3$.

Next, $\omega|_{Q_3}$ is just the regular representation of Q_3. (Recall that ω is an irreducible complex character of degree q of the extraspecial group Q_1 of order pq^2, and Q_3 is an elementary abelian subgroup of order q of Q_1 that trivially intersects $Z(Q_1)$.) In particular, $\omega|_{Q_3}$ contains 1_{Q_3} with multiplicity 1. We can view S_3 as a Borel subgroup of $\mathbf{Sp}(2, q)$, and the construction of the extension ω as specified in [Gr, §13] shows that S_3 acts on the 1-dimensional subspace of Q_3-fixed points inside $\omega|_Q$ via the quadratic character $\delta\ :\ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mapsto \begin{pmatrix} a \\ q \end{pmatrix}$ (with $\begin{pmatrix} a \\ q \end{pmatrix}$ being the Legendre symbol), see [Gr, (13.2), (13.3)].

By its definition, ξ_i is trivial on Q_3, and $\xi_i|_{S_3}$ contains the quadratic character δ with multiplicity 1. (In fact $\xi_1 + \xi_2 = \mathrm{Ind}_{S_3}^{\mathbf{Sp}(2,q)}(\delta)$.)

The above discussion implies that $\omega\xi_i|_{Q_3:S_3}$ contains the trivial character of $Q_3 : S_3$. In other words, $\omega\xi_i|_{R_1}$ contains 1_{R_1}, as desired. By Lemma 3.3, σ_i is irreducible. $\square$

Clearly, $1 = (\sigma_i, \sigma_i)_{\tilde{P}} = (\sigma_i|_{\tilde{R}}, \omega\xi_i)_{\tilde{R}}$. In particular, $\sigma_i|_{Z(Q)}$ contains $\omega\xi_i|_{Z(Q)} = (q(q+1)/2)\lambda_B$. It follows that

$$\sigma_i|_{Z(Q)} = \frac{q(q+1)}{2} \cdot \sum_{\lambda_B \in \mathcal{O}_1} \lambda_B\ .$$

Proposition 7.2. *The permutation character ρ of $\tilde{G} = \mathbf{SO}(2m+1, q)$ on $\mathbf{P}_0(V)$ decomposes into irreducible $\tilde{P}$-constituents as follows:*

$$\rho|_{\tilde{P}} = 2 \cdot 1_{\tilde{P}} + 2\tau + \varsigma + \sigma_1 + \sigma_2$$

where all the above $\tilde{P}$-constituents remain irreducible over P.

Proof. We have shown that $\rho|_{\tilde{P}}$ contains the right-hand side of the decomposition, as the σ_i with $i = 1, 2$ are distinct irreducibles, and $(\rho|_{\tilde{P}}, \sigma_i)_{\tilde{P}} > 0$ by Proposition 7.1. Comparing the restriction to $Z(Q)$ we get the decomposition. $\square$

From Section 2.1 and Tables 1 and 2 we have $\rho = 1 + \psi + \varphi$, where $\psi, \varphi \in \mathrm{Irr}(\tilde{G})$, $\psi(1) = (q^m+1)(q^m-q)/2(q-1)$, $\varphi(1) = (q^m-1)(q^m+q)/2(q-1)$, and ψ, φ are irreducible over G.

Corollary 7.3. *Under the above notation and interchanging σ_1 and σ_2 if necessary, we have $\psi|_{\tilde{P}} = \tau + \sigma_1$ and $\varphi|_{\tilde{P}} = 1_{\tilde{P}} + \tau + \varsigma + \sigma_2$.*

Proof. We will make use of Proposition 7.2. Clearly, $\psi|_{\tilde{P}}$ contains at least one $\tilde{P}$-irreducible character that is nontrivial on $Z(Q)$, so we may assume that $\psi|_{\tilde{P}}$ contains σ_1. It also contains τ by Lemma 2.6, so the Corollary follows by degrees. $\square$

7.2. **Reduction mod ℓ.** Next we study the mod ℓ decomposition of the above characters of $\tilde{P}$ and P. Since we will need the result in the case $\ell = 2$, we formulate the result only for that case. The decomposition of τ has already been given in Lemma 2.5.

Proposition 7.4. *Assume $m \geq 3$, q is odd and $\ell = 2$.*

(i) *$\overline{\varsigma}$ is irreducible over $\tilde{P}$ and over P.*

(ii) *For $i = 1, 2$, $\overline{\sigma}_i = \sigma_{i1} + \overline{\sigma}_0$. Here, $\sigma_{i1} = \mathrm{Ind}_{\tilde{R}}^{\tilde{P}}(\overline{\omega\eta}_i)$ has degree $q(q-1)|\mathcal{O}_1|/2$, and $\overline{\sigma}_0 = \mathrm{Ind}_{\tilde{R}}^{\tilde{P}}(\overline{\omega})$ is of degree $q|\mathcal{O}_1|$. Furthermore, σ_{i1} and $\overline{\sigma}_0$ are irreducible over $\tilde{P}$ and P and $\sigma_{11} \neq \sigma_{21}$.*

Proof. (i) Since Q is an elementary abelian p-group and $\tilde{L}$, L act transitively on $\mathcal{O}_1$, $\overline{\varsigma}$ is irreducible (over both $\tilde{P}$ and P) for any $\ell \neq p$.

(ii) It is well known that $\overline{\xi}_i = 1 + \overline{\eta}_i$, and $\overline{\eta}_i$ is irreducible. Hence the irreducibility follows by Lemmas 3.2 and 3.3. When we restrict σ_{11} and σ_{21} to Q, the respective λ_B-isotypic components afford distinct characters of $\tilde{R}$. Since $\tilde{R}$ is the stabilizer of λ_B, the induced characters σ_{i1} are distinct. □

We can now give the decomposition of $\overline{\varphi}$ and $\overline{\psi}$ into irreducible Brauer characters of G and $\tilde{G}$.

Corollary 7.5. *Assume $m \geq 3$, q is odd and $\ell = 2$.*

(i) $\beta(U') = \begin{cases} \beta(X) \text{ if } m \text{ is odd} \\ 1 + \beta(X) \text{ if } m \text{ is even} \end{cases}$, $\beta(U')|_{\tilde{P}} = \overline{\tau} + \overline{\sigma}_0$, *and X is simple.*

(ii) $\overline{\psi} = \beta(U') + \beta(Y)$, *where Y is simple and $\beta(Y)|_{\tilde{P}} = \sigma_{11}$.*

(iii) $\overline{\varphi} = 1 + \beta(U') + \beta(D)$, *where D is simple and $\beta(D)|_{\tilde{P}} = \overline{\varsigma} + \sigma_{21}$.*

Proof. The first equation in (i) is simply Corollary 2.10. Since $\sigma_{11} \neq \sigma_{21}$, by Proposition 7.4, Lemma 3.1 implies the last part of (i). Then, since $(\overline{\psi} - \beta(U'))|_{\tilde{P}} = \sigma_{11}$ is irreducible, we must have $\sigma_{11} = \beta(Y)|_{\tilde{P}}$ for some simple $\mathbb{F}\tilde{G}$-module Y, proving (ii). By Lemma 2.13(iii), the module $U'^{\perp}/U'$ has at least two trivial G-composition factors, so it follows from (ii) that $\overline{\varphi} - \beta(U') - 1$ is the Brauer character of an $\mathbb{F}\tilde{G}$-module, which we shall denote by D. Now $(\overline{\varphi} - \beta(U') - 1)|_{\tilde{P}} = \overline{\varsigma} + \sigma_{21}$. Since ς has $Z(Q)$ in its kernel it cannot be the restriction of a simple $\mathbb{F}\tilde{G}$-module. Hence D is a simple and so (iii) holds. □

Now we describe the submodule structure. We continue to assume $m \geq 3$ and recall that $\mathbf{G} = \mathbf{GO}(2m+1, q)$. In combination with Corollary 2.10 and Remark 2.12 the $\mathbb{F}\mathbf{G}$-submodule lattice of $\mathbb{F}^{\mathbf{P}_0}$ is given by the following result.

Lemma 7.6. *Suppose $\ell = 2$. Then $U'^{\perp}/U' \cong \mathcal{C}/U' \oplus Y$ as $\mathbb{F}\mathbf{G}$-modules, where Y is simple and $\mathcal{C}/U'$ is uniserial with series $\mathbb{F}$, D, $\mathbb{F}$.*

Proof. We know by Corollary 7.5 that the composition factors of $U'^{\perp}/U'$ for $\mathbf{G}$ are $\mathbb{F}$, $\mathbb{F}$, D and Y and we have the dimensions. The modules D and Y must be self-dual since their dimensions differ. The commutator subgroup of $\mathbf{G}$ acts transitively on maximal singular subspaces, so by Lemma 2.3(iii), the module $\mathcal{C}/\mathcal{C} \cap \mathcal{C}^{\perp}$ has dimension at least 3 and it must have a nontrivial composition factor. This composition factor is also a constituent of $\overline{\varphi}$, so this must be D by Corollary 7.5(iii). Thus the quotient $\mathcal{C}/\mathcal{C} \cap \mathcal{C}^{\perp}$ of $\mathcal{C}/U'$ has composition factors among $\mathbb{F}$, D, $\mathbb{F}$ and is self-dual. Moreover, it is a quotient of $\mathcal{C}$ so has no quotient isomorphic to the simple module D, since $\tilde{P}$ has no fixed points on D. By self-duality it also has no submodules isomorphic to D. It follows that $\mathcal{C}/\mathcal{C} \cap \mathcal{C}^{\perp}$ must have at least three composition factors and since $\mathcal{C}/U'$ has at most three, the two modules are equal and the composition factors are $\mathbb{F}$

(twice) and D. Moreover, since D is neither a submodule nor a quotient, the module is uniserial with series $\mathbb{F}$, D, $\mathbb{F}$. This establishes the structure of $\mathcal{C}/U'$. Finally, since the simple module Y is self-dual and occurs with multiplicity one in the self-dual module ${U'}^{\perp}/U'$, it must be isomorphic to a direct summand. This completes the proof. □

The submodule structure we have determined is depicted in Fig. 6.

The case when $m = 2$ and $\ell = 2$ is slightly different with respect to the commutator subgroup of $\mathbf{GO}(5, q)$ and has been described fully in [LST].

8. The non-split orthogonal groups in even dimensions

8.1. **Character restriction.** Let $V = \mathbb{F}_q^{2m}$ be endowed with a nondegenerate non-split orthogonal form $\mathfrak{q}$. Unless otherwise stated, we will assume throughout this section that $m \geq 4$. Since the main results are true for $(m, q) = (4, 2)$, $(5, 2)$ by using [Atlas], [JLPW], we will assume that $(m, q) \neq (4, 2), (5, 2)$. We assume that the associated bilinear form has Gram matrix $\begin{pmatrix} 0 & I_{m-1} & 0 \\ I_{m-1} & 0 & 0 \\ 0 & 0 & D \end{pmatrix}$ in a basis $(e_1, \dots, e_{m-1}, f_1, \dots, f_{m-1}, e_m, f_m)$, and we choose this basis so that the form $\mathfrak{q}$ is totally singular on $\langle e_1, \dots, e_{m-1}\rangle_{\mathbb{F}_q}$ and on $\langle f_1, \dots, f_{m-1}\rangle_{\mathbb{F}_q}$, and anisotropic on $\langle e_m, f_m\rangle_{\mathbb{F}_q}$. In particular, if q is odd, we can choose $D = \begin{pmatrix} 1 & 0 \\ 0 & -\kappa \end{pmatrix}$ with $\kappa \in \mathbb{F}_q$ such that the polynomial $x^2 - \kappa$ is irreducible over $\mathbb{F}_q$. If q is even, we can choose $D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\mathfrak{q}(e_m) = 1$, and $\mathfrak{q}(f_m) = \kappa \in \mathbb{F}_q$ such that $x^2 + x + \kappa$ is irreducible over $\mathbb{F}_q$. Let $\tilde{G} = \mathbf{SO}^-(2m, q)$ be the group of all linear transformations of V of determinant 1 that preserve $\mathfrak{q}$, and let $G = \Omega^-(2m, q)$. Fix $W' := \langle e_1, \dots, e_{m-1}\rangle_{\mathbb{F}_q}$, $W = \langle f_1, \dots, f_{m-1}\rangle_{\mathbb{F}_q}$. Let $\tilde{P} := \mathrm{Stab}_{\tilde{G}}(W') = Q : \tilde{L}$ and $P := \tilde{P} \cap G = Q : L$, where $Q := O_p(P)$. We have

$$Q = \left\{ [I, X, a] := \begin{pmatrix} I_{m-1} & X & -aD \\ 0 & I_{m-1} & 0 \\ 0 & {}^t a & I_2 \end{pmatrix} \;\middle|\; \begin{array}{c} X \in M_{m-1}(\mathbb{F}_q), a \in M_{m-1,2}(\mathbb{F}_q), \\ X + {}^tX + aD \cdot {}^ta = 0 \end{array} \right\}$$

if q is odd. If q is even, one needs the extra condition that for any $j < m$, $x_{jj} + a_{j1}a_{j2} + a_{j1}^2 + \kappa a_{j2}^2 = 0$ if $X = (x_{ij})$ and $a = (a_{ij})$. In particular, $|Q| = q^{(m-1)(m+2)/2}$. Furthermore,

$$Z(Q) = \{[I, X, 0] \mid X \in M_{m-1}(\mathbb{F}_q), X + {}^tX = 0, \mathrm{diag}(X) = 0\}$$

is elementary abelian of order $q^{(m-1)(m-2)/2}$. Fix a primitive p^{th} root ϵ of 1 in $\mathbb{C}$. As in §6, let $H_m(q) := \{X \in M_{m-1}(\mathbb{F}_q) \mid X + {}^tX = 0\}$ if q is odd, and let $H_m(q)$ be the quotient space of $M_{m-1}(\mathbb{F}_q)$ by the subspace of symmetric matrices if q is even.

Then any linear character (over $\mathbb{C}$ or over $\mathbb{F}$) of $Z(Q)$ is of the form $\lambda_B : [I, X, 0] \mapsto \epsilon^{\mathrm{tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mathrm{Tr}(BX))}$ for some $B \in H_m(q)$. Furthermore,

$$\tilde{L} = \{[A, C] := \mathrm{diag}({}^t\!A^{-1}, A, C) \mid A \in \mathbf{GL}(W), C \in \mathbf{SO}(\mathfrak{q}|_{\langle e_m, f_m\rangle})\} \\ \simeq \mathbf{GL}(m-1, q) \times \mathbf{SO}^-(2, q).$$

Clearly $L \geq \mathbf{SL}(m-1, q)$.

If q is odd, let $\mathcal{O}_1$ be the set of all λ_B with $\mathrm{rank}(B) = 2$. If q is even, let $\mathcal{O}_1$ be the set of all λ_B, where B is chosen so that $\mathrm{rank}(B - {}^t\!B) = 2$. Then $|\mathcal{O}_1| = (q^{m-1}-1)(q^{m-2}-1)/(q^2-1)$. Since $m > 3$, the subgroup L acts transitively on $\mathcal{O}_1$. For the rest of this section, we choose $B = \mathrm{diag}(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \underbrace{0, \ldots, 0}_{m-3})$ and let $B_1 = -(1/2)B$ if q is odd. If q is even, we choose $B = \mathrm{diag}(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \underbrace{0, \ldots, 0}_{m-3})$ and $B_1 = \mathrm{diag}(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \underbrace{0, \ldots, 0}_{m-3})$. Then $\mathrm{Tr}(BB_1) = 1$. Let $N := \{[I, X, 0] \in Q \mid \mathrm{Tr}(BX) = 0\}$. Then $N < Z(Q)$, so $N \lhd Q$, and $N \leq \mathrm{Ker}(\lambda_B)$. One can check that $Q/N = \tilde{Q}_1 \times Q_2$, with $\tilde{Q}_1$ a group of symplectic type of order q^{1+4}, and Q_2 is elementary abelian of order $q^{2(m-3)}$. In fact, we can write

$$\tilde{Q}_1 = \left\{[I, X, a]N \mid [I, X, a] \in Q, {}^t\!a = \begin{pmatrix} * & * & 0 & \ldots & 0 \\ * & * & 0 & \ldots & 0 \end{pmatrix}\right\},$$

$$Q_2 = \left\{[I, 0, b]N \mid [I, 0, b] \in Q, {}^t\!b = \begin{pmatrix} 0 & 0 & * & \ldots & * \\ 0 & 0 & * & \ldots & * \end{pmatrix}\right\}.$$

Let $K := \mathrm{Ker}(\lambda_B)$. Then $Q/K = Q_1 \times Q_2$, with Q_1 an extraspecial group of order pq^4. So any Q-irreducible character that contains λ_B when restricted to $Z(Q)$ has degree q^2. This is also true for any $\lambda_{B'} \in \mathcal{O}_1$.

Let ρ be the permutation character of $\tilde{G}$ on $\mathbf{P}_0(V)$. Let V_1 be the subspace of V spanned by $e_1, \ldots, e_{m-1}, f_1, \ldots, f_{m-1}$. Since $(V_1, Z(Q))$ plays the role of (V, Q) in §6, the $Z(Q)$-permutation character on $\mathbf{P}_0(V_1)$ is $\frac{2(q^{m-1}-1)}{q-1} \cdot 1_{Z(Q)} + (q+1)\sum_{\lambda_B \in \mathcal{O}_1} \lambda_B$. Also from §6 we see that for each $l = \langle f\rangle \in \mathbf{P}(W)$, $Z(Q)$ acts transitively on the set Ω_l of q^{m-2} singular 1-spaces $\langle f + u\rangle$ with $u \in W'$. Let g be any nonzero vector in $\langle e_m, f_m\rangle_{\mathbb{F}_q}$. Then for any $f \in W \setminus \{0\}$, $Z(Q)$ acts transitively on the set Ω'_f of q^{m-2} singular 1-spaces $\langle g + f + u\rangle$ with $u \in W'$, with the permutation character $\mathrm{Ind}_{Z(Q)_{f+u}}^{Z(Q)}(1_{Z(Q)_{f+u}})$, where $Z(Q)_{f+u} := \mathrm{Stab}_{Z(Q)}(f + u)$. Setting $l := \langle f\rangle_{\mathbb{F}_q}$, we see that $Z(Q)_{f+u}$ plays the role of Q_l in §6; in particular, $\mathrm{Ind}_{Z(Q)_{f+u}}^{Z(Q)}(1_{Z(Q)_{f+u}})$ consists of $1_{Z(Q)}$ and $q^{m-2}-1$ distinct characters $\lambda_B \in \mathcal{O}_1$. Thus the only characters of $Z(Q)$ that occur in $\rho|_{Z(Q)}$ are the trivial one and the ones from $\mathcal{O}_1$. Since $\tilde{G}$ acts on $\mathbf{P}_0(V)$ and $\tilde{P}$ acts transitively on $\mathcal{O}_1$, the multiplicity of λ_B in $\rho|_{Z(Q)}$ is the same for all

$\lambda_B \in \mathcal{O}_1$. It follows that

$$\rho|_{Z(Q)} = (q^2+1) \cdot \frac{q^{m-1}-1}{q-1} \cdot 1_{Z(Q)} + q^2(q+1) \sum_{\lambda_B \in \mathcal{O}_1} \lambda_B \ . \tag{34}$$

Direct computation shows that Q has $2(q^{m-1}-1)/(q-1)$ orbits on $\mathbf{P}_0(V)$. Since $Q/Z(Q)$ is abelian, we see that the linear Q-characters in $\rho|_Q$ constitute

$$\frac{2(q^{m-1}-1)}{q-1} \cdot 1_Q + \varsigma \tag{35}$$

where ς is a sum of $(q+1)(q^{m-1}-1)$ nontrivial linear characters of $Q/Z(Q)$. Remark 2.7 tells us that the Q-trivial component of $\rho|_{\tilde{P}}$ affords the character $2.1_{\tilde{P}} + 2\tau$. Since $\tilde{P}$ acts on the $Z(Q)$-fixed points, ς extends to a $\tilde{P}$-character which we also denote by ς (and recall that $(\varsigma|_Q, 1_Q)_Q = 0$). We have shown that the Q-component listed in (35) gives rise to the $\tilde{P}$-character

$$2 \cdot 1_{\tilde{P}} + 2\tau + \varsigma \ . \tag{36}$$

Next we analyze the nonlinear Q-characters inside $\rho|_Q$. For the chosen B, the stabilizer $\tilde{S} := \{a \in \tilde{L} \mid a \circ \lambda_B = \lambda_B\}$ equals

$$\left\{ [A, C] \mid A = \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix} \in \mathbf{GL}(m-1, q), X \in \mathbf{Sp}(2, q), C \in \mathbf{SO}^-(2, q) \right\}. \tag{37}$$

Let $\tilde{R} := Q : \tilde{S} = \mathrm{Stab}_{\tilde{P}}(\lambda_B)$.

From Section 2.1 and Tables 1 and 2 we have $\rho = 1 + \psi + \varphi$, where $\psi, \varphi \in \mathrm{Irr}(\tilde{G})$, $\psi(1) = (q^m+1)(q^{m-1}-q)/(q^2-1)$, $\varphi(1) = (q^{2m}-q^2)/(q^2-1)$, and ψ, φ are irreducible over G. It is clear that some irreducible constituent of $\psi|_{\tilde{P}}$ has to be nontrivial on $Z(Q)$. The formula (34) and the above discussion show that the $Z(Q)$-nontrivial part of $\psi|_{\tilde{P}}$ contains the $Z(Q)$-character $\sum_{\lambda_B \in \mathcal{O}_1} \lambda_B$. Recall that any $\lambda_B \in \mathcal{O}_1$ gives rise to a Q-irreducible character of degree q^2, and moreover $\psi(1) < 2q^2|\mathcal{O}_1|$. Hence the $Z(Q)$-nontrivial part of $\psi|_{\tilde{P}}$ affords the $Z(Q)$-character $q^2 \sum_{\lambda_B \in \mathcal{O}_1} \lambda_B$, and, as a $\tilde{P}$-character, it is equal to $\mathrm{Ind}_{\tilde{R}}^{\tilde{P}}(\omega)$, where ω is an $\tilde{R}$-irreducible character of degree q^2 (afforded by the λ_B-homogeneous component of $\psi|_{Z(Q)}$). As we mentioned above, $\omega|_Q$ is irreducible.

In the notation of (37), the map $[A, C] \mapsto X$ is a surjective homomorphism $\pi : \tilde{S} \to \mathbf{Sp}(2, q)$. Let μ_0 and μ_1 denote the trivial and the Steinberg character of $\mathbf{Sp}(2, q)$, respectively. Using π, we inflate μ_i to $\tilde{S}$, and then to an $\tilde{R}$-character that we also denote by μ_i, for $i = 0, 1$. By Lemma 3.2, $\omega\mu_i$ is irreducible over $\tilde{R}$ and also over R for any i.

Proposition 8.1. *In the above notation, there is an $\tilde{R}$-character ξ of degree 1 such that for $i = 0$, 1 we have $(\rho|_{\tilde{P}}, \sigma_i)_{\tilde{P}} > 0$, where $\sigma_i = \mathrm{Ind}_{\tilde{R}}^{\tilde{P}}(\omega\xi\mu_i)$. Also for $i = 0$, 1 we have $(\rho|_P, \mathrm{Ind}_R^P(\omega\xi\mu_i|_R))_P > 0$. Furthermore, σ_i is irreducible over $\tilde{P}$ and over P.*

Proof. The proofs are the same for $\tilde{P}$ and for P, so we will give the details for $\tilde{P}$. Denoting $\tilde{P}_1 := \mathrm{Stab}_{\tilde{G}}(\langle f_1\rangle)$ and $P_1 = G \cap \tilde{P}_1$, we have $\rho = \mathrm{Ind}_{\tilde{P}_1}^{\tilde{G}}(1_{\tilde{P}_1})$.

1) It suffices to prove that there is a $\tilde{P}$-character ξ such that $\omega\xi|_{\tilde{R}\cap\tilde{P}_1}$ contains $1_{\tilde{R}\cap\tilde{P}_1}$. Indeed, as in the proof of Proposition 4.1, this containment implies that $(\rho|_{\tilde{P}}, \sigma_0)_{\tilde{P}} > 0$. Similarly to (12), we also have

$$\tilde{R} \cap \tilde{P}_1 = (Q \cap P_1) \cdot (\tilde{S} \cap \tilde{P}_1). \tag{38}$$

For short we denote $R_1 := \tilde{R} \cap \tilde{P}_1$.

Recall that μ_1 is chosen to be the Steinberg character of $\mathbf{Sp}(2, q)$ pulled back to $\tilde{R}$. So μ_1 is trivial on $Q \cap P_1$ and it reduces to the restriction of the Steinberg representation of $\mathbf{Sp}(2, q)$ to its Borel subgroup $\{\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix} \mid a \in \mathbb{F}_q^\times, b \in \mathbb{F}_q\}$. It follows that $\mu_1|_{\tilde{R}\cap\tilde{P}_1}$ contains $1_{\tilde{R}\cap\tilde{P}_1}$. Hence, $\omega\xi\mu_1|_{\tilde{R}\cap\tilde{P}_1}$ contains $1_{\tilde{R}\cap\tilde{P}_1}$, and so $(\rho|_{\tilde{P}}, \sigma_1)_{\tilde{P}} > 0$ as in the case of σ_0. The irreducibility of σ_i now follows from Lemmas 3.2 and 3.3.

2) Recall that $Q/K = Q_1 \times Q_2$ for $K = \mathrm{Ker}(\lambda_B)$. Let $\tilde{S}_1 := \{[A, I_2] \in \tilde{S}\}$ and $\tilde{S}_2 := \{[I_{m-1}, C] \in \tilde{S}\} \simeq \mathbf{SO}^-(2, q)$; in particular, $\tilde{S} = \tilde{S}_1 \times \tilde{S}_2$. Then $Q_2 \simeq Q_{21} \oplus Q_{22}$ as $\tilde{S}_1$-modules. Here, $\tilde{S}_1$ acts on each Q_{2j} as $\mathbf{GL}(m-3, q)$ on its natural module. In particular, $\tilde{S}_1$ permutes the $q^{m-3} - 1$ nontrivial linear characters of Q_{2j}. Since $\omega|_{Q_1}$ is irreducible and Q_{2j} centralizes Q_1, $\omega|_{Q_{2j}}$ is scalar. But $q^{m-3} - 1 > 1$ by our assumption. So Clifford's theorem implies that $\omega|_{Q_{2j}}$ is trivial for $j = 1, 2$. Consequently, $Q_2 \le \mathrm{Ker}(\omega)$.

Next we further decompose $\tilde{S}_1 = \tilde{S}_{12} : \tilde{S}_{11}$ with

$$\tilde{S}_{11} := \left\{[A, I_2] \in \tilde{S} \mid A = \begin{pmatrix} X & 0 \\ 0 & I_{m-3}\end{pmatrix}\right\} \simeq \mathbf{Sp}(2, q)\ ,$$

$$\tilde{S}_{12} := \left\{[A, I_2] \in \tilde{S} \mid A = \begin{pmatrix} I_2 & Y \\ 0 & Z\end{pmatrix}\right\} \simeq \mathbb{F}_q^{2(m-3)} : \mathbf{GL}(m-3, q)\ .$$

We have already shown that $\mathrm{Ker}(\omega)$ contains K and Q_2. Direct computation now shows that $\tilde{S}_{12}$ centralizes Q_1 (modulo $\mathrm{Ker}(\omega)$). Since $\omega|_{Q_1}$ is irreducible, $\omega|_{\tilde{S}_{12}}$ is scalar; in particular, we may write $\tilde{S}_1 = \tilde{S}_{12} \times \tilde{S}_{11}$ modulo $\mathrm{Ker}(\omega)$. Denote by ν_{12} the central character of $\tilde{S}_{12}$ on ω.

3) Now we focus on the action (via conjugation) of $\tilde{S}_{11} \simeq \mathbf{Sp}(2, q)$ on the extraspecial p-group Q_1 of order pq^4. Notice that $\tilde{S}_{11}$ acts trivially on $Z(Q_1)$. (Indeed, $Z(\tilde{Q}_1)$ consists of $[I, xB_1, 0]$ with $x \in \mathbb{F}_q$, and ${}^tAB_1A = B_1$ for all $[A, C] \in \tilde{S}$.) We can identify $Q_1/Z(Q_1)$ with $M_2(\mathbb{F}_q)$ and $\tilde{S}_{11}$ with $\mathbf{Sp}(2, q)$ in such a way that $X \in \mathbf{Sp}(2, q)$ acts on $a \in M_2(\mathbb{F}_q)$ via $X \circ a = {}^tX^{-1}a$. We also consider the symplectic form $(a, b) \mapsto [a, b]$ on $M_2(\mathbb{F}_q)$ induced by taking commutators in $\tilde{Q}_1$.

Assume q is even. Fix the basis

$$u_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ u_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ v_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ v_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

of $M_2(\mathbb{F}_q)$. Then this is also a symplectic basis for $[\cdot,\cdot]$, i.e. $[u_i, u_j] = [v_i, v_j] = 0$, $[u_i, v_j] = \delta_{ij}$ (just notice that $[a, b] = a_{12}b_{21} + a_{11}b_{22} - a_{22}b_{11} - a_{21}b_{12}$ for matrices $a = (a_{ij})$ and $b = (b_{ij})$ in $M_2(\mathbb{F}_q)$). Furthermore, $X \in \mathbf{Sp}(2, q)$ acts on $\langle u_1, u_2\rangle$ as ${}^tX^{-1} \in \mathbf{GL}(2, q)$ acting on the column space $\mathbb{F}_q^2$, and X acts on $\langle v_1, v_2\rangle$ as $X \in \mathbf{GL}(2, q)$ acting on the column space $\mathbb{F}_q^2$. Thus the action of $\tilde{S}_{11}$ on Q_1 is just the restriction of the standard embedding $\mathbf{GL}(2, q) \hookrightarrow \mathbf{Sp}(4, q) \le \mathrm{Out}(Q_1)$ to the subgroup $\mathbf{Sp}(2, q) \simeq \mathbf{SL}(2, q)$. (For later use we let $T := M_2(\mathbb{F}_q)$ if q is even.)

Next assume q is odd. Then $[a, b] = 2(a_{11}b_{21} - a_{21}b_{11} - \kappa a_{12}b_{22} + \kappa a_{22}b_{12})$ for $a = (a_{ij})$ and $b = (b_{ij})$ in $M_2(\mathbb{F}_q)$. Let e_{ij} be the matrix in $M_2(\mathbb{F}_q)$ with the (i, j)-entry equal 1 and all others being 0, $1 \le i, j \le 2$. Then the Gram matrix of $[\cdot,\cdot]$ in the basis $\mathcal{E} = (e_{12}, e_{11}, e_{21}, e_{22})$ is $\begin{pmatrix} 0 & 0 & 0 & -2\kappa \\ 0 & 0 & 2 & 0 \\ 0 & -2 & 0 & 0 \\ 2\kappa & 0 & 0 & 0 \end{pmatrix}$. Choose $\theta \in \mathbb{F}_{q^2}$ such that $\theta^2 = \kappa$. Since $x^2 - \kappa$ is irreducible over $\mathbb{F}_q$, $\theta^{q-1} = -1$. Consider the 2-dimensional $\mathbb{F}_{q^2}$-space $T = \langle e, f\rangle$ with the hermitian form $u \circ v$, where $e \circ e = f \circ f = 0$, $e \circ f = 1$. Then the form $(u, v) \mapsto \mathrm{tr}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(-\theta(u \circ v))$ is nondegenerate symplectic, and it has the same Gram matrix $\begin{pmatrix} 0 & 0 & 0 & -2\kappa \\ 0 & 0 & 2 & 0 \\ 0 & -2 & 0 & 0 \\ 2\kappa & 0 & 0 & 0 \end{pmatrix}$ in the basis $\mathcal{B} = (e, \theta^{-1}e, f, \theta f)$ of the $\mathbb{F}_q$-space T. So we can identify $M_2(\mathbb{F}_q)$ with T, $\mathcal{E}$ with $\mathcal{B}$, and $[\cdot,\cdot]$ with this form. Under this identification, the action of $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ on $\mathcal{B}$ is given by the matrix $\begin{pmatrix} d & -c\theta^{-1} \\ -b\theta & a \end{pmatrix}$. Consequently, the action of $\tilde{S}_{11}$ on Q_1 is just the restriction of the standard embedding $\mathbf{GU}(2, q^2) \hookrightarrow \mathbf{Sp}(4, q) \le \mathrm{Out}(Q_1)$ to the subgroup $\mathbf{Sp}(2, q) \simeq \mathbf{SU}(2, q^2)$.

4) Now we study the restriction of ω to $Q_1 : (\tilde{S}_{11} \times \tilde{S}_2)$. Observe that if $[I, X, a] \in R_1$, then the first column of X and the first row of a are 0. It follows that $Z(Q) \cap R_1 \le K = \mathrm{Ker}(\lambda_B)$. Recall that ω is trivial on K and on Q_2. Hence, modulo $\mathrm{Ker}(\omega\mu_i)$, $Q \cap P_1$ is elementary abelian and it is just the subspace $Q_3 = \langle e_{21}, e_{22}\rangle_{\mathbb{F}_q}$ in $M_2(\mathbb{F}_q)$; and $\tilde{S}_{11} \cap \tilde{P}_1$ is just $S_3 = \{\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{F}_q^\times, b \in \mathbb{F}_q\}$. Under the identifications we made in 3), Q_3 is a maximal totally singular subspace of T that is stable under $S_3 \times \tilde{S}_2$.

Notice that $\omega|_{Q_3}$ is the regular representation of Q_3. (Recall ω is an irreducible complex character of degree q^2 of the extraspecial group Q_1 of order pq^4, and Q_3 is an elementary abelian subgroup of order q of Q_1 that trivially intersects $Z(Q_1)$.) In particular, $\omega|_{Q_3}$ contains 1_{Q_3} with multiplicity 1. Since $S_3\times\tilde{S}_2$ normalizes Q_3, $S_3\times\tilde{S}_2$ acts on the 1-dimensional subspace F of Q_3-fixed points inside ω.

Assume q is odd. For the extension ω_0 of $\omega|_{Q_1}$ to $Q_1:\mathbf{Sp}(4,q)$ as specified in [Gr, §13] we see that S_3 acts trivially on F. Indeed, the action of $\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix}\in S_3$ on the symplectic basis $(e_{21},e_{22},-\frac{1}{2}e_{11},\frac{1}{2\kappa}e_{12})$ is given by $\begin{pmatrix} a & 0 & b/2 & 0 \\ 0 & a & 0 & -b/2a \\ 0 & 0 & a^{-1} & 0 \\ 0 & 0 & 0 & a^{-1}\end{pmatrix}$, so it acts on the Q_3-fixed points inside $\omega|_Q$ via the quadratic character $\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix}\mapsto\left(\frac{a^2}{q}\right)=1$ (with $\left(\frac{\cdot}{q}\right)$ being the Legendre symbol), see [Gr, (13.2), (13.3)]. Our representation $\omega|_{Q_1:\tilde{S}_{11}}$ is $\omega_0|_{Q_1:\tilde{S}_{11}}$ tensored with a linear character say ν_3 of $\tilde{S}_{11}$. So we see that, on F, Q_3 acts trivially, S_3 acts via $\nu_3|_{S_3}$, and $\tilde{S}_2=\mathbf{SO}^-(2,q)\simeq \mathrm{Z}_{q+1}$ acts via some linear character say ν_2.

Assume q is even. For the extension ω_0 of $\omega|_{Q_1}$ to $Q_1:\mathbf{Sp}(4,q)$ as specified in [Gr, §12] we see that S_3 acts trivially on F. Indeed, the action of $\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix}\in S_3$ on the symplectic basis $(e_{21},e_{22},e_{12},e_{11})$ is given by $\begin{pmatrix} a & 0 & 0 & b \\ 0 & a & b & 0 \\ 0 & 0 & a^{-1} & 0 \\ 0 & 0 & 0 & a^{-1}\end{pmatrix}$, so it acts on the Q_3-fixed points inside $\omega|_Q$ trivially, see [Gr, (12.3)]. Our representation $\omega|_{Q_1:\tilde{S}_{11}}$ is $\omega_0|_{Q_1:\tilde{S}_{11}}$ tensored with a linear character say ν_3 of $\tilde{S}_{11}$. So we see that, on F, Q_3 acts trivially, S_3 acts via $\nu_3|_{S_3}$, and $\tilde{S}_2=\mathbf{SO}^-(2,q)\simeq \mathrm{Z}_{q+1}$ acts via some linear character say ν_2.

5) Now we define the linear character ξ as follows. First, ξ is trivial on Q. Recall that $\tilde{R}/Q=\tilde{S}=(\tilde{S}_{12}:\tilde{S}_{11})\times\tilde{S}_2$. Then we define ξ equal ν_{12}^{-1} on $\tilde{S}_{12}$, ν_3^{-1} on $\tilde{S}_{11}$, and ν_2^{-1} on $\tilde{S}_2$. The above discussion now implies that $\omega\xi|_{R_1}$ contains the trivial character 1_{R_1}, and so we are done. $\square$

Clearly, $1=(\sigma_i,\sigma_i)_{\tilde{P}}=(\sigma_i|_{\tilde{R}},\omega\xi\mu_i)_{\tilde{R}}$. In particular, $\sigma_0|_{Z(Q)}$ contains $\omega|_{Z(Q)}=q^2\lambda_B$ and $\sigma_1|_{Z(Q)}$ contains $\omega\mu_1|_Q=q^3\lambda_B$. It follows that

$$\sigma_0|_{Z(Q)}=q^2\sum_{\lambda_B\in\mathcal{O}_1}\lambda_B\ ,\sigma_1|_{Z(Q)}=q^3\sum_{\lambda_B\in\mathcal{O}_1}\lambda_B\ .$$

Proposition 8.2. *Assume $m \geq 4$, $(m,q) \neq (4,2), (5,2)$. The permutation character ρ of $\tilde{G} = \mathbf{SO}^-(2m,q)$ on $\mathbf{P}_0(V)$ decomposes into $\tilde{P}$-constituents as follows:*

$$\rho|_{\tilde{P}} = 2 \cdot 1_{\tilde{P}} + 2\tau + \varsigma + \sigma_0 + \sigma_1$$

where all the above $\tilde{P}$-constituents, except possibly ς, are irreducible over P. Furthermore, $\varsigma|_P$ is trivial on $Z(Q)$ and all P-irreducible constituents of $\overline{\varsigma}|_P$ have degree $\geq q^{m-1} - 1$.

Proof. We have shown that $\rho|_{\tilde{P}}$ contains the right-hand side of the decomposition, as σ_i is irreducible and $(\rho|_{\tilde{P}}, \sigma_i)_{\tilde{P}} > 0$ by Proposition 8.1. Comparing the restriction to $Z(Q)$ we get the decomposition. We have already noticed that $\varsigma|_Q$ is the sum of $(q+1)(q^{m-1}-1)$ nontrivial linear characters of $Q/Z(Q)$. Notice that P contains a subgroup $\mathbf{SL}(m-1,q)$, and $Q/Z(Q)$ is the direct sum of two natural $\mathbb{F}_q$-modules A_1, A_2 for this group. Assume γ is an irreducible P-constituent of $\overline{\varsigma}|_P$. Then γ must contain a linear character which is nontrivial on some A_i. Since $\mathbf{SL}(m-1,q)$ transitively permutes $q^{m-1}-1$ nontrivial linear characters of A_i, $\gamma(1) \geq q^{m-1}-1$. $\square$

Corollary 8.3. *Under the above notation we have $\psi|_{\tilde{P}} = \tau + \sigma_0$ and $\varphi|_{\tilde{P}} = 1_{\tilde{P}} + \tau + \varsigma + \sigma_1$.*

Proof. We will make use of Proposition 8.2. Clearly, $\psi|_{\tilde{P}}$ contains at least one $\tilde{P}$-irreducible character that is nontrivial on $Z(Q)$. Since $\psi(1) < \sigma_1(1)$, we see that $\psi|_{\tilde{P}}$ contains σ_0. It also contains τ by Lemma 2.6, so the corollary follows by degrees. $\square$

8.2. **Reduction mod ℓ.** Next we study the mod ℓ decomposition of the above characters of $\tilde{P}$ and P. Since we will need the result in the case $\ell|(q+1)$, we formulate the result only for that case. The decomposition of $\overline{\tau}$ has already been given in Lemma 2.5. We do not know the irreducible constituents of $\overline{\varsigma}$; however, as we mentioned in Proposition 8.2, all P-irreducible constituents of $\overline{\varsigma}|_P$ have degree $\geq q^{m-1}-1$.

Proposition 8.4. *Assume that $\ell|(q+1)$.*

(i) *$\overline{\sigma}_0$ is irreducible over $\tilde{P}$ and over P.*
(ii) *If $\ell \neq 2$, $\overline{\sigma}_1 = \sigma_{11} + \overline{\sigma}_0$, where $\sigma_{11} \in \mathrm{IBr}_\ell(\tilde{P})$, $\sigma_{11}(1) = (q-1)q^2|\mathcal{O}_1|$, and σ_{11} is irreducible over P.*
(iii) *If $\ell = 2$ then $\overline{\sigma}_1 = \sigma_{11}^+ + \sigma_{11}^- + \overline{\sigma}_0$, where $\sigma_{11}^\pm \in \mathrm{IBr}_\ell(\tilde{P})$, $\sigma_{11}^\pm(1) = (q-1)q^2|\mathcal{O}_1|/2$, and $\sigma_{11}^\pm$ is irreducible over P.*

Proof. (i) Note that $\overline{\omega\xi}$ is irreducible over $\tilde{R}$ and R. Since Q is an elementary abelian p-group and $\tilde{L}$, L act transitively on $\mathcal{O}_1$, $\overline{\sigma}_0$ is irreducible (over both $\tilde{P}$ and P) for any $\ell \neq p$.

(ii) Assume $\ell \neq 2$. Since $\ell|(q+1)$, $\overline{\mu}_1 = \mu_{11} + \overline{\mu}_0$, where μ_{11} is an irreducible Brauer character of degree $q-1$ for $j=1$. (Recall that μ_1 is the Steinberg character of $\mathbf{SL}(2,q)$ inflated to $\tilde{R}$.) By Lemma 3.3, $\sigma_{11} := \mathrm{Ind}_{\tilde{R}}^{\tilde{P}}(\overline{\omega\xi}\mu_{11})$ is irreducible, and $\overline{\sigma}_1 = \sigma_{11} + \overline{\sigma}_0$.

(iii) Assume $\ell = 2$ (in particular q is odd). Then $\overline{\mu}_1 = \mu_{11}^+ + \mu_{11}^- + \overline{\mu}_0$, where $\mu_{11}^\pm$ are irreducible Brauer characters of degree $(q-1)/2$. Now the same argument as above applies to give the irreducibility of the $\sigma_{11}^\pm$. $\square$

We can now give the decomposition of $\overline{\psi}$ into irreducible Brauer characters of G and $\tilde{G}$.

Corollary 8.5. *Assume $m \geq 4$ and $\ell|(q+1)$.*

(i) $\beta(U') = \begin{cases} \beta(X) \text{ if } m \text{ is even} \\ 1 + \beta(X) \text{ if } m \text{ is odd} \end{cases}$, $\beta(U')|_{\tilde{P}} = \overline{\tau} + \overline{\sigma}_0$, *and X is simple.*

(ii) $\overline{\psi} = \beta(U')$.

Proof. Part (i) is Corollary 2.10 and Lemma 3.1. Then (ii) is immediate from Proposition 8.4. $\square$

Remark 8.6. Let $\kappa_{2,m}$ be equal 1 if $2|m$ and 0 otherwise as in §6. Then Corollary 8.5 implies that the simple G-module X has dimension

$$\psi(1) - 1 + \kappa_{2,m} = \frac{(q^m+1)(q^{m-1}-q)}{q^2-1} - 1 + \kappa_{2,m}.$$

According to [Ho], this dimension is exactly the smallest dimension of nontrivial irreducible projective representations of $\Omega^-(2m,q)$ (if one assumes $m \geq 4$, $\ell|(q+1)$, and $(m,q) \neq (4,4)$, $(5,3)$).

To proceed further, we must examine the $\mathbb{F}\mathbf{G}$-submodule lattice of $\mathbb{F}^{\mathbf{P}_0}$. There are three cases to be treated in the next three lemmas. When combined with Corollary 2.10 and Remark 2.12 these results yield the complete picture in each case.

Lemma 8.7. *Suppose m is even and ℓ is an odd prime divisor of $q+1$. Then $\mathbb{F}^{\mathbf{P}_0} = \mathbb{F}\mathbf{1} \oplus (\mathbb{F}^{\mathbf{P}_0})'$ and $(\mathbb{F}^{\mathbf{P}_0})'$ is a uniserial $\mathbb{F}\mathbf{G}$-module with series X, D, X, with $U' \cong X$. Moreover $\mathcal{C} = U'^{\perp}$.*

Proof. First note that since $|\mathbf{P}_0|$ is coprime to $q+1$ we have

$$\mathbb{F}^{\mathbf{P}_0} = \mathbb{F}\mathbf{1} \oplus (\mathbb{F}^{\mathbf{P}_0})' \tag{39}$$

and this decomposition is orthogonal. So we must determine the structure of $(\mathbb{F}^{\mathbf{P}_0})'$. Corollary 8.5 shows that $\beta(U') = \beta(X) = \overline{\psi}$, so by Proposition 8.4 the restriction to $\tilde{P}$ of $U'^{\perp}/U'$ has Brauer character $2 + \sigma_{11} + \overline{\varsigma}$, where σ_{11} is irreducible and every constituent of $\overline{\varsigma}$ is nonlinear and has $Z(Q)$ in its kernel. Using (39), the self-duality of $U'^{\perp}/U'$ and the fact that a simple $\mathbb{F}\mathbf{G}$-module cannot have $Z(Q)$ in its kernel, it follows that the possiblities for $U'^{\perp}/U'$ are $\mathbb{F}\oplus\mathbb{F}\oplus D$, with D simple and $\beta(D)|_{\tilde{P}} = \sigma_{11}+\overline{\varsigma}$, or $\mathbb{F} \oplus D$, with D simple and $\beta(D)|_{\tilde{P}} = 1 + \sigma_{11} + \overline{\varsigma}$. We shall show that the second alternative holds. Consider $\mathcal{C}/U'$. Since $U' \subseteq \mathcal{C} \cap \mathcal{C}^{\perp}$, and since the commutator subgroup of $\mathbf{G}$ acts transitively on maximal singular subspaces, Lemma 2.3(iii) implies that $\mathcal{C}/U'$ must have D as a summand. Now $\mathcal{C}/U'$ is a quotient of $\mathbb{F}^{\Omega_{max}}$, hence so is D. But then Frobenius reciprocity implies that $\tilde{P}$ has a fixed points on D, contradiction. This shows that $U'^{\perp}/U' \cong \mathbb{F}\mathbf{1} \oplus D$. It remains to show that $\mathcal{C} = U'^{\perp}$, for which it suffices to show $\mathcal{C}/U' = U'^{\perp}/U'$. We have shown that $U'^{\perp}/U'$ has three composition factors $X \cong U'$, D and $\mathbb{F}$. We have already seen that $\mathcal{C}/U'$ has D as a composition factor. By Lemma 2.4(i), and the fact that $U' \subset \mathcal{C}'$ we also see that $\mathcal{C}/U'$ has a trivial quotient, so we are done. □

Lemma 8.8. *Suppose m is odd and ℓ is an odd prime divisor of $q+1$. Then the $\mathbb{F}\mathbf{G}$-module $U'^{\perp}/U' = \mathcal{C}/U' \cong \mathbb{F} \oplus D$ where D is a nontrivial simple module.*

Proof. Corollary 8.5 shows that $\beta(U') = 1 + \beta(X) = \overline{\psi}$, so by Proposition 8.4 the restriction to $\tilde{P}$ of $U'^{\perp}/U'$ has Brauer character $2 + \sigma_{11} + \overline{\varsigma}$, where σ_{11} is irreducible and every constituent of $\overline{\varsigma}$ is nonlinear and has $Z(Q)$ in its kernel. Therefore, the $\mathbf{G}$-composition factors of $U'^{\perp}/U'$ are $\mathbb{F}$ (once or twice), and a simple module D. We wish to show that there is only one trivial factor. If there are two, then $\tilde{P}$ has no fixed points on D. By Lemma 2.3(iii), we know that $\mathcal{C}/U'$ has a nontrivial composition factor D. Suppose $U'^{\perp}/U'$ has 3 composition factors $\mathbb{F}$, $\mathbb{F}$ and D. Then since $\tilde{P}$ fixes no points on D, we see that D is not a quotient of $\mathcal{C}/U'$, so in any ascending composition series of $\mathcal{C}/U'$, hence of $U'^{\perp}/U'$, there is a trivial factor after D. By the self-duality of $U'^{\perp}/U'$, the same is true of any descending composition series. Thus, if $U'^{\perp}/U'$ has 3 composition factors then it is uniserial with series $\mathbb{F}$, D, $\mathbb{F}$. But this implies that the whole module is uniserial and as a consequence we obtain $U \subset U^{\perp}$, contrary to Table 3. Thus, $U'^{\perp}/U'$ has just two composition factors $\mathbb{F}$ and D, and by self-duality, we have $U'^{\perp}/U' \cong \mathbb{F} \oplus D$. Finally, by Lemma 2.4(i), this module is also equal to $\mathcal{C}/U'$. □

Lemma 8.9. *Suppose $\ell = 2$. Then the $\mathbb{F}\mathbf{G}$-module $U'^{\perp}/U' = \mathcal{C}/U'$ has radical and socle series $\mathbb{F}$, $D_1 \oplus D_2$, $\mathbb{F}$. The conformal group $\hat{G} = \mathbf{CO}^{-}(2m,q)$ acts irreducibly on the sum $D_1 \oplus D_2$.*

Proof. Table 3 shows that $U \subset U^{\perp}$ in this case. Then, by Proposition 8.4 and Corollary 8.5 we see that restriction to $\tilde{P}$ of the Brauer character of $U^{\perp}/U$ is $\sigma_{11}^{+} + \sigma_{11}^{-} + \overline{\varsigma}$. We can show that the stabilizer in $\hat{G} = \mathbf{CO}^{-}(2m,q)$ of the maximal totally singular subspace W' contains an element which induces a non-inner automorphism of the $\mathbf{Sp}(2,q)$ factor of $\tilde{R}$ (in the notation of Proposition 8.1), whence this element fuses σ_{11}^{+} and σ_{11}^{-}. Since each constituent of $\overline{\varsigma}$ is nonlinear and has $Z(Q)$ in its kernel, it now follows that $U^{\perp}/U$ must be a simple module for $\hat{G}$, which we will denote by D. It is proved in [T] that when restricted to $\mathbf{G}$, the module D splits into two simple summands D_1 and D_2. These two summands remain irreducible for G, since by Proposition 8.4 the Brauer characters $\sigma_{11}^{\pm}$ do. By Lemma 2.13(iii), $U^{\perp}/U$ has composition factors D_1 and D_2. Since σ_{11}^{+} and σ_{11}^{-} are fused by the conformal group $\hat{G}$, D_1 and D_2 are conjugate under $\hat{G}$. Because $\hat{G}$ acts on $U^{\perp}/U$, this module is simple for $\hat{G}$, hence semisimple for $\mathbf{G}$. Consider the subquotient $\mathcal{C}/\mathcal{C}\cap\mathcal{C}^{\perp}$ of $U'^{\perp}/U'$. By Lemma 2.3(iii), its dimension is at least 3, since the commutator subgroup of $\mathbf{G}$ is transitive on maximal singular subspaces. Therefore it must have a nontrivial composition factor and since it is a module for the conformal group, it must have both D_1 and D_2 as composition factors. Since $\tilde{P}$ has no fixed points on D_i, there are no quotients of $\mathcal{C}/\mathcal{C}\cap\mathcal{C}^{\perp}$ isomorphic to D_i. By duality, we see there are no submodules of $\mathcal{C}/\mathcal{C}\cap\mathcal{C}^{\perp}$ isomorphic to D_i. Therefore $\mathcal{C}/\mathcal{C}\cap\mathcal{C}^{\perp}$ must have at least two other composition factors than the D_i. This forces $\mathcal{C}/\mathcal{C}\cap\mathcal{C}^{\perp} = U'^{\perp}/U'$ and also proves that this module has the stated radical and socle series. □

The mod ℓ decomposition of φ is easily read off from the above three lemmas. We summarize the information here.

Corollary 8.10. (i) *If ℓ is odd, then $\overline{\varphi} = \beta(U') + \beta(D)$, where D is simple and $\beta(D)|_{\tilde{P}} = 1 + \overline{\varsigma} + \sigma_{11}$.*

(ii) *If $\ell = 2$ then $\overline{\varphi} = 1 + \beta(U') + \beta(D)$, where D is simple for the conformal group $\hat{G} := \mathbf{CO}^{-}(2m,q)$ and decomposes as the direct sum $D_1 \oplus D_2$ for any subgroup between $G = \Omega^{-}(2m,q)$ and $\mathbf{GO}^{-}(2m,q)Z(\hat{G})$.* □

The submodule structures we have determined are depicted in Figures 7 and 8.

8.3. **m=3.** In the case $m = 3$ we will make use of the well-known correspondence between the geometry of a 6-dimensional space V over $\mathbb{F}_q$ carrying a nonsplit nonsingular quadratic form and that of a 4-dimensional hermitian space E over $\mathbb{F}_{q^2}$. In order to avoid a clash of notation, we will use a superscript "u" for unitary group objects, which had previously been denoted by the unadorned symbols in Section 4, reserving the unadorned symbols for orthogonal group objects. Thus, the action of $E = V^{\mathrm{u}}$ and the action of $\mathbf{G}^{\mathrm{u}} = \mathbf{GU}(E)$ on $\mathbf{P}_0^{\mathrm{u}} = \mathbf{P}_0(E)$ has permutation character $\rho^{\mathrm{u}} = 1 + \varphi^{\mathrm{u}} + \psi^{\mathrm{u}}$. It will be helpful to recall how the above correspondence arises (when q is odd). Exterior multiplication defines a nonsingular $\mathbf{SL}(E)$-invariant symmetric bilinear form on $\wedge^2(E)$. This defines a mapping $\mathbf{GL}(E) \to \mathbf{CO}(\wedge^2(E))$ in which the conformal scalar of an element in $\mathbf{GL}(E)$ is its determinant. The hermitian form on E defines an isomorphism between its Galois conjugate $E^{(q)}$ and its dual $\mathrm{Hom}_{\mathbb{F}_{q^2}}(E, \mathbb{F}_{q^2})$, which induces an $\mathbb{F}_{q^2}\mathbf{SU}(E)$-isomorphism

$$j : (\wedge^2(E))^{(q)} \to \wedge^2(E). \tag{40}$$

Viewing j as an $\mathbb{F}_q$-endomorphism, we have $j^2 = \mathrm{id}$, whence $\mathbf{SU}(E)$ has an 6-dimensional orthogonal representation on the $\mathbb{F}_q$-subspace $V \subset \wedge^2(E)$ of vectors fixed by j. If we use the notation of Section 4 and identify $\wedge^2(E)$ with skew-symmetric matrices in the usual way, then, by tracing through the isomorphisms above, one sees that V is the set of matrices

$$\begin{pmatrix} 0 & a & \alpha & b\varepsilon \\ -a & 0 & c\varepsilon & \alpha^q \\ -\alpha & -c\varepsilon & 0 & d \\ -b\varepsilon & -\alpha^q & -d & 0 \end{pmatrix}, \quad (a,\, b,\, c,\, d \in \mathbb{F}_q,\, \alpha \in \mathbb{F}_{q^2}) \tag{41}$$

where ε is a fixed nonzero element of $\mathbb{F}_{q^2}$ such that $\varepsilon^q + \varepsilon = 0$. It is easily seen using (41) that the quadratic form on V is of Witt index 2, so $\mathbf{CO}(V) \simeq \mathbf{CO}^-(6,q)$. Also, using (41) one can check that the image of $\mathbf{GU}(V)$ in $\mathbf{CO}(\wedge^2(E))$ lies in $\mathbf{CO}(V)\mathbb{F}_{q^2}^\times$, so its image in $\mathbf{PCO}(\wedge^2(E))$ lies in $\mathbf{PCO}(V)$. The group $\mathbf{PCO}(V)$ contains the simple group $\mathbf{P}\Omega(V)$ as a subgroup of index $2.(4, q+1)$ and the quotient is dihedral of order 4 or 8, depending on the congruence of q mod 4. The image of $\mathbf{PGU}(E)$ in $\mathbf{PCO}(V)$ is a subgroup of index 2 which does not interchange the two classes of maximal totally singular subspaces in $\wedge^2(E)$, while the two classes are interchanged by $\mathbf{GO}(V)$. Therefore the image of $\mathbf{PGU}(E)$ and $\mathbf{PGO}(V)$ are distinct subgroups of index 2 in $\mathbf{PCO}(V)/\mathbf{P}\Omega(V)$. The first is cyclic and the second is of exponent 2.

Let $\hat{G} = \mathbf{CO}(V) \simeq \mathbf{CO}^-(6,q)$ and let ρ be the permutation character of $\hat{G}$ on $\mathbf{P}_0(V)$. Let $\mathbf{G} = \mathbf{GO}(V)$ and $G = \Omega(V)$.

Corollary 8.11. *Assume $\ell|(q+1)$.*

(i) $\beta(U') = 1 + \beta(X) = \overline{\psi}$, *where X is simple for G.*

(ii) $\overline{\varphi} = \beta(U') + \beta(D)$, *where D has the following properties. If $\ell \neq 2$ then D is simple for G. If $\ell = 2$, then D is simple for $\hat{G}$. If K is a subgroup of $\hat{G}$ containing G, then D splits into a direct sum $D_1 \oplus D_2$ of simple modules if and only if $K \leq Z(\hat{G})\mathbf{G}$.*

(iii) ${U'}^\perp/U' = \mathbb{F} \oplus D$. *The submodule lattice is the same as when $m \geq 4$ and depicted in Figures 7 and 8.*

Proof. Under the identification of $G/Z(G) \simeq \mathbf{P}\Omega^-(6,q)$ with $G^{\mathrm{u}}/Z(G^{\mathrm{u}}) \simeq \mathbf{PSU}(4,q^2)$ described above, the character ψ is identified with the unipotent Weil character of G^{u}. Since $\ell|(q+1)$, it is known (cf. [HM] for instance) that over G^{u}, $\overline{\psi}$ is the sum of the trivial character and an irreducible character. Then (i) follows using Corollary 2.10. To prove (ii), we note that $\varphi = \varphi^{\mathrm{u}}$ and $X^{\mathrm{u}} = D$ under the above correspondence. In view of the fact that $\mathbf{PGU}(E)$ acts irreducibly on X^{u}, its image in $\mathbf{PCO}(V)$ acts irreducibly on D, hence so does $\mathbf{PCO}(V)$. Therefore, the mod ℓ decomposition of φ with respect to $\hat{G}$ is given by Corollary 4.5(iii), (iv). Finally, it is proved in [T] that D splits into two as an $\mathbb{F}\mathbf{G}$-module when $\ell = 2$. Since $\mathbf{PGO}(V)$ has index 2 in $\mathbf{PCO}(V)$, we have (ii). Then (iii) follows from self-duality and Remark 2.12. $\square$

Remark 8.12. As we mentioned in an earlier remark (4.7), our results in this subsection are related to the topic of *generalized quadrangles* [Ti]. The singular 1-spaces and 2-spaces of V, with the natural notion of incidence, form the *points* and *lines* of an incidence system $\mathcal{H}(q)$ known as a generalized quadrangle. One can also form the *dual* generalized quadrangle $\mathcal{H}^{\mathrm{u}}(q)$ by interchanging the roles of the points and lines. It is well known that $\mathcal{H}^{\mathrm{u}}(q)$ is isomorphic to the incidence system obtained by taking the points and lines to be the singular 1-spaces and 2-spaces respectively of E. The automorphism group Γ of $\mathcal{H}(q)$ (and $\mathcal{H}^{\mathrm{u}}(q)$) contains $\mathbf{PCO}(V)$ and inside this subgroup are copies of $\mathbf{PGU}(E)$, $\mathbf{PGO}(V)$ and $\mathbf{P}\Omega(V) \simeq \mathbf{PSU}(E)$ as described above. The duality of quadrangles tells us that the $\mathbb{F}\Gamma$-permutation module on the set $\mathbf{P}_0^{\mathrm{u}} = \mathbf{P}_0(E)$ of points of $\mathcal{H}^{\mathrm{u}}(q)$ is isomorphic to the $\mathbb{F}\Gamma$-permutation module on the lines of $\mathcal{H}(q)$ and vice versa. Therefore, it is natural to ask for the $\mathbb{F}\Gamma$-submodule lattices of these permutation modules on points and lines.

For $\mathcal{H}(q)$, one also wishes to find the submodule lattices with respect to subgroups of $\mathbf{PCO}(V)$ containing $\mathbf{P}\Omega(V)$ and for $\mathcal{H}^{\mathrm{u}}(q)$ to find the submodule lattice with respect to $\mathbf{PGU}(E)$ and $\mathbf{PSU}(E)$. The complete answer for Γ, $\mathbf{PGU}(E)$, $\mathbf{PCO}(V)$ can be read off the results we have proved in Sections 4 and 8 (Figures 1, 7 and 8). We observe that nearly all modules shown in these figures are in fact modules for Γ, as can be seen from the geometric nature of their definitions. Thus, the main question is whether the variants of the unitary and orthogonal groups act irreducibly on the $\mathbb{F}\Gamma$-composition factors. When ℓ is odd, our results show that all groups between Γ and $\mathbf{P}\Omega(V)$ act irreducibly on every $\mathbb{F}\Gamma$-composition factor, and the submodule structures are the same for all variants. When $\ell = 2$, $\mathbf{PGU}(E)$ acts irreducibly on $X^{\mathrm{u}} = D$, while D decomposes as a sum of two simple submodules D_1 and D_2 for subgroups of $\mathbf{PCO}(V)$ between $\mathbf{PGO}(V)$ and $\mathbf{P}\Omega(V)$. In addition the module Y^{u} is simple for $\mathbf{PGU}(E)$, hence also for $\mathbf{PCO}(V)$, but we know only that it has at most two simple summands for $\mathbf{PGO}(V)$.

In [BWH] the module $\mathbb{F}^{\mathbf{P}_0}$ for the case $\ell = 2$, $m = 2$ is investigated and a conjecture made about its submodule lattice as a module for $\mathbf{PGU}(E)$. The above shows that the conjecture is correct. In the same paper, a conjecture is also made about the module $\mathbb{F}^{\mathbf{P}_0^{\mathrm{u}}}$ as a module for $\mathbf{PGO}(V)$. We see that this conjecture is correct for the action of $\mathbf{PGU}(E)$ but that, strictly speaking, it is wrong for $\mathbf{PGO}(V)$, where we have to enlarge the group to $\mathbf{PCO}(V)$ to obtain the given submodule structure.

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